MATHEMATICA, Tome 55 (78), N° 2, 2013, pp. 142–158

REAL-VALUED FUNCTIONS OF FINITE ENERGY ON THE SIERPINSKI GASKET

BRIGITTE E. BRECKNER

Abstract. We present, including all necessary computation, the harmonic extension procedure on the Sierpinski gasket in the n-dimensional Euclidean space. Thus we complete the results of [10] where this procedure is performed only in the cases $n \in \{1, 2\}$. Moreover, we derive from this procedure certain properties of real-valued functions of finite energy defined on the Sierpinski gasket. We stress on the Hölder continuity, since we haven't found in the literature a proof for it in the case $n \geq 3$.

MSC 2010. 28A80, 49J52.

Key words. Sierpinski gasket, harmonic function on the Sierpinski gasket, harmonic extension procedure, energy form on the Sierpinski gasket.

1. INTRODUCTION

The domain of *analysis on fractals* has emerged in the last three decades, motivated by Mandelbrot's book [9], where fractals are proposed as models for different physical phenomena. Over the years, there have been developed suitable instruments, allowing the study of these models, i.e., of differential equations and of partial differential equations (PDEs) on fractals. There have been taken different approaches (for instance, for defining differential operators) that are adapted to certain classes of fractals. An overview of these researches can be found in the introduction of R.S. Strichartz's book [10]. Here we only point out the important contributions of J. Kigami (e.g., [5]–[8]) who developed a suitable framework for studying PDEs on the so-called post-critically-finite (p.c.f.) fractals. Kigami's pioneering paper in this direction is [5], where he founded his theory in the case of the Sierpinski gasket, which is typical for the more general class of p.c.f. fractals. Having the definition of the Laplacian on the unit interval of $\mathbb{R}$ as a model, Kigami introduced in [5] the Laplacian on the Sierpinski gasket in the n-dimensional Euclidean spaces. Kigami's work has considerably influenced subsequent papers devoted to PDEs on the Sierpinski gasket. A list of them, including also several recent contributions, may be found in the introduction of the paper [1].

A central concept in Kigami's approach is that of *harmonic functions on fractals*. Kigami introduced in [5] the harmonic functions on the Sierpinski gasket through *harmonic differences*. The ideas developed by Kigami in [5] have been used by M. Fukushima and T. Shima to define in [4] a certain *energy form* on the Sierpinski gasket. In [10] there has been taken another approach as in the previously mentioned papers [4] and [5] to introduce harmonic functions and the energy form on the Sierpinski gasket. Both concepts arise in a natural way via the *harmonic extension procedure.* This procedure is presented in detail, including all necessary computation, in Section 1.3 of [10] only for the Sierpinski gasket in $\mathbb{R}$ (i.e., for the unit interval) and for the Sierpinski gasket in $\mathbb{R}^2$ (i.e., for the Sierpinski triangle). In Section 3 of the present paper we perform all computation involved in the harmonic extension procedure for the Sierpinski gasket in $\mathbb{R}^n$, where n is an arbitrary nonzero natural number. In Section 4 we then prove several properties of real-valued functions of finite energy on the Sierpinski gasket. Among them, the Hölder continuity is the most prominent. Although the exponent for the Hölder condition is mentioned, for instance, in [3], we haven't found in the literature a proof for this fact in Euclidean spaces of dimension $n \geq 3$. The proof given in Theorem 4.4 below (for arbitrary nonzero natural dimensions n) reveals the importance of the symmetric structure of the Sierpinski gasket.

By presenting in detail all aspects involved in the harmonic extension procedure for the Sierpinski gasket in Euclidean spaces of arbitrary dimension, the paper represents an important contribution to the theory of harmonic functions defined on p.c.f. fractals.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$, by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of positive naturals, by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$, and by $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the inner product which gives rise to the norm $|\cdot|$. The spaces $\mathbb{R}^n$ are endowed, throughout the paper, with the Euclidean topology induced by $|\cdot|$. If M is a nonempty subset of $\mathbb{R}^n$, then $\operatorname{diam} M$ stands for the diameter of M, i.e., $\operatorname{diam} M := \sup\{|x-y| \mid x, y \in M\}$. If M is a subset of $\mathbb{R}^n$, then $\overline{M}$ denotes the closure of M, and $\operatorname{card}(M)$ the cardinality of M.

2. THE SIERPINSKI GASKET

Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for $i \in \{1, \dots, N\}$, the map $S_i \colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}\,x + \frac{1}{2}\,p_i\,.$$

Denote by $S\colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$S(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known (see, for example, Theorem 9.1 in [2]) that there is a unique nonempty compact subset V of $\mathbb{R}^{N-1}$ such that $S(V) = V$ (that is, V is a fixed point of the map S). The set V is called the *Sierpinski gasket* (SG for short) in $\mathbb{R}^{N-1}$. In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$.

In order to construct the SG, put

$$V_0 := \{p_1, \dots, p_N\}, \; V_{m+1} := S(V_m), \text{ for } m \in \mathbb{N}, \text{ and } V_* := \bigcup_{m \geq 0} V_m. \tag{2.1}$$

It can be proved that $\overline{V_*}$ is compact and a fixed point of S, thus

$$V = \overline{V_*}.$$

Denoting by C the convex hull of the set $\{p_1, \dots, p_N\}$ and observing that

$$S_i(C) \subseteq C, \text{ for all } i \in \{1, \dots, N\}, \tag{2.2}$$

we get that $V_m \subseteq C$ for every $m \in \mathbb{N}$, so $V \subseteq C$.

REMARK 2.1. In the particular case $N = 2$ the SG coincides with the compact interval of $\mathbb{R}$ determined by p_1 and p_2, i.e., with C. If $N = 3$ the SG becomes the *Sierpinski triangle* whose construction goes back to the Polish mathematician W. Sierpinski.

Since $|p_i - p_j| = 1$ for $i, j \in \{1, \dots, N\}$, $i \neq j$, we get that for $i, j, k \in \{1, \dots, N\}$ with $i \neq j$ the following equalities hold

$$\langle p_k - p_j, p_i - p_j \rangle = \begin{cases} 0, & \text{if } k = j \\ 1, & \text{if } k = i \\ \frac{1}{2}, & \text{if } k \in \{1, \dots, N\} \setminus \{i, j\}. \end{cases} \tag{2.3}$$

It follows that, for $i \in \{1, \dots, N\}$ fixed, the set $\{p_k - p_i \mid k \in \{1, \dots, N\} \setminus \{i\}\}$ is linearly independent in $\mathbb{R}^{N-1}$. This yields the following basic fact concerning the set C.

LEMMA 2.2. *Let $i, j \in \{1, \dots, N\}$ with $i \neq j$. Then*

$$S_i(C) \cap S_j(C) = \{S_i(p_j)\} = \{S_j(p_i)\}.$$

For every $m \in \mathbb{N}^*$ let $\mathfrak{W}_m := (\{1, \dots, N\})^m$. An element $w \in \mathfrak{W}_m$ is called a *word* of *length* m. For $w = (w_1, \dots, w_m) \in \mathfrak{W}_m$ put $S_w := S_{w_1} \circ \cdots \circ S_{w_m}$. The inclusions (2.2) yield that

$$S_w(C) \subseteq C, \text{ for every } w \in \mathfrak{W}_m.$$

Using a straightforward induction argument, we thus obtain from Lemma 2.2 the following result.

PROPOSITION 2.3. *Let $m \in \mathbb{N}^*$ and $w, w' \in \mathfrak{W}_m$ with $w \neq w'$. Then*

$$\operatorname{card}(S_w(C) \cap S_{w'}(C)) \leq 1.$$

We determine in the next proposition the diameter of the sets $S_w(C)$.

PROPOSITION 2.4. *Let $m \in \mathbb{N}^*$ and $w \in \mathfrak{W}_m$. Then*

$$\operatorname{diam} S_w(C) = \frac{1}{2^m}.$$

Proof. We prove first that

$$\operatorname{diam} C = 1. \tag{2.4}$$

For this, pick $x, y \in C$, and consider $t_i, s_i \in [0,1]$, $i \in \{1, \ldots, N\}$, such that $\sum_{i=1}^{N} t_i = \sum_{i=1}^{N} s_i = 1$, $x = \sum_{i=1}^{N} t_i p_i$ and $y = \sum_{i=1}^{N} s_i p_i$. Since, for every $j \in \{1, \ldots, N\}$, we have that

$$|x - p_j| = \left| \sum_{i=1}^{N} t_i (p_i - p_j) \right| \leq \sum_{i=1}^{N} t_i |p_i - p_j| \leq \sum_{i=1}^{N} t_i = 1,$$

we get that

$$|x - y| = \left| \sum_{i=1}^{N} s_i (x - p_i) \right| \leq \sum_{i=1}^{N} s_i |x - p_i| \leq \sum_{i=1}^{N} s_i = 1,$$

showing that $\operatorname{diam} C \leq 1$. On the other hand, $|p_1 - p_2| = 1$ yields that $\operatorname{diam} C \geq 1$, thus (2.4) holds.

Using induction, it can be readily verified that

$$S_w(x) - S_w(y) = \frac{1}{2^m}(x - y), \text{ for all } x, y \in C. \tag{2.5}$$

The statement follows now from (2.4). □

Let $m \in \mathbb{N}^*$. The equality $V = S(V)$ clearly yields

$$V = \bigcup_{w \in \mathfrak{W}_m} S_w(V). \tag{2.6}$$

Equation (2.6) is the *level m decomposition of V*, and each $S_w(V)$, $w \in \mathfrak{W}_m$, is called a *cell of level m*, or, for short, an m-cell. We refer to V as the 0-cell. Since $V \subseteq C$, Proposition 2.3 implies the following result concerning these m-cells.

Corollary 2.5. *Let $m \in \mathbb{N}^*$. Then every two distinct m-cells are either disjoint or intersect at a single point.*

Definition 2.6. Let $m \in \mathbb{N}^*$. Two m-cells that intersect at a single point are said to be *adjacent*.

Remark 2.7. Let $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Since $V_0 \subseteq V \subseteq C$, Lemma 2.2 implies that

$$S_i(V) \cap S_j(V) = \{S_i(p_j)\} = \{S_j(p_i)\}.$$

Hence every two distinct 1-cells are adjacent.

Remark 2.8. The inclusions $V_0 \subseteq V \subseteq C$ imply, according to (2.4) and to Proposition 2.4, that $\operatorname{diam} V = 1$ and that

$$\operatorname{diam} S_w(V) = \frac{1}{2^m}, \text{ for every } w \in \mathfrak{W}_m,\ m \in \mathbb{N}^*.$$

We state, for later use, the following results, which can be proved easily.

PROPOSITION 2.9. *Let $m \in \mathbb{N}^*$ and $w, w' \in \mathfrak{W}_m$ be such that $S_w(V)$ and $S_{w'}(V)$ are adjacent. Then there exist $i, j \in \{1, \ldots, N\}$ with $i \neq j$ such that*

$$S_w(V) \cap S_{w'}(V) = \{S_w(p_i)\} = \{S_{w'}(p_j)\}.$$

PROPOSITION 2.10. *Let $m \in \mathbb{N}^*$, $w \in \mathfrak{W}_m$, and $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Then $S_w(S_i(p_j)) \notin V_m$.*

3. THE HARMONIC EXTENSION PROCEDURE

In order to describe this procedure, consider the sets V_m, $m \in \mathbb{N}$, defined in (2.1). The equalities $p_i = S_i(p_i)$, $i \in \{1, \ldots, N\}$, imply the inclusion $V_0 \subseteq V_1$. It follows inductively that $V_m \subseteq V_{m+1}$ for every $m \in \mathbb{N}$.

We introduce now a binary relation on the sets V_m, $m \in \mathbb{N}$. For $x, y \in V_m$ we set $x \underset{m}{\sim} y$ if there is a cell of level m containing both x and y. Note that $x \underset{0}{\sim} y$ for every $x, y \in V_0$. If $m \geq 1$, then, for $x, y \in V_m$, we have that $x \underset{m}{\sim} y$ if and only if there exist $w \in \mathcal{W}_m$ and $i, j \in \{1, \ldots, N\}$ with

$$x = F_w(p_i) \text{ and } y = F_w(p_j).$$

Obviously, if $x, y \in V_m$ with $x \neq y$, then $x \underset{m}{\sim} y \Leftrightarrow |x - y| = \frac{1}{2^m}$.

The key tools in the harmonic extension procedure are certain energy forms E_m attached to the sets V_m. More exactly, given $m \in \mathbb{N}$, define for every $u \colon V_m \to \mathbb{R}$

$$E_m(u) := \sum_{\substack{x,y \in V_m \\ x \underset{m}{\sim} y}} (u(x) - u(y))^2. \tag{3.1}$$

The *harmonic extension procedure* consists in the following: Given $m \in \mathbb{N}$ and the map $u \colon V_m \to \mathbb{R}$, find a *harmonic extension* $\widetilde{u} \colon V_{m+1} \to \mathbb{R}$ of u to V_{m+1}, i.e., an extension of u to V_{m+1} (hence $\widetilde{u}|_{V_m} = u$) that minimizes the energy E_{m+1} for all extensions of u to V_{m+1}. Thus, for every other extension $u' \colon V_{m+1} \to \mathbb{R}$ of u to V_{m+1}, the inequality

$$E_{m+1}(\widetilde{u}) \leq E_{m+1}(u')$$

has to hold.

THEOREM 3.1. *Let $m \in \mathbb{N}$. Then every $u \colon V_m \to \mathbb{R}$ has a unique harmonic extension $\widetilde{u} \colon V_{m+1} \to \mathbb{R}$. Moreover, the following equality holds*

$$E_{m+1}(\widetilde{u}) = \frac{N}{N+2} E_m(u).$$

Proof. The key element in the proof is to show that the statement holds for $m = 0$. Thus assume that $u \colon V_0 \to \mathbb{R}$ is given. For simplicity denote by

$a_i := u(p_i)$, for $i \in \{1, \ldots, N\}$. Then

$$E_0(u) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(u(p_i) - u(p_j))^2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(a_i - a_j)^2. \tag{3.2}$$

Note that

$$V_1 \setminus V_0 = \{S_i(p_j) \mid \ i, j \in \{1, \ldots, N\}, \ i \neq j\}.$$

For an extension $u'\colon V_1 \to \mathbb{R}$ of u we have

$$E_1(u') = \sum_{k=1}^{N} E_0(u' \circ S_k),$$

where, for every $k \in \{1, \ldots, N\}$,

$$E_0(u' \circ S_k) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\big((u'(S_k(p_i)) - u'(S_k(p_j))\big)^2.$$

For $i, j \in \{1, \ldots, N\}$ set

$$x_{ij} := \begin{cases} u'(S_i(p_j)), & \text{if } i \neq j \\ a_i, & \text{if } i = j. \end{cases}$$

Then

$$x_{ij} = x_{ji}, \text{ for all } i, j \in \{1, \ldots, N\},$$

and

$$E_1(u') = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_{ki} - x_{kj})^2. \tag{3.3}$$

Thus $E_1(u')$ is a (real-valued) expression depending on the (real) variables x_{ij}, $1 \leq i < j \leq N$. Denote by $f\colon \mathbb{R}^{\frac{N(N-1)}{2}} \to \mathbb{R}$ the function that assigns to these variables the value $E_1(u')$ according to (3.3). To find a harmonic extension of u is equivalent to determine a global minimum of f. Since f is convex and differentiable, the global minima of f are exactly the stationary points of f. The latter are the solutions of the following system of linear equations

$$\sum_{i=1}^{N}(x_{k\ell} - x_{ki}) + \sum_{i=1}^{N}(x_{\ell k} - x_{li}) = 0, \forall\ k, \ell \in \{1, \ldots, N\} \text{ with } k \neq \ell. \tag{3.4}$$

Note that $x_{k\ell}$ and $x_{\ell k}$ represent the same variable, but, for symmetry reasons and in order to simplify computation, we write both of them in the above system. The system (3.4) is equivalent to

$$2Nx_{k\ell} - \sum_{i=1}^{N} x_{ki} - \sum_{i=1}^{N} x_{\ell i} = 0, \forall\ k, \ell \in \{1, \ldots, N\} \text{ with } k \neq \ell. \tag{3.5}$$

Fix now $k \in \{1, \ldots, N\}$. By summing up all equations containing the first term $2Nx_{kj}$, for $j \in \{1, \ldots, N\} \setminus \{k\}$, we obtain from (3.5) that

$$2N \sum_{\substack{j=1 \\ j\neq k}}^{N} x_{kj} - (N-1) \sum_{i=1}^{N} x_{ki} - \sum_{\substack{j=1 \\ j\neq k}}^{N} \sum_{i=1}^{N} x_{ji} = 0.$$

A straightforward computation implies that this equation is equivalent to

$$(N+2) \sum_{i=1}^{N} x_{ki} - 2Nx_{kk} - \sum_{j=1}^{N} \sum_{i=1}^{N} x_{ji} = 0,$$

thus

$$\sum_{i=1}^{N} x_{ki} = \frac{2N}{N+2} x_{kk} + \frac{1}{N+2} \sum_{j=1}^{N} \sum_{i=1}^{N} x_{ji}, \forall\ k \in \{1, \ldots, N\}. \tag{3.6}$$

Adding the equations (3.6) for all $k \in \{1, \ldots, N\}$, we get

$$\sum_{k=1}^{N} \sum_{i=1}^{N} x_{ki} = \frac{2N}{N+2} \sum_{k=1}^{N} x_{kk} + \frac{N}{N+2} \sum_{j=1}^{N} \sum_{i=1}^{N} x_{ji},$$

hence

$$\sum_{j=1}^{N} \sum_{i=1}^{N} x_{ji} = N \sum_{i=1}^{N} x_{ii}. \tag{3.7}$$

From (3.6) and (3.7) we derive

$$\sum_{i=1}^{N} x_{ki} = \frac{2N}{N+2} x_{kk} + \frac{N}{N+2} \sum_{i=1}^{N} x_{ii}, \forall\ k \in \{1, \ldots, N\}. \tag{3.8}$$

From (3.5) and (3.8) we finally obtain

$$x_{k\ell} = \frac{x_{kk} + x_{\ell\ell}}{N+2} + \frac{1}{N+2} \sum_{i=1}^{N} x_{ii}, \forall\ k,\ \ell \in \{1, \ldots, N\} \text{ with } k \neq \ell.$$

Denoting by

$$\sigma := \sum_{i=1}^{N} a_i, \tag{3.9}$$

we thus get

$$x_{k\ell} = \frac{a_k + a_\ell + \sigma}{N+2}, \forall\ k,\ \ell \in \{1, \ldots, N\} \text{ with } k \neq \ell. \tag{3.10}$$

A simple computation yields that these values are indeed solutions of the system (3.4). We conclude that the values given by (3.10) determine uniquely the harmonic extension $\widetilde{u}\colon V_1 \to \mathbb{R}$ of u.

We compute now $E_1(\widetilde{u})$. Consider that $x_{k\ell}$, $k \neq \ell$, are given by (3.10). Recall that

$$E_1(\widetilde{u}) = \sum_{k=1}^{N} E_0(\widetilde{u} \circ S_k) = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (x_{ki} - x_{kj})^2.$$

Taking into account that for $i,\, j,\, k \in \{1, \ldots, N\}$

$$x_{ki} - x_{kj} = \begin{cases} \frac{a_i - a_j}{N+2}, & \text{if } i \neq k \text{ and } j \neq k \\ \\ \frac{a_i - a_k}{N+2} + \frac{\sigma - N a_k}{N+2}, & \text{if } i \neq k \text{ and } j = k, \end{cases}$$

we obtain that

$$E_0(\widetilde{u} \circ S_k) = \frac{E_0(u)}{(N+2)^2} + \frac{N+1}{(N+2)^2}(\sigma - N a_k)^2, \forall\, k \in \{1, \ldots, N\},$$

so

$$E_1(\widetilde{u}) = \frac{N}{(N+2)^2} E_0(u) + \frac{N+1}{(N+2)^2} \sum_{k=1}^{N} (\sigma - N a_k)^2. \tag{3.11}$$

On the other hand, denoting by

$$s := \sum_{i=1}^{N} a_i^2,$$

we have, by (3.2) and (3.9), that

$$E_0(u) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i^2 - 2 a_i a_j + a_j^2) = Ns - \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j = Ns - \sigma^2$$

and

$$\sum_{k=1}^{N} (\sigma - N a_k)^2 = \sum_{k=1}^{N} (\sigma^2 - 2N\sigma a_k + N^2 a_k^2) = N^2 s - N\sigma^2 = N E_0(u).$$

Using (3.11), we conclude that

$$E_1(\widetilde{u}) = \frac{N}{(N+2)} E_0(u). \tag{3.12}$$

Consider now $m \in \mathbb{N}^*$ and $u \colon V_m \to \mathbb{R}$. Note that each m-cell leads to $\frac{(N-1)N}{2}$ points belonging to $V_{m+1} \setminus V_m$. More exactly, if $w \in \mathfrak{W}_m$, then $S_w(S_i(p_j))$, $i,\, j \in \{1, \ldots, N\}$, $i \neq j$, are exactly the points in $V_{m+1} \setminus V_m$ which lie in the m-cell $S_w(V)$ (see also Proposition 2.10). Moreover, $S_w(V)$ is the only m-cell containing the points $S_w(S_i(p_j))$, $i,\, j \in \{1, \ldots, N\}$, $i \neq j$.

If $u'\colon V_{m+1} \to \mathbb{R}$ is an extension of u, then $E_{m+1}(u')$ is the sum of contributions from each m-cell $S_w(V)$, $w \in \mathfrak{W}_m$. Since the contribution from the m-cell $S_w(V)$ is just the energy E_1 of $u' \circ S_w\colon V_1 \to \mathbb{R}$, we get

$$E_{m+1}(u') = \sum_{w\in\mathfrak{W}_m} E_1(u' \circ S_w). \tag{3.13}$$

Thus the problem of minimizing $E_{m+1}(u')$ can be reduced to minimize each term $E_1(u' \circ S_w)$, $w \in \mathfrak{W}_m$, on the right side of the equality (3.13). But minimizing each of these terms is exactly a problem of the sort we have solved at the beginning of the proof. Hence we get that $\widetilde{u}\colon V_{m+1} \to \mathbb{R}$, defined by $\widetilde{u}|_{V_m} = u$ and, for all $w \in \mathfrak{W}_m$ and all $i, j \in \{1, \dots, \mathrm{N}\}$ with $i \neq j$, by

$$\widetilde{u}(S_w(S_i(p_j))) = \frac{u(S_w(p_i)) + u(S_w(p_j)) + \displaystyle\sum_{k=1}^{N} u(S_w(p_k))}{N+2},$$

is the unique harmonic extension of u. Moreover, we deduce from (3.12) that

$$E_1(\widetilde{u} \circ S_w) = \frac{N}{(N+2)} E_0(u \circ S_w), \forall\, w \in \mathfrak{W}_m.$$

This implies, according to (3.13), that

$$E_{m+1}(\widetilde{u}) = \frac{N}{N+2} \sum_{w\in\mathfrak{W}_m} E_0(u \circ S_w) = \frac{N}{N+2} E_m(u),$$

which finishes the proof. □

Given $m \in \mathbb{N}$, we introduce now the renormalization W_m of the energy function E_m, defined in (3.1), by

$$W_m(u) = \left(\frac{N+2}{N}\right)^m E_m(u), \text{ for every } u\colon V_m \to \mathbb{R}. \tag{3.14}$$

From Theorem 3.1 we now immediately derive the following result.

Corollary 3.2. *Let $m \in \mathbb{N}$ and let $u\colon V_m \to \mathbb{R}$. If $\widetilde{u}\colon V_{m+1} \to \mathbb{R}$ is the harmonic extension of u and if $u'\colon V_{m+1} \to \mathbb{R}$ is an arbitrary extension of u then the following relations hold*

$$W_m(u) = W_{m+1}(\widetilde{u}) \le W_{m+1}(u').$$

Recall from (2.1) that V_* is the union of the sets V_m, $m \in \mathbb{N}$. For a function $u\colon V_* \to \mathbb{R}$ consider now its restrictions $u|_{V_m}$ to the sets V_m, $m \in \mathbb{N}$. For simplicity we denote by

$$W_m(u) := W_m\left(u|_{V_m}\right), \forall\, m \in \mathbb{N}.$$

Corollary 3.2 yields then the following result.

Corollary 3.3. *For every $u\colon V_* \to \mathbb{R}$ the sequence $\left(W_m(u)\right)_{m\in\mathbb{N}}$ is increasing.*

According to Corollary 3.3, it makes sense to define for a function $u\colon V_* \to \mathbb{R}$ its energy $W(u)$ by

$$W(u) := \lim_{m\to\infty} W_m(u). \tag{3.15}$$

Denote by

$$\text{dom } W := \{u\colon V_* \to \mathbb{R} \mid W(u) < \infty\}.$$

A map $u \in \text{dom } W$ is said to be a *function of finite energy.*

Remark 3.4. Let $u\colon V_* \to \mathbb{R}$. Using the definition of W_m and Corollary 3.3, we get that

$$0 \leq W_m(u) \leq W(u), \forall\ m \in \mathbb{N}.$$

Thus $W(u) = 0$ if and only if u is constant.

Definition 3.5. Let $m \in \mathbb{N}$. A function $h\colon V_* \to \mathbb{R}$ is called a *harmonic function of level* m if h is obtained by specifying the values of h on V_m arbitrarily and then extending harmonically to V_k for each $k > m$. Denote by $\mathcal{H}_m$ the set of all harmonic functions of level m.

Remark 3.6. Let $m \in \mathbb{N}$. By Corollary 3.2 we have that

$$W(u) = W_m(u), \forall\ u \in \mathcal{H}_m,$$

thus $\mathcal{H}_m \subseteq \text{dom } W$.

The next result follows directly from the harmonic extension procedure as described in Theorem 3.1.

Lemma 3.7. *Let $m \in \mathbb{N}$, $u\colon V_m \to \mathbb{R}$ be a function and $\widetilde{u}\colon V_{m+1} \to \mathbb{R}$ its harmonic extension. Then*

$$\max_{V_{m+1}} \widetilde{u} = \max_{V_m} u \text{ and } \min_{V_{m+1}} \widetilde{u} = \min_{V_m} u.$$

Proof. Since $V_m \subseteq V_{m+1}$ and $\widetilde{u}|_{V_m} = u$, the following inequalities hold

$$\max_{V_m} u \leq \max_{V_{m+1}} \widetilde{u} \text{ and } \min_{V_{m+1}} \widetilde{u} \leq \min_{V_m} u. \tag{3.16}$$

We first prove the statement in the case $m = 0$. Pick $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Then (3.10) implies that $\widetilde{u}(S_i(p_j))$ is a convex combination of the reals $u(p_1), \ldots, u(p_N)$, hence

$$\min_{V_0} u \leq \widetilde{u}(S_i(p_j)) \leq \max_{V_0} u,$$

so

$$\max_{V_1} \widetilde{u} \leq \max_{V_0} u \text{ and } \min_{V_0} u \leq \min_{V_1} \widetilde{u}. \tag{3.17}$$

From (3.16) and (3.17) we derive

$$\max_{V_1} \widetilde{u} = \max_{V_0} u \text{ and } \min_{V_1} \widetilde{u} = \min_{V_0} u. \tag{3.18}$$

Assume now that $m \in \mathbb{N}^*$ and choose an arbitrary $w \in \mathfrak{W}_m$. We know from the proof of Theorem 3.1 that $\widetilde{u} \circ S_w : V_1 \to \mathbb{R}$ is the harmonic extension of $u \circ S_w : V_0 \to \mathbb{R}$. So, applying (3.18), we obtain that

$$\max_{V_1}(\widetilde{u} \circ S_w) = \max_{V_0}(u \circ S_w) \text{ and } \min_{V_1}(\widetilde{u} \circ S_w) = \min_{V_0}(u \circ S_w).$$

Since

$$V_m = \bigcup_{w \in \mathfrak{W}_m} S_w(V_0) \text{ and } V_{m+1} = \bigcup_{w \in \mathfrak{W}_m} S_w(V_1),$$

we get that

$$\max_{V_{m+1}} \widetilde{u} \leq \max_{V_m} u \text{ and } \min_{V_m} u \leq \min_{V_{m+1}} \widetilde{u}. \tag{3.19}$$

Relations (3.16) and (3.19) finally yield the asserted equalities. □

We derive from Lemma 3.7 the following results on harmonic functions which will be used in the subsequent section.

Corollary 3.8. *Let $u : V_0 \to \mathbb{R}$ and denote by $\overline{u} \in \mathcal{H}_0$ the harmonic function of level 0 obtained from u. If $x \in V_*$, then*

$$\min_{V_0} u \leq \overline{u}(x) \leq \max_{V_0} u.$$

Proof. Let $m \in \mathbb{N}$ be so that $x \in V_m$. Then clearly

$$\min_{V_m} \overline{u} \leq \overline{u}(x) \leq \max_{V_m} \overline{u}. \tag{3.20}$$

Applying Lemma 3.7, we get inductively that

$$\max_{V_m} \overline{u} = \cdots = \max_{V_1} \overline{u} = \max_{V_0} u \quad \text{and} \quad \min_{V_m} \overline{u} = \cdots = \min_{V_1} \overline{u} = \min_{V_0} u.$$

Using (3.20), we get the asserted inequalities. □

Corollary 3.9. *Let $m \in \mathbb{N}^*$, $u : V_m \to \mathbb{R}$, and denote by $\overline{u} \in \mathcal{H}_m$ the harmonic function of level m obtained from u. If $w \in \mathfrak{W}_m$ and $x \in S_w(V_*)$, then*

$$\min_{S_w(V_0)} u \leq \overline{u}(x) \leq \max_{S_w(V_0)} u.$$

Proof. Since $\overline{u} \circ S_w : V_* \to \mathbb{R}$ is the harmonic function of level 0 obtained from $u \circ S_w : V_0 \to \mathbb{R}$, the conclusion follows from Corollary 3.8. □

4. PROPERTIES OF FUNCTIONS OF FINITE ENERGY

We first turn to prove that functions of finite energy are Hölder continuous. The first step for this is contained in the statement of the following result. Since its proof involves the same computation and the same arguments as those performed on page 19 in [10] in the cases $N \in \{2, 3\}$, we omit it.

PROPOSITION 4.1. *Let $u \in \operatorname{dom} W$ and $m \in \mathbb{N}$. If $x, y \in V_*$ belong to the same m-cell or to adjacent m-cells, then*

$$|u(x) - u(y)| \leq \frac{2r^{\frac{m}{2}}}{1 - \sqrt{r}} \sqrt{W(u)},$$

where $r = \frac{N}{N+2}$.

In order to derive from Proposition 4.1 the Hölder continuity of functions of finite energy, we need the geometric result contained in Proposition 4.3 below, giving estimates for the distance between disjoint cells. Proposition 4.3 itself is based on the following simple fact expressed in the next lemma.

LEMMA 4.2. *Let a, b, $w \in \mathbb{R}^{N-1}$ be so that $\langle b - a, w\rangle > 0$ and define*

$$H_1 := \left\{v \in \mathbb{R}^{N-1} \mid \langle v - a, w\rangle \leq 0\right\}, \quad H_2 := \left\{v \in \mathbb{R}^{N-1} \mid \langle v - b, w\rangle \geq 0\right\}.$$

If $(v_1, v_2) \in H_1 \times H_2$, then

$$|v_1 - v_2| \geq \frac{\langle b - a, w\rangle}{|w|}.$$

Proof. The inequality $\langle b - a, w\rangle > 0$ implies in particular that $w \neq 0$. From $\langle v_1 - a, w\rangle \leq 0$ and $\langle v_2 - b, w\rangle \geq 0$ it follows that $\langle v_1 - a + b - v_2, w\rangle \leq 0$. Involving the Cauchy-Schwarz inequality, we then obtain

$$\langle b - a, w\rangle \leq \langle v_2 - v_1, w\rangle \leq |v_1 - v_2| \cdot |w|,$$

which yields the asserted inequality. □

PROPOSITION 4.3. *Let $m \in \mathbb{N}^*$, w, $w' \in \mathfrak{W}_m$, and i, $j \in \{1, \dots, N\}$ with $i \neq j$ such that $S_w(C) \cap S_{w'}(C) = \{S_w(p_i)\} = \{S_{w'}(p_j)\}$. If $\ell \in \{1, \dots, N\} \setminus \{i\}$ and $(v_1, v_2) \in S_w \circ S_\ell(C) \times S_{w'}(C)$, then*

$$|v_1 - v_2| > \frac{1}{2^{m+2}}.$$

Proof. Put

$$a := S_w \circ S_j(p_i), \quad b := S_w(p_i) = S_{w'}(p_j), \quad w := 2p_i - p_j - p_\ell = p_i - p_j + p_i - p_\ell.$$

Let H_1 and H_2 be defined as in Lemma 4.2. We show first that

$$S_w \circ S_\ell(C) \subseteq H_1. \tag{4.1}$$

For this consider an arbitrary $x \in C$. Then there exist $t_1, \dots, t_N \in [0, 1]$ such that $t_1 + \dots + t_N = 1$ and

$$x = \sum_{k=1}^{N} t_k p_k. \tag{4.2}$$

Using (2.5), we have that

$$S_w \circ S_\ell(x) - a = S_w \circ S_\ell(x) - S_w \circ S_j(p_i) = \frac{1}{2^{m+1}}(p_\ell - p_j + x - p_i). \tag{4.3}$$

From

$$\langle p_\ell - p_j + x - p_i, w\rangle = \langle p_\ell - p_j, p_i - p_j\rangle + \langle p_\ell - p_j, p_i - p_\ell\rangle + \langle x - p_i, w\rangle\,,$$

we get, applying (2.3),

$$\langle p_\ell - p_j + x - p_i, w\rangle = \langle x - p_i, w\rangle\,. \tag{4.4}$$

Since

$$\langle x - p_i, w\rangle = \sum_{k=1}^{N} t_k \langle p_k - p_i, p_i - p_j\rangle + \sum_{k=1}^{N} t_k \langle p_k - p_i, p_i - p_\ell\rangle\,,$$

(2.3) implies

$$\langle x - p_i, w\rangle = -\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{N} \frac{t_k}{2} - t_j - \sum_{\substack{k=1\\ k\neq i\\ k\neq \ell}}^{N} \frac{t_k}{2} - t_\ell.$$

From (4.3) and (4.4) we thus obtain $\langle S_w \circ S_\ell(x) - a, w\rangle \leq 0$, implying the inclusion (4.1).

We next show that

$$S_{w'}(C) \subseteq H_2. \tag{4.5}$$

Let $x \in C$ be as in (4.2) with $t_1, \dots, t_N \in [0,1]$ and $t_1 + \dots + t_N = 1$. We have by (2.5) that

$$S_{w'}(x) - b = S_{w'}(x) - S_{w'}(p_j) = \frac{1}{2^m}(x - p_j). \tag{4.6}$$

From

$$\langle x - p_j, w\rangle = \sum_{k=1}^{N} t_k \langle p_k - p_j, p_i - p_j\rangle + \sum_{k=1}^{N} t_k \langle p_k - p_j, p_i - p_\ell\rangle$$

we get, applying (2.3),

$$\langle x - p_j, w\rangle = \sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{N} \frac{t_k}{2} + t_i + \sum_{\substack{k=1\\ k\neq i\\ k\neq \ell}}^{N} t_k \langle p_k - p_j, p_i - p_\ell\rangle$$
$$+ t_\ell \langle p_\ell - p_j, p_i - p_\ell\rangle + t_i \langle p_i - p_j, p_i - p_\ell\rangle\,.$$

Involving again (2.3), we obtain

$$\langle p_\ell - p_j, p_i - p_\ell\rangle = \begin{cases} 0, & \text{if } \ell = j \\ -\frac{1}{2}, & \text{if } \ell \neq j, \end{cases}$$

$$\langle p_i - p_j, p_i - p_\ell\rangle = \begin{cases} 1, & \text{if } \ell = j \\ \frac{1}{2}, & \text{if } \ell \neq j \end{cases}$$

and, for $k \in \{1, \ldots, N\} \setminus \{i, \ell\}$,

$$\langle p_k - p_j, p_i - p_\ell \rangle = \langle p_k - p_i, p_i - p_\ell \rangle + \langle p_i - p_j, p_i - p_\ell \rangle = \begin{cases} \frac{1}{2}, & \text{if } \ell = j \\ 0, & \text{if } \ell \neq j. \end{cases}$$

Thus

$$\langle x - p_j, w \rangle = \begin{cases} \displaystyle\sum_{\substack{k=1 \\ k\neq i \\ k \neq j}}^{N} \frac{t_k}{2} + t_i + \sum_{\substack{k=1 \\ k\neq i \\ k \neq \ell}}^{N} \frac{t_k}{2} + t_i, & \text{if } \ell = j \\ \displaystyle\sum_{\substack{k=1 \\ k\neq i \\ k \neq j}}^{N} \frac{t_k}{2} + t_i - \frac{t_\ell}{2} + \frac{t_1}{2}, & \text{if } \ell \neq j. \end{cases}$$

If $\ell \neq j$, the element ℓ belongs to the set $\{1, \ldots, N\} \setminus \{i, j\}$. Thus the term $\frac{t_\ell}{2}$ appears in the sum $\displaystyle\sum_{\substack{k=1 \\ k\neq i \\ k \neq j}}^{N} \frac{t_k}{2}$. We conclude that for sure

$$\langle x - p_j, w \rangle \geq 0. \tag{4.7}$$

From (4.6) and (4.7) we get $\langle S_{w'}(x) - b, w \rangle \geq 0$, implying the inclusion (4.5).

Since, by (2.5),

$$b - a = S_w(p_i) - S_w \circ S_j(p_i) = \frac{1}{2^{m+1}}(p_i - p_j),$$

we compute, applying (2.3),

$$\langle b - a, w \rangle = \frac{1}{2^{m+1}} \langle p_i - p_j, p_i - p_j + p_i - p_\ell \rangle = \begin{cases} \frac{1}{2^m}, & \text{if } \ell = j \\ \frac{3}{2^{m+2}}, & \text{if } \ell \neq j. \end{cases}$$

Involving again (2.3), we get

$$\langle w, w \rangle = 2 + 2 \langle p_i - p_j, p_i - p_\ell \rangle = \begin{cases} 4, & \text{if } \ell = j \\ 3, & \text{if } \ell \neq j, \end{cases}$$

thus

$$\frac{\langle b - a, w \rangle}{|w|} = \begin{cases} \frac{1}{2^{m+1}}, & \text{if } \ell = j \\ \frac{\sqrt{3}}{2^{m+2}}, & \text{if } \ell \neq j. \end{cases}$$

Using (4.1), (4.5) and Lemma 4.2, we finally conclude that for sure $|v_1 - v_2| > \frac{1}{2^{m+2}}$. □

Now everything is prepared in order to prove the Hölder continuity of the functions of finite energy.

THEOREM 4.4. *Let $u \in \text{dom } W$. Then the following inequality holds*

$$|u(x) - u(y)| \leq \frac{2}{r(1-\sqrt{r})}|x-y|^{\alpha}\sqrt{W(u)}, \forall\, x,\, y \in V_*,$$

where $r = \frac{N}{N+2}$ *and* $\alpha = \frac{\ln \frac{1}{r}}{2\ln 2}$.

Proof. Let $x,\, y \in V_*$. Without any loss of generality we may assume that $x \neq y$. Set

$$M := \{k \in \mathbb{N}^* \mid x \text{ and } y \text{ belong to disjoint } k\text{-cells}\}.$$

Assuming that $M = \emptyset$, we get, for every $k \in \mathbb{N}^*$, that x and y belong either to the same k-cell or to adjoint k-cells. It follows, by Remark 2.8, that $|x-y| \leq \frac{1}{2^{k-1}}$, for all $k \in \mathbb{N}^*$, thus $x = y$, a contradiction. Hence $M \neq \emptyset$. Denote by $m := \min M$. Since, by Remark 2.7, two distinct 1-cells are adjacent, we conclude that $m \geq 2$. Also, due to the minimality of m, we have that $m-1 \notin M$. Thus x and y belong to cells of level $m-1$ with common points. We argue by contradiction to show that x and y cannot belong to the same cell of level $m-1$. Assume that $x,\, y \in S_w(V)$ with $w \in \mathfrak{W}_{m-1}$. Then there exist $i,\, j \in \{1, \ldots, N\}$ such that $x \in S_w \circ S_i(V)$ and $y \in S_w \circ S_j(V)$. Since $S_i(V) \cap S_j(V) \neq \emptyset$, we get that x and y lie in m-cells with common points, contradicting the fact that $m \in M$. Thus x and y belong to adjacent $m-1$-cells and to disjoint m-cells. Let $S_w(V)$ and $S_{w'}(V)$ (where $w,\, w' \in \mathfrak{W}_{m-1}$) be adjacent $m-1$-cells containing x, respectively, y. Proposition 2.9 implies the existence of $i,\, j \in \{1, \ldots, N\}$ with $i \neq j$ such that

$$S_w(V) \cap S_{w'}(V) = \{S_w(p_i)\} = \{S_{w'}(p_j)\}.$$

Obviously, $S_w(C) \cap S_{w'}(C) = \{S_w(p_i)\} = \{S_{w'}(p_j)\}$. Let $\ell,\, \ell' \in \{1, \ldots, N\}$ be so that $x \in S_w \circ S_\ell(V)$ and $y \in S_{w'} \circ S_{\ell'}(V)$. From $S_w \circ S_\ell(V) \cap S_{w'} \circ S_{\ell'}(V) = \emptyset$ we conclude that $(\ell, \ell') \neq (i, j)$.

Case 1: $\ell \in \{1, \ldots, N\} \setminus \{i\}$. Since $x \in S_w \circ S_\ell(C)$ and $y \in S_{w'}(C)$, Proposition 4.3 yields

$$|x - y| > \frac{1}{2^{m+1}}. \tag{4.8}$$

Case 2: $\ell = i$. In this case $\ell' \in \{1, \ldots, N\} \setminus \{j\}$. Then $x \in S_w(C)$ and $y \in S_{w'} \circ S_{\ell'}(C)$, so, applying once again Proposition 4.3, we get that (4.8) holds in this case, too.

On the other hand, since x and y belong to adjacent cells of level $m-1$, Proposition 4.1 implies that

$$|u(x) - u(y)| \leq \frac{2r^{\frac{m-1}{2}}}{1-\sqrt{r}}\sqrt{W(u)} = \frac{2r^{\frac{m+1}{2}}}{r(1-\sqrt{r})}\sqrt{W(u)}. \tag{4.9}$$

We determine now the unique positive real α satisfying the condition

$$r^{\frac{m+1}{2}} = \left(\frac{1}{2^{m+1}}\right)^{\alpha} \iff \frac{m+1}{2}\ln r = \alpha(m+1)\ln\frac{1}{2} \iff \alpha = \frac{\ln\frac{1}{r}}{2\ln 2}.$$

Since $\alpha > 0$ we thus get from (4.8) that

$$r^{\frac{m+1}{2}} = \left(\frac{1}{2^{m+1}}\right)^{\alpha} < |x-y|^{\alpha}.$$

From (4.9) we finally derive the inequality to be proved. □

Theorem 4.4 yields the following immediate consequence of it.

COROLLARY 4.5. *Let* $u \in \operatorname{dom} W$. *Then* u *is uniformly continuous, thus* u *admits a unique continuous extension to* V.

According to Corollary 4.5, the set $\operatorname{dom} W$ may be viewed as a subset of $C(V) := \{u\colon V \to \mathbb{R} \mid f \text{ continuous}\}$. The space $C(V)$ is endowed with the usual supremum norm $||\cdot||_{\sup}$.

THEOREM 4.6. *The set* $\operatorname{dom} W$ *is dense in the space* $(C(V), ||\cdot||_{\sup})$.

Proof. Pick arbitrary $f \in C(V)$ and $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \frac{\varepsilon}{2}, \text{ for all } x, y \in V \text{ with } |x-y| < \delta. \tag{4.10}$$

Let $m \in \mathbb{N}^*$ be so that $\frac{1}{2^m} < \delta$ and denote by u_m the continuous extension to V of the harmonic function of level m obtained from $f|_{V_m}$. (Recall from Remark 3.6 that the harmonic functions of level m belong to $\operatorname{dom} W$.) Consider now an arbitrary $x \in V$. Then there exists $w \in \mathfrak{W}_m$ such that $x \in S_w(V)$. Since, by Remark 2.8, $\operatorname{diam} S_w(V) = \frac{1}{2^m} < \delta$, relation (4.10) implies that

$$\max_{S_w(V_0)} f - \min_{S_w(V_0)} f \le \frac{\varepsilon}{2} \tag{4.11}$$

and

$$|f(S_w(p_1)) - f(x)| \le \frac{\varepsilon}{2}. \tag{4.12}$$

Corollary 3.9 and inequality (4.11) imply by continuity that

$$|u_m(x) - u_m(S_w(p_1))| \le \frac{\varepsilon}{2}. \tag{4.13}$$

Keeping in mind that $u_m(S_w(p_1)) = f(S_w(p_1))$ we get, using (4.12) and (4.13),

$$|u_m(x) - f(x)| \le |u_m(x) - u_m(S_w(p_1))| + |f(S_w(p_1)) - f(x)| \le \varepsilon.$$

Since $x \in V$ was arbitrarily chosen, we conclude that $||u_m - f||_{\sup} \le \varepsilon$. □

5. CONCLUSIONS

The paper presents the background of the theory of PDEs on the SG. It shows how the energy form on the SG and certain properties of real-valued functions of finite energy emerge from the harmonic extension procedure. These are the major ingredients for defining the weak Laplacian on the SG and thus for the study of PDEs on the SG (see, for instance, [3]). In this sense we mention only that the Hölder continuity of functions of finite energy (proved in Theorem 4.4) leads to a compact embedding of a certain Hilbert space (where on looks for solutions of PDEs on the SG) in a space of continuous real-valued functions, endowed with the usual supremum norm. This compact embedding is the central element which allows to investigate PDEs on this fractal using variational methods. All papers treating PDEs on the SG and mentioned in the introduction of the paper [1] are actually based on this compact embedding, whose origins lie in the harmonic extension procedure presented in Section 3 of the paper.

REFERENCES

[1] Breckner, B.E. and Varga, Cs., *Multiple solutions of Dirichlet problems on the Sierpinski gasket*, to apper in J. Optim. Theory Appl.; DOI 10.1007/s10957-013-0368-7.

[2] Falconer, K.J., *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edition, John Wiley & Sons, 2003.

[3] Falconer, K.J. and Hu, J.,*Non-linear elliptical equations on the Sierpinski gasket*, J. Math. Anal. Appl., **240** (1999), 552–573.

[4] Fukushima M. and Shima T., *On a spectral analysis for the Sierpinski gasket*, Potential Anal., **1** (1992), 1–35.

[5] Kigami, J., *A harmonic calculus on the Sierpinski spaces*, Japan J. Appl. Math., **6** (1989), 259–290.

[6] Kigami, J., *Harmonic calculus on p.c.f. self-similar sets*, Trans. Am. Math. Soc., **335** (1993), 721–755.

[7] Kigami, J., *In quest of fractal analysis*, in *Mathematics of Fractals* (H. Yamaguti, M. Hata, J. Kigami, eds.), Providence, RI: American Mathematical Society, 1993, 53–73.

[8] Kigami, J., *Analysis on Fractals*, Cambridge University Press, Cambridge, UK, 2001.

[9] Mandelbrot, B.B., *The Fractal Geometry of Nature*, Freeman, New York, 1977.

[10] Strichartz, R.S., *Differential Equations on Fractals. A Tutorial*, Princeton University Press, Princeton, NJ, 2006.

Received November 15, 2013
Accepted March 11, 2014

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. M. Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: brigitte@math.ubbcluj.ro