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OSCILLATION ANALYSIS FOR NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH SEVERAL DELAYS AND FORCING TERM

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Abstract. In this paper, sufficient conditions are obtained for the oscillation of the nonlinear neutral forced differential equations of second-order with several delays of the form

(E)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]\right]+\sum_{i=1}^{m}q_i(t)H\big(x(t-\sigma_i)\big)=f(t),\quad t\geq t_0>0,$$

under the assumptions $\int^{\infty}\frac{1}{r(\eta)}\mathrm{d}\eta=\infty$ and $\int^{\infty}\frac{1}{r(\eta)}\mathrm{d}\eta<\infty$ for various ranges of the bounded neutral coefficient p. Also, an attempt is made to discuss existence of bounded positive solutions of (E). Further, one illustrative example showing the applicability of the new results is included.

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Key words. Contraction principle, delay, existence of positive solution, neutral differential equations, non-linear, nonoscillation, oscillation.

1. INTRODUCTION

The neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines (see e.g. [5]). In this paper, we restrict our attention to study (E), which includes a class of nonlinear functional differential equations of neutral type.

There have been many investigations into the oscillation and nonoscillation of second order nonlinear neutral delay differential equations (see e.g. [1, 2, 6], [7-15], [19-23]. However, the study of oscillatory and asymptotic behaviour of the solutions of (E) has received much less attention, which is mainly due to the technical difficulties arising in its analysis. In what follows, we provide some background details that motivated this study. In [16], Santra has

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considered

$$(E_1) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=f(t),$$

and

$$(E_2) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=0.$$

He has established sufficient conditions for the oscillation of the solutions of (E_1) and (E_2) for $|p(t)| < +\infty$, when H is linear, sublinear and superlinear. Also, he has studied the existence of a bounded positive solution of (E_1). In [17], Santra has studied necessary and sufficient conditions for the asymptotic behaviour of (E_2) for various ranges of the bounded neutral coefficient p. In [14], Pinelas and Santra have established necessary and sufficient conditions for the oscillation of (E_2) for $|p(t)| < \infty$. In [18], Santra has obtained sufficient conditions for the oscillatory and asymptotic behaviour of the homogeneous counterpart of (E) for different ranges of p. In an another paper [8], Karpuz and Santra have studied sufficient conditions for the oscillatory and asymptotic behaviour of the homogeneous counterpart of (E) with variable delays for $|p(t)| < \infty$. Many references regarding some applications of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x(t)+p(t)x(t-\tau)\big]+q(t)H\big(x(t-\sigma)\big)=0, \quad t \ge t_0 > 0,$$

can be found in [3] and [5]. In this direction, we refer the reader to some of the works on equation (E) for single constant delay (see e.g. [6, 7, 9, 12, 19, 20, 23]) or single variable delay (see e.g. [1, 2, 6, 10, 11]). All of them established sufficient conditions for the oscillation of the solutions of equation (E), only under the assumption $\int_0^\infty \frac{\mathrm{d}\eta}{r(\eta)} = \infty$ and only for $0 \le p(t) \le 1$.

Hence, in this work, an attempt is made to study the oscillatory behaviour of the solutions of a class of nonlinear neutral second order delay differential equations of the form

(1)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]\right]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=f(t), \quad t \ge t_0 > 0,$$

where

- (A_1) $\tau, \sigma_i \in \mathbb{R}_+ = (0,+\infty)$, $p \in C([0,\infty),\mathbb{R})$, $q_i, r \in C(\mathbb{R}_+,\mathbb{R}_+)$, $i = 1,2,\ldots,m$, $f \in C(\mathbb{R},\mathbb{R})$;
- (A_2) $H \in C(\mathbb{R},\mathbb{R})$ is nondecreasing with $uH(u) > 0$ for $u \neq 0$.

This investigation on the oscillatory behavior of the solutions of (1) depends on various ranges of the bounded neutral coefficient p and follows two possible conditions:

- (C_1) $\int^\infty \frac{1}{r(\eta)}\mathrm{d}\eta = \infty$,

(C_2) $\int^{\infty} \frac{1}{r(\eta)} \mathrm{d}\eta < \infty$.

By a solution of (1) we understand a function $x \in C([-\rho, \infty), \mathbb{R})$ such that

$$z(t) = x(t) + p(t)x(t-\tau) \tag{2}$$

is twice continuously differentiable, $rz'(t)$ is once continuously differentiable and equation (1) is satisfied for $t \geq t_0 + \rho$, where $\rho = \max\{\tau, \sigma_i\}$ for $i = 1, 2, \ldots, m$, and $\sup\{|x(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

Remark 1.1. When the domain is not specied explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. SUFFICIENT CONDITIONS FOR OSCILLATION

In this section, sufficient conditions are obtained for the oscillation of the solutions for nonlinear second order forced neutral differential equations with several delays of the form (1). We need to work with the following conditions in the sequel:

$(A3)$ there exists $F \in C(\mathbb{R}, \mathbb{R})$ such that $F(t)$ changes sign,

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$$

and $f(t) = \big(rF'\big)'(t)$;

$(A4)$ $F^+(t) = \max\{F(t), 0\}$ and $F^-(t) = \max\{-F(t), 0\}$;

$(A5)$ there exists $\lambda > 0$ such that

$$H(u) + H(v) \geq \lambda H(u+v) \quad \text{for } u, v \geq 0$$

and

$$H(u) + H(v) \leq \lambda H(u+v) \quad \text{for } u, v \leq 0;$$

$(A6)$

$$H(uv) \leq H(u)H(v) \quad \text{for } u, v \geq 0$$

and

$$H(uv) \geq H(u)H(v) \quad \text{for } u, v \leq 0.$$

2.1. Oscillation under the condition (C1)

Throughout this discussion we will assume that

$$w(t) = z(t) - F(t) \quad \text{for} \quad t \geq t_0 > 0. \tag{1}$$

Lemma 2.1. *Assume that $(C1)$, $(A1)$ and $(A2)$ hold. Let x be an eventually positive solution of (1). If w defined by (1) is eventually positive, then w satisfies*

$$w'(t) > 0 \quad \textit{and} \quad (rw')'(t) < 0 \quad \textit{for all large } t. \tag{2}$$

Proof. Suppose that $x(t) > 0$ and $w(t) > 0$ for $t \geq t_1$, where $t_1 \geq t_0 > 0$. So, we may assume without loss of generality that $x(t-\sigma_i) > 0$ for $t \geq t_1$ and $i = 1, .., m$. From (1), (2), (1) and $(A2)$, it follows that

$$(3) \qquad (rw')'(t) = -\sum_{i=1}^{m} q_i(t) H\left(x(t-\sigma_i)\right) < 0 \quad \text{for } t \geq t_1.$$

Consequently, rw' is nonincreasing on $[t_1, \infty)$ and thus either $w'(t) < 0$ or $w'(t) > 0$ for $t \geq t_2$, where $t_2 \geq t_1$. If $w'(t) < 0$, then there exists $\varepsilon > 0$ such that $r(t)w'(t) \leq -\varepsilon$ for $t \geq t_2$, which yields, upon integration over $[t_2, t) \subset [t_2, \infty)$, after dividing through by r, that

$$(4) \qquad w(t) \leq w(t_2) - \varepsilon \int_{t_2}^{t} \frac{1}{r(\eta)} \mathrm{d}\eta \quad \text{for } t \geq t_2.$$

In view of $(C1)$, letting $t \to \infty$ in (4) yields $w(t) \to -\infty$, which is a contradiction. Therefore, $w'(t) > 0$ for $t \geq t_2$. This completes the proof. □

Remark 2.2. It follows from Lemma 2.1 that $\lim_{t\to\infty} w(t) > 0$, i.e., there exists $\varepsilon > 0$ such that $w(t) \geq \varepsilon$ for all large t.

Lemma 2.3. *Assume that* $(C1)$, $(A1)$ *and* $(A2)$ *hold. Let* x *be an eventually positive solution of* (1). *If* w *defined by* (1) *is bounded, then* w *satisfies* (2) *for all large* t.

Proof. The proof can be obtained from the proof of Lemma 2.1. □

Theorem 2.4. *Let* $0 \leq p(t) \leq p < \infty$ *for* $t \in \mathbb{R}_+$, *where* p *is a constant. Assume that* $(C1)$ *and* $(A1)-(A6)$ *hold. Furthermore, assume that the following conditions*

$(A7)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} Q(\eta) H\left(F^+(\eta-\sigma_i)\right) \mathrm{d}\eta = \infty$

and

$(A8)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} Q(\eta) H\left(F^-(\eta-\sigma_i)\right) \mathrm{d}\eta = \infty$

hold, where $Q_i(t) = \min\{q_i(t), q_i(t-\tau)\}$, $t \geq \tau$. *Then every solution of* (1) *oscillates.*

Proof. Suppose the contrary holds, i.e., x is a nonoscillatory solution of (1). Then, there exists $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq t_1$. Assume that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma_i) > 0$ for $t \geq t_1$ and $i = 1, 2, .., m$. Proceeding as in the proof of Lemma 2.1, we see that rw' is nonincreasing and w is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $w(t) < 0$ for $t \geq t_2$. So, $0 < z(t) < F(t)$ for $t \geq t_2$, which is a contradiction.

Case 2. Let $w(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2) holds for $t \geq t_3$, where $t_3 \geq t_2$. Note that $\lim_{t\to\infty}(rw')(t)$ exists for $t \geq t_3$. Ultimately, $z(t) > F(t)$

and hence $z(t) > \max\{0, F(t)\} = F^+(t)$ for $t \geq t_4$, where $t_4 \geq t_3$. Therefore, (3) becomes

$$0 = \big(rw'\big)'(t) + H(p)\big(rw'\big)'(t-\tau) + \sum_{i=1}^{m}\left[q_i(t)H\big(x(t-\sigma_i)\big) + H(p)q_i(t-\tau)H\big(x(t-\tau-\sigma_i)\big)\right]$$

for $t \geq t_4$ and because of $(A5)$, $(A6)$ and $z(t) \leq x(t) + px(t-\tau)$ we find that

$$\begin{aligned} 0 &\geq \big(rw'\big)'(t) + H(p)\big(rw'\big)'(t-\tau) + \sum_{i=1}^{m} Q_i(t)\left[H\big(x(t-\sigma_i)\big) + H\big(px(t-\tau-\sigma_i)\big)\right] \\ &\geq \big(rw'\big)'(t) + H(p)\big(rw'\big)'(t-\tau) + \lambda\sum_{i=1}^{m} Q_i(t)H\big(z(t-\sigma_i)\big) \qquad (5) \\ &\geq \big(rw'\big)'(t) + H(p)\big(rw'\big)'(t-\tau) \\ &\quad + \lambda\sum_{i=1}^{m} Q_i(t)H\big(F^+(t-\sigma_i)\big) \quad \text{for} \quad t \geq t_4. \end{aligned}$$

Integrating (5) over the interval $[t_4, t) \subset [t_4, \infty)$, we get

$$\lambda\left[\int_{t_4}^{t}\sum_{i=1}^{m} Q_i(\eta)H\big(F^+(\eta-\sigma_i)\big)\mathrm{d}\eta\right] \leq \left[(rw')(t_4) + H(p)\big((rw')(t_4-\tau)\big)\right]$$

for all $t \geq t_4$. This contradicts $(A7)$. Thus, $x(t) > 0$ for $t \geq t_1$ cannot hold.

If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) := -x(t)$ for $t \geq t_1$ in (1). Using $(A2)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) + p(t)y(t-\tau)\big]\right] + \sum_{i=1}^{m} q_i(t)G\big(y(t-\sigma_i)\big) = \widetilde{f}(t) \tag{6}$$

for $t \geq t_1$, where $\widetilde{f}(t) = -f(t)$ and $G(u) := -H(-u)$ for $u \in \mathbb{R}$. Clearly, G also satisfies $(A2)$. Let $\widetilde{F}(t) = -F(t)$. Then

$$-\infty < \liminf_{t\to\infty} \widetilde{F}(t) < 0 < \limsup_{t\to\infty} \widetilde{F}(t) < \infty$$

and $\big(r\widetilde{F}'\big)'(t) = -f(t) = \widetilde{f}(t)$ hold. Further, $\widetilde{F}^+(t) = F^-(t)$ and $\widetilde{F}^-(t) = F^+(t)$. Proceeding as above, we find a contradiction to $(A8)$. This completes the proof. □

Theorem 2.5. *Let $-1 \leq -p \leq p(t) \leq 0$ for $t \in \mathbb{R}_+$, where $p > 0$ is a constant. Assume that $(C1)$ and $(A1)$–$(A4)$ hold. Furthermore, assume that the following conditions*

$(A9)$ $\int_{t_0}^{\infty}\sum_{i=1}^{m} q_i(\eta)H\big(F^-(\eta+\tau-\sigma_i)\big)\mathrm{d}\eta = \infty$,

$(A10)$ $\int_{t_0}^{\infty}\sum_{i=1}^{m} q_i(\eta)H\big(F^+(\eta-\sigma_i)\big)\mathrm{d}\eta = \infty$,

$(A11)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(\eta) H\big(F^+(\eta+\tau-\sigma_i)\big)\mathrm{d}\eta = \infty$
and
$(A12)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(\eta) H\big(F^-(\eta-\sigma_i)\big)\mathrm{d}\eta = \infty$
hold. Then the conclusion of Theorem 2.4 is true.

Proof. We proceed as in the proof of the Theorem 2.4 to conclude that w and rw' have constant sign on $[t_2,\infty)$, where $t_2 \geq t_1$. We have following possible cases.

Case 1. Let $w(t) < 0$ for $t \geq t_2$. Note that in this case, we have $z(t) \geq p(t)y(t-\tau)$ and $z(t) \leq x(t)$ for $t \geq t_2$.

(a) Let $w'(t) < 0$ for $t \geq t_2$. By Lemma 2.1, $w(t) < 0$ and $\lim_{t\to\infty} w(t) = -\infty$ for $t \geq t_3$, where $t_3 \geq t_2$. On the other hand, we claim that $x(t)$ is bounded. If not, there exists $\{\xi_n\}$ such that $\xi_n \to \infty$ as $n \to \infty$, $x(\xi_n) \to \infty$ as $n \to \infty$ and

$$x(\xi_n) = \max\{x(\eta) : t_3 \leq \eta \leq \zeta_n\}.$$

Therefore,

$$\begin{aligned} w(\xi_n) &= x(\xi_n) + p(\xi_n)x(\xi_n-\tau) - F(\xi_n) \\ &\geq (1-p)x(\xi_n-\tau) - F(\xi_n) \\ &\to +\infty, \ \ as \ \ n \to \infty, \end{aligned}$$

which is in contradiction with the fact that $w(t) < 0$. So, our claim holds. Consequently, $\lim_{t\to\infty} w(t)$ exists, which is again in contradiction with the fact that $\lim_{t\to\infty} w(t) = -\infty$.

(b) Let $w'(t) > 0$ for $t \geq t_2$. So, $\lim_{t\to\infty}(rw')(t)$ exists for $t \geq t_2$. $-x(t-\tau) \leq p(t)x(t-\tau) \leq z(t) < F(t)$ imply $x(t) > -F(t+\tau)$ for $t \geq t_3$, where $t_3 \geq t_2$. Clearly, $x(t) \geq F^-(t+\tau)$ for $t \geq t_3$. Therefore, (3) becomes

$$\big(rw'\big)'(t) + \sum_{i=1}^{m} q_i(t) H\big(F^-(t+\tau-\sigma_i)\big) \leq 0 \quad \text{for} \quad t \geq t_3.$$

Integrating the last inequality over the interval $[t_3,t) \subset [t_3,\infty)$, we obtain

$$\int_{t_3}^{t} \sum_{i=1}^{m} q_i(\eta) H\big(F^-(\eta+\tau-\sigma_i)\big)\mathrm{d}\eta < (rw')(t_3) \quad \text{for all} \quad t \geq t_3.$$

This contradicts $(A9)$.

Case 2. Let $w(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2) holds for $t \geq t_3$, where $t_3 \geq t_2$. So, $\lim_{t\to\infty}(rw')(t)$ exists for $t \geq t_3$. Note that in this case, we have $z(t) \leq x(t)$ for $t \geq t_3$. Consequently, $F(t) < z(t) \leq x(t)$ and hence $x(t) > F^+(t)$ for $t \geq t_4$, where $t_4 \geq t_3$. Therefore, (3) can be viewed as

$$\big(rw'\big)'(t) + \sum_{i=1}^{\infty} q_i(t) H\big(F^+(t-\sigma_i)\big) \leq 0 \quad \text{for} \quad t \geq t_4.$$

Integrating the preceding inequality over the interval $[t_4, t) \subset [t_4, \infty)$, we get

$$\int_{t_4}^{t} \sum_{i=1}^{\infty} q_i(\eta) H\big(F^+(\eta - \sigma_i)\big) \mathrm{d}\eta < (rw')(t_4) \quad \text{for all} \quad t \geq t_4.$$

This contradicts $(A10)$. Thus, $x(t) > 0$ for $t \geq t_1$ cannot hold.

The case when x is an eventually negative solution is very similar and we omit it. Thus the theorem is proved. □

THEOREM 2.6. *Let* $-\infty < -p \leq p(t) \leq -1$ *for* $t \in \mathbb{R}_+$, *where* $p > 0$ *is a constant. Assume that* $(C1)$, $(A1)$*–*$(A4)$, $(A10)$ *and* $(A12)$ *hold. Furthermore, assume that the following conditions*

$(A13)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(\eta) H\left(\frac{1}{p} F^-(\eta + \tau - \sigma_i)\right) \mathrm{d}\eta = \infty$

and

$(A14)$ $\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(\eta) H\left(\frac{1}{p} F^+(\eta + \tau - \sigma_i)\right) \mathrm{d}\eta = \infty$

hold. Then every bounded solution of (1) *oscillates.*

Proof. The proof of the theorem can be carried out as in the proof of Theorem 2.5. Hence the details are omitted. □

2.2. OSCILLATION UNDER THE CONDITION (C2)

REMARK 2.7. If we set

$$R(t) := \int_{t}^{\infty} \frac{1}{r(\eta)} \mathrm{d}\eta \quad \text{for} \quad t \geq t_0, \tag{7}$$

then $(C2)$ implies that $R(t) \to 0$ as $t \to \infty$.

LEMMA 2.8. *Assume that* $(C2)$, $(A1)$ *and* $(A2)$ *hold. Let* x *be an eventually positive solution of* (1). *If* w *defined by* (1) *is eventually decreasing and positive, then there exists* $\varepsilon > 0$ *such that* w *satisfies*

$$\varepsilon R(t) \leq w(t) \quad \textit{for all large} \quad t, \tag{8}$$

where R *is defined in* (7).

Proof. Suppose that $x(t)$, $w(t) > 0$ and $w'(t) < 0$ for $t \geq t_1$, where $t_1 \geq t_0$. So, we may assume without loss of generality that $x(t - \sigma_i) > 0$ for $t \geq t_1$ and $i = 1, 2, .., m$. From (1) and $(A2)$, we get (3). Consequently, rw' is nonincreasing on $[t_1, \infty)$. Therefore, $r(s)w'(s) \leq r(t)w'(t)$ for $s \geq t \geq t_1$, which implies

$$w'(s) \leq \frac{r(t)w'(t)}{r(s)} \quad \text{for} \quad s \geq t \geq t_1.$$

Consequently,

$$w(s) \leq w(t) + r(t)w'(t) \int_{t}^{s} \frac{1}{r(\eta)} \mathrm{d}\eta \quad \text{for} \quad s \geq t \geq t_1.$$

As rw' is nonincreasing, we can find a constant $\varepsilon > 0$ such that $r(t)w'(t) \leq -\varepsilon$ for $t \geq t_1$. As a result $w(s) \leq w(t) - \varepsilon \int_t^s \frac{1}{r(\eta)}\mathrm{d}\eta$ for $s \geq t \geq t_1$. By letting $s \to \infty$, we get $0 \leq w(t) - \varepsilon R(t)$ for $t \geq t_1$, which proves (8). □

THEOREM 2.9. *Let $0 \leq p(t) \leq p < \infty$ for $t \in \mathbb{R}_+$, where p is a constant. Assume that $(C2)$ and $(A1)-(A8)$ hold. Furthermore, assume that the following conditions*

$(A15)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} Q_i(\zeta) H\big(F^+(\zeta - \sigma_i)\big)\mathrm{d}\zeta\right]\mathrm{d}\eta = \infty$

and

$(A16)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} Q_i(\zeta) H\big(F^-(\zeta - \sigma_i)\big)\mathrm{d}\zeta\right]\mathrm{d}\eta = \infty$

hold, where $Q_i(t)$ is defined in Theorem 2.4. Then the conclusion of Theorem 2.4 is true.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Proceeding as in Theorem 2.4, we get that rw' and w are monotonic functions on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following possible cases.

Case 1. Let $w(t) < 0$ for $t \geq t_2$. Proceeding as in Case 1 in the proof of Theorem 2.4, we get a contradiction.

Case 2. Let $w(t) > 0$ for $t \geq t_2$.

(a) Let $w'(t) > 0$ for $t \geq t_2$. Then, we proceed as in Case 2 in the proof of Theorem 2.4 to get a contradiction to $(A7)$.

(b) Let $w'(t) < 0$ for $t \geq t_2$. By Lemma 2.8, we get (8) for $t \geq t_3$ where $\varepsilon > 0$ and $t_3 \geq t_2$. Therefore, $z(t) \geq F(t) + CR(t)$ imply $z(t) - CR(t) \geq F(t)$ for $t \geq t_3$. If $z(t) - CR(t) < 0$ for $t \geq t_3$, then $F(t) < 0$, which is a contradiction. Hence, $z(t) - CR(t) > 0$ for $t \geq t_3$ and clearly, $z(t) - CR(t) \geq F^+(t)$, that is, $z(t) \geq CR(t) + F^+(t) \geq F^+(t)$ for $t \geq t_4$, where $t_4 \geq t_3$. Consequently, (5) reduce to

$$(rw')'(t) + H(p)(rw')'(t-\tau) + \lambda \sum_{i=1}^{m} Q_i(t) H\big(F^+(t - \sigma_i)\big) \leq 0 \quad \text{for} \quad t \geq t_4.$$

Integrating the above inequality over the interval $[t_4, t) \subset [t_4, \infty)$, we obtain

$$\begin{aligned} \lambda\left[\int_{t_4}^{t} \sum_{i=1}^{m} Q_i(\eta) H\big(F^+(\eta - \sigma_i)\big)\mathrm{d}\eta\right] &\leq -[(rw')(t) + H(p)(rw')(t-\tau)] \\ &\leq -\big(1 + H(p)\big) r(t) w'(t), \end{aligned}$$

which implies

$$\frac{\lambda}{1 + H(p)} \frac{1}{r(t)} \left[\int_{t_4}^{t} \sum_{i=1}^{m} Q_i(\eta) H\big(F^+(\eta - \sigma)\big)\mathrm{d}\eta\right] \leq -w'(t) \quad \text{for} \quad t \geq t_4.$$

Again, integrating the last inequality over the interval $[t_4, t) \subset [t_4, \infty)$, we get

$$\frac{\lambda}{1+H(p)} \int_{t_4}^{t} \frac{1}{r(\eta)} \left[\int_{t_4}^{\eta} \sum_{i=1}^{m} Q_i(\zeta) H\big(F^{+}(\zeta-\sigma_i)\big) \mathrm{d}\zeta \right] \mathrm{d}\eta \le w(t_4).$$

This contradicts $(A15)$. Thus, $x(t) > 0$ for $t \ge t_1$ cannot hold.

The case where x is eventually negative can be dealt similarly, and we omit the details here. Thus, the proof of the theorem is complete. □

Theorem 2.10. *Let $-1-p \le \ \le p(t) \le 0$ for $t \in \mathbb{R}_+$, where $p > 0$ is a constant. Assume that $(C2)$, $(A1)$–$(A4)$ and $(A9)$–$(A12)$ hold. Furthermore, assume that the following conditions*

$(A17)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\left(F^{-}(\zeta+\tau-\sigma_i)\right) \mathrm{d}\zeta \right] \mathrm{d}\eta = \infty,$

$(A18)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\left(F^{+}(\zeta+\tau-\sigma_i)\right) \mathrm{d}\zeta \right] \mathrm{d}\eta = \infty,$

$(A19)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\left(F^{+}(\zeta-\sigma_i)\right) \mathrm{d}\zeta \right] \mathrm{d}\eta = \infty$

and

$(A20)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \left[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\left(F^{-}(\zeta-\sigma_i)\right) \mathrm{d}\zeta \right] \mathrm{d}\eta = \infty$

hold. Then the conclusion of Theorem 2.4 is true.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Then proceeding as in Theorem 2.5 we obtain that w and rw' are of one sign on $[t_2, \infty)$, where $t_2 \ge t_1$. We have the following possible cases.

Case 1. Let $w(t) < 0$ for $t \ge t_2$. Note that in this case, we have $z(t) \ge p(t)y(t-\tau)$ for $t \ge t_2$.

(a) Let $w'(t) < 0$ for $t \ge t_2$. We claim that $x(t)$ is bounded for $t \ge t_3$, where $t_3 \ge t_2$. If not, there exists $\{\xi_n\}$ such that $\xi_n \to \infty$ as $n \to \infty$, $x(\xi_n) \to \infty$ as $n \to \infty$ and

$$x(\xi_n) = \max\{x(\eta) : t_3 \le \eta \le \zeta_n\}.$$

Therefore,

$$\begin{aligned} w(\xi_n) &= x(\xi_n) + p(\xi_n)x(\xi_n-\tau) - F(\xi_n) \\ &\ge (1-p)x(\xi_n-\tau) - F(\xi_n) \\ &\to +\infty, \ \ as \ \ n \to \infty, \end{aligned}$$

which is in contradiction with the fact that $w(t) < 0$. So, our claim holds. Consequently, $\lim_{t\to\infty} w(t)$ exists for $t \ge t_3$. $-x(t-\tau) \le p(t)x(t-\tau) \le z(t) < F(t)$ imply $x(t) > -F(t+\tau)$ for $t \ge t_4$, where $t_4 \ge t_3$. Clearly, $x(t) \ge F^{-}(t+\tau)$ for $t \ge t_5$, where $t_5 \ge t_4$. Therefore, (3) becomes

$$\big(rw'\big)'(t) + \sum_{i=1}^{m} q_i(t) H\big(F^{-}(t+\tau-\sigma_i)\big) \le 0 \quad \text{for} \quad t \ge t_5.$$

Integrating the last inequality over the interval $[t_5, t) \subset [t_5, \infty)$, we obtain

$$\int_{t_5}^{t} \sum_{i=1}^{m} q_i(\eta) H\big(F^-(\eta + \tau - \sigma_i)\big) \mathrm{d}\eta < -(rw')(t),$$

which implies

$$\frac{1}{r(t)} \bigg[\int_{t_5}^{t} \sum_{i=1}^{m} q_i(\eta) H\big(F^-(\eta + \tau - \sigma_i)\big) \mathrm{d}\eta \bigg] < -w'(t).$$

Again, integration on the last inequality over the interval $[t_5, t) \subset [t_5, \infty)$ yields

$$\int_{t_5}^{t} \frac{1}{r(\eta)} \bigg[\int_{t_5}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\big(F^-(\zeta + \tau - \sigma_i)\big) \mathrm{d}\zeta \bigg] \mathrm{d}\eta < -w(t) \quad \text{for} \quad t \geq t_5.$$

This contradicts $(A17)$.

(b) Let $w'(t) > 0$ for $t \geq t_2$. Proceeding as in Case 1 in the proof of Theorem 2.5, we get a contradiction to $(A9)$.

Case 2. Let $w(t) > 0$ for $t \geq t_2$. Note that in this case, we have $z(t) \leq x(t)$ for $t \geq t_2$.

(a) Let $w'(t) > 0$ for $t \geq t_2$. Then, we proceed as in Case 2 in the proof of Theorem 2.5 to get a contradiction to $(A10)$.

(b) Let $w'(t) < 0$ for $t \geq t_2$. Proceeding as in Case 2 in the proof of Theorem 2.9, we get $z(t) \geq CR(t) + F^+(t) \geq F^+(t)$. Consequently, $x(t) \geq F^+(t)$. For the rest of proof, we can proceed as in Case 2 in the proof of Theorem 2.9 to get a contradiction to $(A19)$. Hence, the details are omitted.

The case when x is an eventually negative solution is very similar and we omit it. This completes the proof of the theorem. □

THEOREM 2.11. *Let $-\infty < -p \leq p(t) \leq -1$ for $t \in \mathbb{R}_+$, where $p > 0$ is a constant. Assume that $(C2)$, $(A1)$–$(A4)$, $(A9)$–$(A12)$, $(A19)$ and $(A20)$ hold. Furthermore, assume that the following conditions*

$(A21)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \Big[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_i(\zeta) H\Big(\frac{1}{p} F^+(\zeta + \tau - \sigma_i)\Big) \mathrm{d}\zeta \Big] \mathrm{d}\eta = \infty$

and

$(A22)$ $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \Big[\int_{t_0}^{\eta} \sum_{i=1}^{m} q_(\zeta) H\Big(\frac{1}{p} F^-(\zeta + \tau - \sigma_i)\Big) \mathrm{d}\zeta \Big] \mathrm{d}\eta = \infty$

hold. Then the conclusion of Theorem 2.6 is true.

Proof. The proof of the theorem can be carried out as the proof of the Theorem 2.10. Hence the details are omitted. □

3. EXISTENCE OF POSITIVE SOLUTION

In this section, sufficient conditions are obtained to show that equation (1) admits a positive bounded solution for various ranges of the bounded neutral coefficient p.

THEOREM 3.1. *Let $p \in C\big(\mathbb{R}_+, [-1, 0]\big)$ and assume that $(A1)$–$(A3)$ hold. If*

$(A23)$ $\int_0^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right]\mathrm{d}\eta < \infty,$

then (1) *admits a positive bounded solution.*

Proof. Let (i) $-1 < -p \le p(t) \le 0$ for $t \in \mathbb{R}_+$, where $p > 0$ is a constant. Due to $(A23)$, it is possible to find a $T > \rho$ such that

$$\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right]\mathrm{d}\eta < \frac{1-p}{10H(1)}.$$

We consider the set

$$M = \Big\{x : x \in C\big([T-\rho,+\infty),\mathbb{R}\big), \quad x(t) = 0 \quad \text{for} \quad t \quad \in [T-\rho, T] \quad \textit{and} \quad \frac{1-p}{20} \le x(t) \le 1\Big\}$$

and define $\Phi : M \to C\big([T-\rho,+\infty),\mathbb{R}\big)$ by the formula

$$(\Phi x)(t) = \begin{cases} 0, & t \in [T-\rho, T) \\ -p(t)x(t-\tau) + \int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)H\big(x(\zeta-\sigma_i)\big)\mathrm{d}\zeta\right]\mathrm{d}\eta + & \\ \qquad F(t) + \frac{1-p}{10}, & t \ge T, \end{cases}$$

where $F(t)$ is such that $|F(t)| \le \frac{1-p}{20}$. For every $x \in M$,

$$\begin{aligned}(\Phi x)(t) &\le -p(t)x(t-\tau) + H(1)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\right]\mathrm{d}\eta + \frac{1-p}{20} + \frac{1-p}{10} \\ &\le p + \frac{1-p}{10} + \frac{1-p}{20} + \frac{1-p}{10} \\ &\le \frac{1+3p}{4} < 1,\end{aligned}$$

and

$$\begin{aligned}(\Phi x)(t) &\ge F(t) + \frac{1-p}{10} \\ &\ge -\frac{1-p}{20} + \frac{1-p}{10} = \frac{1-p}{20}\end{aligned}$$

implies that $(\Phi x)(t) \in M$. Define $u_n : [T-\rho,+\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = (\Phi u_{n-1})(t), \quad n \ge 1$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [T-\rho, T) \\ \frac{1-p}{20}, & t \ge T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{1-p}{20} \le u_{n-1}(t) \le u_n(t) \le 1$$

for $t \ge T$. Therefore for $t \ge T-\rho$, $\lim_{n\to\infty} u_n(t)$ exists. Let $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \ge T-\rho$. By the Lebesgue's dominated convergence theorem $u \in M$ and $(\Phi u)(t) = u(t)$, where $u(t)$ is a solution of (1) on $[T-\rho,\infty)$ such that $u(t) > 0$.

(ii) If $p(t) \equiv -1$ for $t \in \mathbb{R}_+$, we choose $-1 < p_2 < 0$ such that $p_2 \ne -\frac{1}{2}$. In this case, we can apply the above method. Here, we note that

$$\int_T^t \frac{1}{r(\eta)}\Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\Big]\mathrm{d}\eta < \frac{1+2p_2}{10H(-p_2)},$$

$-\frac{1+2p_2}{40} \le F(t) \le \frac{1+2p_2}{20}$ and the set

$$M = \Big\{x : x \in C\big([T-\rho,+\infty),\mathbb{R}\big), \quad x(t) = 0 \quad \mathit{for} \quad t \in [T-\rho, T] \quad \mathit{and} \\ \frac{7+2p_2}{40} \le x(t) \le \, -p_2\Big\}.$$

Also, we define $\Phi : M \to C\big([T-\rho,+\infty),\mathbb{R}\big)$ by

$$(\Phi x)(t) = \begin{cases} x(T), & t \in [T-\rho, T) \\ x(t-\tau) + \int_T^t \frac{1}{r(\eta)}\Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) H\big(x(\zeta-\sigma_i)\big)\mathrm{d}\zeta\Big]\mathrm{d}\eta + F(t) + \\ \qquad \frac{2+p_2}{10}, & t \ge T. \end{cases}$$

This completes the proof of the theorem. □

Theorem 3.2. *Let $p \in C[\mathbb{R}_+,[0,1)]$. Let H be Lipchitzian on the interval $[a,b]$, $0 < a < b < \infty$. If (A1)–(A3) and (A23) hold, then* (1) *admits a positive bounded solution.*

Proof. Let $0 \le p(t) \le p_3 < 1$. It is possible to find $t_1 > 0$ such that

$$\int_{t_1}^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\Big]\mathrm{d}\eta < \frac{1-p_3}{5L},$$

where $L = \max\{L_1, H(1)\}$, L_1 is the Lipschitz constant of H on $\big[\frac{3}{5}(1-p_3),1\big]$. Let $F(t)$ be such that $|F(t)| < \frac{1-p_3}{10}$ for $t \ge t_2$. For $t_3 > \max\{t_1,t_2\}$, we set $Y = BC([T,\infty),\mathbb{R})$, the space of real valued continuous functions on $[t_3,\infty]$. Clearly, Y is a Banach space with respect to the sup-norm defined by

$$\|y\| = \sup\{|y(t)| : t \ge t_3\}.$$

Let

$$S = \{u \in Y : \frac{3}{5}(1-p_3) \le u(t) \le 1, \quad t \ge t_3\}.$$

We notice that S is a closed and convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(t_3+\rho), \quad t \in [t_3, t_3+\rho] \\ -p(t)x(t-\tau) + \frac{9+p_3}{10} + F(t) - \\ \qquad \int_t^\infty \frac{1}{r(\eta)} \Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) H\big(x(\zeta-\sigma_i)\big) \mathrm{d}\zeta \Big] \mathrm{d}\eta, \quad t \geq t_3+\rho. \end{cases}$$

For every $x \in Y$, $(\Phi x)(t) \leq F(t) + \frac{9+p_3}{10} \leq 1$ and

$$\begin{aligned}(\Phi x)(t) &\geq -p(t)x(t-\tau) - H(1)\int_t^\infty \frac{1}{r(\eta)} \Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) \mathrm{d}\zeta \Big] \mathrm{d}\eta \\ &\quad + F(t) + \frac{9+p_3}{10} \\ &\geq -p_3 - \frac{1-p_3}{5} - \frac{1-p_3}{10} + \frac{9+p_3}{10} = \frac{3}{5}(1-p_3)\end{aligned}$$

imply that $(\Phi x) \in S$. Now, for x_1 and $x_2 \in S$, we have

$$\begin{aligned}&|(\Phi x_1)(t) - (\Phi x_2)(t)| \leq p_3 |x_1(t-\tau) - x_2(t-\tau)| \\ &+ \int_t^\infty \frac{1}{r(\eta)} \Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) |H\big(x_1(\zeta-\sigma_i)\big) - H\big(x_2(\zeta-\sigma_i)\big)| \mathrm{d}\zeta \Big] \mathrm{d}\eta,\end{aligned}$$

that is,

$$\begin{aligned}|(\Phi x_1)(t) - (\Phi x_2)(t)| &\leq p_3 \|x_1 - x_2\| \\ &\quad + \|x_1 - x_2\| L_1 \int_t^\infty \frac{1}{r(\eta)} \Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) \mathrm{d}\zeta \Big] \mathrm{d}\eta \\ &\leq \left(p_3 + \frac{1-p_3}{5} \right) \|x_1 - x_2\| \\ &= \frac{4p_3+1}{5} \|x_1 - x_2\|.\end{aligned}$$

Therefore, $\|\Phi x_1 - \Phi x_2\| \leq \frac{4p_3+1}{5} \|x_1 - x_2\|$ implies that Φ is a contraction. By using Banach's fixed point theorem, it follows that Φ has a unique fixed point $x(t)$ in $\big[\frac{3}{5}(1-p_3), 1\big]$. Hence, $\Phi x = x$ and the proof of the theorem is complete. $\square$

REMARK 3.3. We can not apply Lebesgue's dominated convergence theorem for other ranges of $p(t)$, except $-1 \leq p(t) \leq 0$, due to the technical difficulties arising in the method. However, we can apply Banach's fixed point theorem to other ranges of $p(t)$ similar to those in Theorem 3.2.

4. FINAL COMMENTS AND EXAMPLES

In this section, we will be giving two remark and one example to close the paper.

Remark 4.1. In Theorems 2.4–2.11, H is allowed to be linear, sublinear or superlinear.

Remark 4.2. A prototype of the function H satisfying $(A2)$, $(A5)$, $(A6)$ is

$$(1+\alpha|u|^{\beta})|u|^{\gamma}\operatorname{sgn}(u) \quad \text{for} \quad u \in \mathbb{R},$$

where $\alpha \geq 1$ or $\alpha = 0$ and $\beta, \gamma > 0$ are reals. For verifying $(A5)$, we may take help of the well-known inequality (see [4, p. 292]):

$$u^p + v^p \geq h(p)(u+v)^p, \text{ for } u, v > 0, \text{ where } h(p) := \begin{cases} 1, & 0 \leq p \leq 1, \\ \dfrac{1}{2^{p-1}}, & p \geq 1. \end{cases}$$

Example 4.3. Consider the differential equation

$$\big((x(t)+x(t-\pi))\big)'' + x\left(t-\frac{\pi}{2}\right) + x\left(t-\frac{5\pi}{2}\right) = 2\cos(t), \tag{9}$$

where $p(t) = q_1(t) = q_2(t) = 1$, $\tau = \pi$, $m = 2$, $\sigma_1 = \frac{\pi}{2}$, $\sigma_2 = \frac{5\pi}{2}$, $H(x) = x$ and $f(t) = 2\cos(t)$. Indeed, if we choose $F(t) = -2\cos(t)$, then $(F')'(t) = f(t)$. We have

$$F^+(t) = \begin{cases} -2\cos(t), & 2n\pi + \frac{\pi}{2} \leq t \leq 2n\pi + \frac{3\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-(t) = \begin{cases} 2\cos t, & 2n\pi + \frac{3\pi}{2} \leq t \leq 2n\pi + \frac{5\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$F^+\left(t-\frac{\pi}{2}\right) = \begin{cases} -2\sin(t), & (2n+1)\pi \leq t \leq 2(n+1)\pi \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-\left(t-\frac{\pi}{2}\right) = \begin{cases} 2\sin(t), & 2(n+1)\pi \leq t \leq (2n+3)\pi \\ 0, & \text{otherwise.} \end{cases}$$

Also

$$F^+\left(t-\frac{5\pi}{2}\right) = \begin{cases} -2\sin(t), & (2n+3)\pi \leq t \leq 2(n+2)\pi \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F^-\left(t-\frac{5\pi}{2}\right) = \begin{cases} 2\sin(t), & 2(n+2)\pi \leq t \leq (2n+5)\pi \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\int_{\frac{5\pi}{2}}^{\infty} \left[Q_1(\eta)F^+\left(\eta-\frac{\pi}{2}\right) + Q_2(\eta)F^+\left(\eta-\frac{5\pi}{2}\right)\right] \mathrm{d}\eta = I_1 + I_2,$$

where for $n = 0, 1, 2...$, thus we get

$$I_1 = \int_{\frac{5\pi}{2}}^{\infty} F^+ \left(\eta - \frac{\pi}{2}\right) \mathrm{d}\eta = \sum_{n=0}^{\infty} \int_{(2n+1)\pi}^{2(n+1)\pi} [-2\sin(\eta)]\mathrm{d}\eta$$
$$= 2\sum_{n=0}^{\infty} [\cos(t)]_{(2n+1)\pi}^{2(n+1)\pi} = +\infty,$$

$$I_2 = \int_{\frac{5\pi}{2}}^{\infty} F^+ \left(\eta - \frac{5\pi}{2}\right) \mathrm{d}\eta = \sum_{n=0}^{\infty} \int_{(2n+3)\pi}^{2(n+2)\pi} [-2\sin(t)]\mathrm{d}\eta$$
$$= 2\sum_{n=0}^{\infty} [\cos(t)]_{(2n+3)\pi}^{2(n+2)\pi} = +\infty.$$

Clearly, conditions $(C1)$ and $(A1)$–$(A8)$ are satisfied. Hence, by Theorem 2.4, every solution of (9) is oscillatory. Thus, in particular, $x(t) = \sin(t)$ is an oscillatory solution of equation (9).

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