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NEW INTEGRAL RESULTS ON HOLDER TYPE INEQUALITIES

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Abstract. In this paper, using fractional integration, we present new fractional integral inequalities related to Holder inequality. We generalise a Wu's sharpness of Holder inequality for p,q integration. Then, as an application, we propose another way to derive the Holder inequality which is already established by Z. Dahmani on 2012 in General Math. Journal. Also, for our results, the classical Holder inequality is deduced as a special case.

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1. INTRODUCTION

The fractional integral inequalities are of great importance in differential equations, probability and applied sciences. For some applications, one can consult the papers [1, 2, 4, 5, 6, 9, 10]. The idea to develop the present paper is motivated by the well known "positive" Holder inequality which states that if f and g are two functions defined on $[a,b]$, such that $f \geq 0, g \geq 0, f \in L^p([a,b]), g \in L^q([a,b])$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_a^b f(x)g(x)\mathrm{d}x \leq \Big(\int_a^b f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}. \tag{1}$$

It is also motivated by the fractional integration version of Holder inequality proved in [3].

Another paper that motivates the present work is [11], where S.H. Wu established a new (and a nice) sharp version of Holder inequality as follows:

THEOREM 1.1. *Let f, g and e be three integrable functions defined on $[a,b]$, with $f \geq 0, g > 0, 1-e(x)+e(y) \geq 0$, for all $x,y \in [a,b]$, and let $p \geq q > 0$ such that $\frac{1}{p}+\frac{1}{q} \leq 1$. Then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ &\leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}}\Big(\int_a^b g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}-\frac{1}{p}}\Big[\Big(\int_a^b g^q(x)\mathrm{d}x\int_a^b f^p(x)\mathrm{d}x\Big)^2\\ &\quad -\Big(\int_a^b g^q(x)e(x)\mathrm{d}x\int_a^b f^p(x)\mathrm{d}x-\int_a^b g^q(x)\mathrm{d}x\int_a^b f^p(x)e(x)\mathrm{d}x\Big)^2\Big]^{\frac{1}{2p}}.\end{aligned} \tag{2}$$

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The aim of this paper is to establish new generalized versions of some results in [11] by means of Riemann-Liouville fractional integral operator. Other particular generalizations are also derived. Our results have some relationship with those of [3, 11].

2. PRELIMINARIES

We recall the definition of Riemann-Liouville integral operator and some of its properties [7].

DEFINITION 2.1. The *Riemann-Liouville fractional integral operator of order* $\alpha > 0$, for a continuous function f on $[a, b]$ is defined as:

$$J_a^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \alpha > 0, a < t \leq b. \tag{3}$$

For the convenience of establishing the results, we need the properties:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{4}$$

and

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{5}$$

For more details on Riemann-Liouville fractional integration, we refer the reader to [7].

3. MAIN RESULTS

In this section, we prove two main results. The first one generalises a theorem in [11]. The second main result is another way to obtain the fractional Holder inequality [3]. We begin by presenting the following auxiliary lemma [8].

LEMMA 3.1. *Let f and g be integrable functions defined on $[a, b]$. If $f \geq 0$, $g > 0$ and $0 < p < 1$, then*

$$\int_a^t f(x) g^p(x) \mathrm{d}x \leq \Big(\int_a^t f(x) \mathrm{d}x \Big)^{1-p} \Big(\int_a^t f(x) g(x) \mathrm{d}x \Big)^p. \tag{6}$$

Proof. Let us introduce the functions $u := (fg)^p$ and $v := f^q$, with $p + q = 1$. Then, thanks to (1), we can write

$$\begin{aligned} \int_a^b f(x) g^p(x) \mathrm{d}x &= \int_a^b u(x) v(x) \mathrm{d}x \\ &\leq \Big(\int_a^b u^{\frac{1}{p}}(x) \mathrm{d}x \Big)^p \Big(\int_a^b v^{\frac{1}{q}}(x) \mathrm{d}x \Big)^q \\ &= \Big(\int_a^t f(x) g(x) \mathrm{d}x \Big)^p \Big(\int_a^t f(x) \mathrm{d}x \Big)^{1-p}. \end{aligned} \tag{7}$$

□

LEMMA 3.2. *Let* f_i $(i = 1, 2, \dots, m)$ *be integrable functions defined on* $[a, b]$ *such that* $f_i \geq 0$, $p_i > 0$ $(i = 1, 2, \dots, m)$ *and* $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$. *Then we have:*

$$\int_a^t \prod_{i=1}^m f_i(x)\mathrm{d}x \leq \prod_{i=1}^m \left(\int_a^t f_i^{p_i}(x)\mathrm{d}x\right)^{\frac{1}{p_i}}. \tag{8}$$

Our first main result is given by the following theorem.

THEOREM 3.3. *Let* f, g *and* e *be three functions defined on* $[a, b]$ *that satisfy* $f \geq 0, g > 0, 1 - e(x) + e(y) \geq 0$, *for all* $x, y \in [a, b]$, *such that* $f^p, g^q, e \in L^1([a, b]), p \geq q > 0, \frac{1}{p} + \frac{1}{q} \leq 1$. *Then, for any* $\alpha > 0, t \in]a, b]$, *we have:*

$$\begin{aligned} J^\alpha[f(t)g(t)] \leq (t-a)^{1-\frac{1}{p}-\frac{1}{q}} \Big(J^\alpha g^q(t)\Big)^{\frac{1}{q}-\frac{1}{p}} \Big[\Big(J^\alpha g^q(t) J^\alpha f^p(t)\Big)^2 \\ - \Big(J^\alpha[g^q(t)e(t)] J^\alpha f^p(t) - J^\alpha g^q(t) J^\alpha [f^p(t)e(t)]\Big)^2\Big]^{\frac{1}{2p}}. \end{aligned} \tag{9}$$

Proof. To prove this result, we proceed on two main steps:
(i): Suppose $\frac{1}{p} + \frac{1}{q} = 1$. So, we can write

$$\begin{aligned} &\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \\ &\times \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\left(1 - e(\tau) + e(\rho)\right)^{\frac{1}{p}+\frac{1}{q}} \mathrm{d}\rho \\ &= \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\left(1 - e(\tau) + e(\rho)\right)\mathrm{d}\rho \\ &= \int_a^t\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\mathrm{d}\tau\mathrm{d}\rho \\ &- \int_a^t\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)e(\tau)\,\mathrm{d}\tau\mathrm{d}\rho \\ &+ \int_a^t\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)e(\rho)\,\mathrm{d}\tau\mathrm{d}\rho \\ &= \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\right)^2. \end{aligned} \tag{10}$$

Hence, it yields that

$$
\begin{aligned}
\left(J^{\alpha}\left(f(t)g(t)\right)\right)^2 &= \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau\right)^2 \\
&= \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \\
&\times \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\left(1-e(\tau)+e(\rho)\right)^{\frac{1}{p}+\frac{1}{q}}\mathrm{d}\rho.
\end{aligned} \tag{11}
$$

On the other hand, according to $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $\left(\dfrac{1}{q}-\dfrac{1}{p}\right)+\dfrac{1}{p}+\dfrac{1}{p}=1$, thanks to Lemma 3.2, we get

$$
\begin{aligned}
&\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\left(1-e(\tau)+e(\rho)\right)^{\frac{1}{p}+\frac{1}{q}}\mathrm{d}\rho \\
&= \int_a^t \frac{(t-\tau)^{\alpha-1}}{(\Gamma(\alpha))} f(\tau)g(\tau)\mathrm{d}\tau \int_a^t \left(\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{1}{p}+\frac{1}{q}} \\
&\times f(\rho)g(\rho)\left(1-e(\tau)+e(\rho)\right)^{\frac{1}{p}+\frac{1}{q}}\mathrm{d}\rho \\
&\le \int_a^t \frac{(t-\tau)^{\alpha-1}}{(\Gamma(\alpha))} f(\tau)g(\tau)\mathrm{d}\tau \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{p}} \\
&\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{q}} \\
&= \int_a^t \left(\frac{(t-\tau)^{\alpha-1}}{(\Gamma(\alpha))}\right)^{\frac{1}{p}+\frac{1}{q}} \Bigg[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau)f^p(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{p}} \\
&\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau)g^q(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{q}-\frac{1}{p}} \\
&\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau)g^q(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{p}}\Bigg]\mathrm{d}\tau
\end{aligned}
$$

$$\leq \left(\int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) f^p(\rho) \left(1 - e(\tau) + e(\rho)\right) \mathrm{d}\rho \mathrm{d}\tau \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) g^q(\rho) \left(1 - e(\tau) + e(\rho)\right) \mathrm{d}\rho \mathrm{d}\tau \right)^{\frac{1}{q}-\frac{1}{p}}$$

$$\times \left(\int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) g^q(\rho) \left(1 - e(\tau) + e(\rho)\right) \mathrm{d}\rho \mathrm{d}\tau \right)^{\frac{1}{p}}$$

$$= \Bigg(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\rho) \mathrm{d}\rho$$

$$- \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) e(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\rho) \mathrm{d}\rho$$

$$+ \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f^p(\rho) e(\rho) \mathrm{d}\rho \Bigg)^{\frac{1}{p}}$$

$$\times \Bigg(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) \mathrm{d}\rho$$

$$- \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) e\left(\tau\right) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) \mathrm{d}\rho$$

$$+ \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) e(\rho) \mathrm{d}\rho \Bigg)^{\frac{1}{q}-\frac{1}{p}}$$

$$\times \Bigg(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) \mathrm{d}\rho$$

$$- \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) e(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) \mathrm{d}\rho$$

$$+ \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) e(\rho) \mathrm{d}\rho \Bigg)^{\frac{1}{p}}$$

$$= \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \right)^{\frac{2}{q}-\frac{2}{p}}$$

$$\times \Bigg\{ \Bigg[\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) \mathrm{d}\tau \right) \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \right) \Bigg]^2$$

$$- \Bigg[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho) e(\rho) \mathrm{d}\rho \right) \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) \mathrm{d}\tau \right)$$

$$- \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau) \mathrm{d}\tau \right) \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau) e(\tau) \mathrm{d}\tau \right) \Bigg]^2 \Bigg\}^{\frac{1}{p}}$$

$$= \left(J^\alpha g^q(t) \right)^{\frac{2}{q}-\frac{2}{p}} \Bigg\{ \Bigg[\left(J^\alpha f^p(t) \right) \left(J^\alpha g^q(t) \right) \Bigg]^2$$

$$- \Bigg[\left(J^\alpha \left(g^q(t) e(t) \right) \right) \left(J^\alpha f^p(t) \right) - \left(J^\alpha g^q(t) \right) \left(J^\alpha \left(f^p(t) e(t) \right) \right) \Bigg]^2 \Bigg\}^{\frac{1}{p}}.$$

Using (11) and the above inequality, we get (9).

(ii): Now, suppose $\frac{1}{p} + \frac{1}{q} < 1$. So, $\frac{1}{p} + \frac{1}{q} = r$, and then $\frac{1}{pr} + \frac{1}{qr} = 1$. In this case, we observe that

$$\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) g(\tau) \mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho) g(\rho) \left(1 - e(\tau) + e(\rho)\right) \mathrm{d}\rho$$

$$= \int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho) g(\rho) \mathrm{d}\tau \mathrm{d}\rho$$

$$- \int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho) g(\rho) e(\tau) \mathrm{d}\tau \mathrm{d}\rho \tag{12}$$

$$+ \int_a^t \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) g(\tau) \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho) g(\rho) e(\rho) \mathrm{d}\tau \mathrm{d}\rho$$

$$= \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) g(\tau) \right)^2.$$

Thanks to Holder inequality, we can write

$$\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho)\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho$$

$$= \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau$$

$$\times \int_a^t \left(\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{1}{pr}+\frac{1}{qr}} f(\rho)g(\rho)\left(1-e(\tau)+e(\rho)\right)^{\frac{1}{pr}+\frac{1}{qr}}\mathrm{d}\rho$$

$$\leq \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau)\mathrm{d}\tau \left[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} (f(\rho))^{pr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{pr}}\right.$$

$$\left.\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} (g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{qr}}\right]$$

$$= \int_a^t \left[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (f(\tau)f(\rho))^{pr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{pr}}\right.$$

$$\left.\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (g(\tau)g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{qr}}\right]\mathrm{d}\tau$$

$$= \int_a^t \left[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (g(\tau)g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{qr}-\frac{1}{pr}}\right.$$

$$\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (f(\tau))^{pr}(g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{pr}}$$

$$\left.\times \left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f((\rho))^{pr}(g(\tau))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\right)^{\frac{1}{pr}}\right]\mathrm{d}\tau$$

$$\leq \left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (g(\tau))^{qr}(g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{qr}-\frac{1}{pr}}$$

$$\times \left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (f(\tau))^{pr}(g(\rho))^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{pr}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left((\rho)\right)^{pr}\left(g(\tau)\right)^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{pr}}.$$

Using Lemma 3.1 together with $0<r<1$, we find

$$\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^{qr}\left(g(\rho)\right)^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{qr}-\frac{1}{pr}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^{pr}\left(g(\rho)\right)^{qr}\left(1-e\left(\tau\right)+e\left(\rho\right)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{pr}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left((\rho)\right)^{pr}\left(g(\tau)\right)^{qr}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{pr}}$$

$$\leq\left(\int_a^t\int_a^t \left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{(1-r)\left(\frac{1}{qr}-\frac{1}{pr}\right)}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^{q}\left(g(\rho)\right)^{q}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{q}-\frac{1}{p}}$$

$$\times\left(\int_a^t\int_a^t \left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{(1-r)\frac{1}{pr}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^{p}\left(g(\rho)\right)^{q}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^t\int_a^t \left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{(1-r)\frac{1}{pr}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left((\rho)\right)^{p}\left(g(\tau)\right)^{q}\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{p}}$$

$$=\left(\int_a^t\int_a^t \left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{(1-r)}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q\left(g(\rho)\right)^q\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{q}-\frac{1}{p}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^p\left(g(\rho)\right)^q\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^t\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f\left((\rho)\right)^p\left(g(\tau)\right)^q\left(1-e(\tau)+e(\rho)\right)\mathrm{d}\rho\mathrm{d}\tau\right)^{\frac{1}{p}}$$

$$=(t-a)^{2-2r}\Bigg[\Bigg(\Bigg(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q\mathrm{d}\tau$$

$$-\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q e(\tau)\mathrm{d}\tau$$

$$+\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q e\left(\rho\right)\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q\mathrm{d}\tau\Bigg)^{\frac{1}{q}-\frac{1}{p}}$$

$$\times\Bigg(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^p\mathrm{d}\tau$$

$$-\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^p e(\tau)\mathrm{d}\tau$$

$$+\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\rho)\right)^q e(\rho)\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(f(\tau)\right)^p\mathrm{d}\tau\Bigg)^{\frac{1}{p}}$$

$$\times\Bigg(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}f\left((\rho)\right)^p\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q\mathrm{d}\tau$$

$$-\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}f\left((\rho)\right)^p\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q e(\tau)\mathrm{d}\tau$$

$$+\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}f\left((\rho)\right)^p e(\rho)\mathrm{d}\rho\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(g(\tau)\right)^q\mathrm{d}\tau\Bigg)^{\frac{1}{p}}\Bigg]$$

$$
\begin{aligned}
&= (t-a)^{2\left(1-\frac{1}{p}-\frac{1}{q}\right)}\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau)\mathrm{d}\tau\right)^{\frac{2}{q}-\frac{2}{p}} \\
&\times \Bigg\{\Bigg[\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau)\mathrm{d}\tau\right)\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau)\mathrm{d}\tau\right)\Bigg]^2 \\
&- \Bigg[\left(\int_a^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} g^q(\rho)e(\rho)\mathrm{d}\rho\right)\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau)\mathrm{d}\tau\right) \\
&- \left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g^q(\tau)\mathrm{d}\tau\right)\left(\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f^p(\tau)e(\tau)\mathrm{d}\tau\right)\Bigg]^2\Bigg\} \\
&= (t-a)^{2(1-\frac{1}{p}-\frac{1}{q})}(J^\alpha g^q(t))^{\frac{2}{q}-\frac{2}{p}} \\
&\times \Big\{[(J^\alpha f^p(t))(J^\alpha g^q(t))]^2 - [(J^\alpha(g^q(t)e(t)))(J^\alpha f^p(t)) \\
&- (J^\alpha g^q(t))(J^\alpha(f^p(t)e(t)))]^2\Big\}^{\frac{1}{p}}.
\end{aligned} \tag{13}
$$

Thanks to (12), (13), we obtain (9). Theorem 3.3 is thus proved. □

REMARK 3.4. Taking $t = b, \alpha = 1$, and under conditions of p, q integrability, Theorem 1.1 would follow as a special case of Theorem 3.3.

Another way to derive the fractional Holder inequality, which is already established in [3], is given by the following result.

COROLLARY 3.5. *Let f, g be two functions defined on $[a, b]$ and $f \geq 0, g \geq 0$, such that $f^p, g^q \in L^1([a, b]), p \geq q > 0, \frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\alpha > 0, t \in]a, b]$, we have:*

$$
J^\alpha[f(t)g(t)] \leq \Big(J^\alpha g^q(t)\Big)^{\frac{1}{q}}\Big(J^\alpha f^p(t)\Big)^{\frac{1}{p}}. \tag{14}
$$

Proof. Applying Theorem 3.3 with $\frac{1}{p}+\frac{1}{q} = 1$ and $e = 1$ over $[a, b]$, we obtain

$$
\begin{aligned}
J^\alpha[f(t)g(t)] &\leq \Big(J^\alpha g^q(t)\Big)^{\frac{1}{q}-\frac{1}{p}}\Big[\Big(J^\alpha g^q(t) J^\alpha f^p(t)\Big)^2\Big)\Big]^{\frac{1}{2p}} \\
&= \Big(J^\alpha g^q(t)\Big)^{\frac{1}{q}}\Big(J^\alpha f^p(t)\Big)^{\frac{1}{p}}.
\end{aligned}
$$

□

REMARK 3.6. Applying Corollary 3.5 for $t = b, \alpha = 1$, we obtain the classical inequality of Holder (1).

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