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ULAM STABILITY OF A CUBIC FUNCTIONAL EQUATION IN VARIOUS SPACES

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Abstract. We prove the Hyers-Ulam-Rassias stability of the cubic functional equation

$$f(x+my)+f(x-my) = 2(2\cos(\tfrac{m\pi}{2})+m^2-1)f(x) - \tfrac{1}{2}(\cos(\tfrac{m\pi}{2})+m^2-1)f(2x)+m^2(f(x+y)+f(x-y))$$

in various spaces.

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1. INTRODUCTION

In 1940, Ulam [32] proposed the following question concerning the stability of group homomorphisms: Under what condition does there exist an additive mapping near an approximately additive mapping between a group and a metric group? In the next year, Hyers [16] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [26]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instance, [1], [6], [9], [10], [17], and [24]).

According to the considerable influence of Ulam, Hyers and Rassias on the investigation of stability problems of functional equations, the stability phenomenon that was introduced and proved by Th. M. Rassias [26] in the year 1978 is called the Hyers-Ulam-Rassias stability.

In [25], Rassias introduced the cubic functional equation $f(x+2y)+3f(x) = 3f(x+y)+f(x-y)+6f(y)$ for the first time. He also solved the Ulam stability problem for this functional equation. Other results regarding the stability of various forms of the cubic functional equation in miscellaneous spaces can be found in [2], [5], [7], [11], [12], [13], [18], [23], [27], and [33].

The cubic function $f(x) = cx^3$ satisfies the functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x). \tag{1}$$

This is the reason for calling equation (1) a *cubic functional equation* and every solution of it a *cubic function*. In [18], Jun and Kim obtained a stability result

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of equation (1). In [22], Najati obtained the general solution and investigated the Ulam stability problem for the following cubic functional equation

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y). \tag{2}$$

He established the generalized Hyers-Ulam-Rassias stability problem for the functional equation (2) in quasi-Banach spaces. The stability of the functional equation (1) in random normed spaces is investigated in [3] (see also [20]). Recently, in [8], Bodaghi et al. introduced the following new form of cubic functional equations

$$\begin{aligned} f(x+my)+f(x-my)&=2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(x)\\ &-\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(2x)+m^2(f(x+y)+f(x-y)), \end{aligned} \tag{3}$$

where m is an integer with $m\geq 2$. They studied the Hyers-Ulam-Rassias stability of (3). Note that equation (3) holds for all integers.

In this paper, we prove the Hyers-Ulam-Rassias stability of the cubic functional equation (3) in non-Archimedean normed spaces, quasi-Banach spaces and random normed spaces. Since the equation (3) is equivalent to the equations (1) and (2), our proofs provide better estimations in the mentioned spaces.

2. STABILITY OF (3) IN NON-ARCHIMEDEAN NORMED SPACES

By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\leq\max\{|r|,|s|\}$, for all $r,s\in\mathbb{K}$. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:\mathcal{X}\longrightarrow\mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\| \quad (x\in\mathcal{X},\ r\in\mathbb{K})$;

(iii) the strong triangle inequality (ultrametric), i.e.,

$$\|x+y\|\leq\max\{\|x\|,\|y\|\} \qquad (x,y\in\mathcal{X}).$$

Then $(\mathcal{X},\|\cdot\|)$ is called a *non-Archimedean space*. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Due to the fact that

$$\|x_n-x_m\|\leq\max\{\|x_{j+1}-x_j\|;\ m\leq j\leq n-1\} \quad (n\geq m),$$

a sequence $\{x_n\}$ in a non-Archimedean normed space $\mathcal{X}$ is a Cauchy sequence if and only if $\{x_{n+1}-x_n\}$ converges to zero.

In [15], Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting examples of non-Archimedean spaces are provided by the spaces of p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$. In [21], the stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$ and of the quadratic functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ are investigated in non-Archimedean normed spaces.

Let m be an integer number. Given the mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$, we use, throughout the paper, the following notation:

$$\begin{aligned}\mathcal{D}_m f(x,y) &:= f(x+my) + f(x-my) - 2(2\cos(\frac{m\pi}{2}) + m^2 - 1)f(x) \\ &+ \frac{1}{2}(\cos(\frac{m\pi}{2}) + m^2 - 1))f(2x) - m^2\{f(x+y) + f(x-y)\},\end{aligned}$$

for all $x, y \in \mathcal{X}$.

Throughout this section, we assume that $\mathcal{G}$ is an additive semigroup and $\mathcal{X}$ is a complete non-Archimedean space, unless otherwise stated explicitly. In the next theorem, we prove the stability of the functional equation (3) in non-Archimedean spaces. In what follows, we denote the value $4(\cos(\frac{m\pi}{2}) + m^2 - 1)$ by λ.

THEOREM 1. *Let $\phi : \mathcal{G} \times \mathcal{G} \longrightarrow [0, \infty)$ be a function such that*

$$\lim_{k\to\infty} \frac{1}{|8|^k} \phi(2^k x, 2^k y) = 0 \tag{4}$$

for all $x, y \in \mathcal{G}$. Suppose that, for each $x \in \mathcal{G}$, the limit

$$\Phi(x) = \sup\left\{\frac{\phi(2^j x, 0)}{|8|^j} : 0 \le j < n\right\} \tag{5}$$

exists. Assume that $f : \mathcal{G} \longrightarrow \mathcal{X}$ is a mapping satisfying the inequality

$$\|\mathcal{D}_m f(x,y)\| \le \phi(x,y) \tag{6}$$

for all $x, y \in \mathcal{G}$, where m is an integer with $m \ne 0, \pm 1$. If

$$\lim_{l\to\infty} \sup\left\{\frac{\phi(2^j x, 0)}{|8|^j} : l \le j < n + l\right\} = 0 \quad (x \in \mathcal{G}), \tag{7}$$

then there exists a unique cubic mapping $\mathcal{C} : \mathcal{G} \longrightarrow \mathcal{X}$ such that

$$\|f(x) - \mathcal{C}(x)\| \le \frac{1}{|\lambda|}\Phi(x) \tag{8}$$

for all $x \in \mathcal{G}$.

Proof. Putting $y = 0$ in (6), we have

$$\left\|\frac{\lambda}{8} f(2x) - \lambda f(x)\right\| \le \phi(x, 0)$$

for all $x \in \mathcal{G}$. Thus

$$\left\|\frac{1}{8} f(2x) - f(x)\right\| \le \frac{\phi(x,0)}{|\lambda|} \tag{9}$$

for all $x \in \mathcal{G}$. Replacing x by $2^n x$ in (7) and dividing both sides by $|8|^n$, we get

$$\left\| \frac{1}{8^{n+1}} f(2^n x) - \frac{1}{8^n} f(2^n x) \right\| \leq \frac{\phi(x,0)}{|8|^n |\lambda|} \tag{10}$$

for all $x \in \mathcal{G}$ and all non-negative integers n. Thus, by (4) and (10), the sequence $\left\{ \frac{f(2^n x)}{8^n} \right\}$ is Cauchy. The completeness of the non-Archimedean space $\mathcal{X}$ yields the existence of a map $\mathcal{C}$ so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{8^n} = \mathcal{C}(x) \qquad (x \in \mathcal{G}). \tag{11}$$

For each $x \in \mathcal{X}$ and all non-negative integers n, we have

$$\begin{aligned} \left\| \frac{f(2^n x)}{8^n} - f(x) \right\| &= \left\| \sum_{j=0}^{n-1} \frac{f(2^{j+1} x)}{8^{j+1}} - \frac{f(2^j x)}{8^j} \right\| \\ &\leq \max \left\{ \left\| \frac{f(2^{j+1} x)}{8^{j+1}} - \frac{f(2^j x)}{8^j} \right\| : 0 \leq j < n \right\} \\ &\leq \frac{1}{|\lambda|} \max \left\{ \frac{\phi(2^j x, 0)}{|8|^j} : 0 \leq j < n \right\}. \end{aligned} \tag{12}$$

Taking n to approach infinity in (12) and applying (11), we conclude that the inequality (6) holds. It follows from (4), (5) and (10) that, for all $x, y \in \mathcal{G}$,

$$\|\mathcal{D}_m \mathcal{C}(x,y)\| = \lim_{n\to\infty} \frac{1}{|8|^n} \|\mathcal{D}_m f(x,y)\| \leq \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{|8|^n} = 0.$$

Hence the mapping $\mathcal{C}$ is cubic. Assume now that $\mathcal{C}' : \mathcal{G} \longrightarrow \mathcal{X}$ is another cubic map that satisfies (8). Then we have, for all $x \in \mathcal{G}$, that

$$\begin{aligned} \|\mathcal{C}(x) - \mathcal{C}'(x)\| &= \lim_{k\to\infty} \frac{1}{|8|^k} \|\mathcal{C}(2^k x) - \mathcal{C}'(2^k x)\| \\ &\leq \lim_{k\to\infty} \frac{1}{|8|^k} \max \left\{ \|\mathcal{C}(2^k x) - f(2^k x)\|, \|f(2^k x) - \mathcal{C}'(2^k x)\| \right\} \\ &\leq \frac{1}{|\lambda|} \lim_{k\to\infty} \lim_{n\to\infty} \max \left\{ \frac{\phi(2^j x, 0)}{|2|^j} : k \leq j < n + k \right\} \\ &= \frac{1}{|\lambda|} \lim_{k\to\infty} \sup \left\{ \frac{\phi(2^j x, 0)}{|2|^j} : k \leq j < \infty \right\} = 0. \end{aligned}$$

This shows the uniqueness of $\mathcal{C}$. $\square$

Corollary 1. *Let* $\Gamma : [0, \infty) \longrightarrow [0, \infty)$ *be a function satisfying* $\Gamma(|r|s) \leq \Gamma(|r|)\Gamma(s)$, *for all* $r, s \in [0, \infty)$, *and* $\Gamma(|2|) < |8|$. *Suppose that* $\mathcal{G}$ *is a normed space and that* $f : \mathcal{G} \longrightarrow \mathcal{X}$ *is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha \left(\Gamma(\|x\|) + \Gamma(\|y\|) \right) \tag{13}$$

for all $x, y \in \mathcal{G}$, where m is an integer with $m \neq 0, \pm 1$ and $\alpha > 0$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{G} \longrightarrow \mathcal{X}$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{\alpha \Gamma(\|x\|)}{|\lambda|} \tag{14}$$

for all $x \in \mathcal{G}$.

Proof. We show that all conditions required in Theorem 1 are satisfied. Defining $\phi : \mathcal{G} \times \mathcal{G} \longrightarrow [0, \infty)$ via $\phi(x, y) = \alpha\left(\Gamma(\|x\|) + \Gamma(\|y\|)\right)$, we obtain

$$\lim_{n\to\infty} \frac{1}{|8|^n} \phi(2^n x, 2^n y) \leq \lim_{n\to\infty} \left(\frac{\Gamma(|2|)}{|8|}\right)^n \phi(x, y) = 0$$

for all $x, y \in \mathcal{G}$. Also

$$\Phi(x) = \sup\left\{\frac{\phi(2^j x, 0)}{|8|^j} : 0 \leq j < n\right\} = \phi(x, 0) = \alpha\left(\Gamma(\|x\|)\right)$$

for all $x \in \mathcal{G}$. On the other hand,

$$\lim_{l\to\infty} \sup\left\{\frac{\phi(2^j x, 0)}{|8|^j} : l \leq j < n + l\right\} = 0 \quad (x \in \mathcal{G}).$$

Theorem 1 implies now the asserted result. $\square$

Remark 1. An example of a function Γ, satisfying the assumptions of Corollary 1, is the mapping $\Gamma(t) = t^p$ $(t \in [0, \infty))$, where $p \in \mathbb{R}$ is such that $|2| = 1$.

We have the following analogous result to Theorem 1 for the cubic equation (3). The proof is similar, but, for sake of completeness, we include it here.

Theorem 2. *Let $\phi : \mathcal{G} \times \mathcal{G} \longrightarrow [0, \infty)$ be a function such that*

$$\lim_{k\to\infty} |8|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \tag{15}$$

for all $x, y \in \mathcal{G}$. Suppose that, for each $x \in \mathcal{G}$, the limit

$$\Phi(x) = \sup\left\{|8|^j \phi\left(\frac{x}{2^j}, 0\right) : 0 \leq j < n\right\} \tag{16}$$

exists. Assume that $f : \mathcal{G} \longrightarrow \mathcal{X}$ is a mapping satisfying the inequality

$$\|\mathcal{D}_m f(x, y)\| \leq \phi(x, y) \tag{17}$$

for all $x, y \in \mathcal{G}$, where m is an integer with $m \neq 0, \pm 1$. If

$$\lim_{l\to\infty} \sup\left\{|8|^j \phi\left(\frac{x}{2^j}, 0\right) : l \leq j < n + l\right\} = 0 \quad (x \in \mathcal{G}), \tag{18}$$

then there exists a unique cubic mapping $\mathcal{C} : \mathcal{G} \longrightarrow \mathcal{X}$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \left|\frac{64}{\lambda}\right| \Phi(x) \tag{19}$$

for all $x \in \mathcal{G}$.

Proof. Similar to the proof of Theorem 1, we have

$$\left\|\frac{\lambda}{4}f(2x)-2\lambda f(x)\right\|\leq |2|\phi(x,0) \tag{20}$$

for all $x\in\mathcal{G}$. If we replace x by $\frac{x}{2^{n+1}}$ in the above inequality and multiply both sides of (20) by $|8|^n$, then we get

$$\left\|8^n f\left(\frac{x}{2^n}\right)-8^{n+1}f\left(\frac{x}{2^{n+1}}\right)\right\|\leq \frac{1}{|\lambda|}|8|^{n+1}\phi\left(\frac{x}{2^{n+1}},0\right) \tag{21}$$

for all $x\in\mathcal{X}$ and all non-negative integers n. Thus we conclude from (15) and (21) that $\left\{8^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence. Since the non-Archimedean space $\mathcal{Y}$ is complete, this sequence leads to the mapping $\mathcal{C}$, i.e.,

$$\mathcal{C}(x)=\lim_{n\to\infty}8^n f\left(\frac{x}{2^n}\right)\qquad (x\in\mathcal{G}). \tag{22}$$

It follows from (21) that

$$\begin{aligned}\left\|8^n f\left(\frac{x}{2^n}\right)-8^l f\left(\frac{x}{2^l}\right)\right\| &= \left\|\sum_{j=l}^{n}8^{j+1}f\left(\frac{x}{2^{j+1}}\right)-8^j f\left(\frac{x}{2^j}\right)\right\|\\ &\leq \max\left\{\left\|8^{j+1}f\left(\frac{x}{2^{j+1}}\right)-8^j f\left(\frac{x}{2^j}\right)\right\| : l\leq j<n\right\}\\ &\leq \left|\frac{64}{\lambda}\right|\max\left\{|8|^j\phi\left(\frac{x}{2^j},0\right) : l\leq j<n\right\} \end{aligned}\tag{23}$$

for all $x\in\mathcal{G}$ and all non-negative integers n,l with $n>l\geq 0$. Letting $l=0$ and taking n to approach infinity in (23), and applying (18), we obtain that inequality (19) holds. The final part of the proof can be performed as in case of Theorem 1. □

COROLLARY 2. *Let $\Gamma:[0,\infty)\longrightarrow[0,\infty)$ be a function satisfying $\Gamma(|r|s)\leq\Gamma(|r|)\Gamma(s)$, for all $r,s\in[0,\infty)$, and $\Gamma(|2|^{-1})<|8|^{-1}$. Suppose that $\mathcal{G}$ is a normed space and that $f:\mathcal{G}\longrightarrow\mathcal{X}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\|\leq\alpha\left(\Gamma(\|x\|)+\Gamma(\|y\|)\right) \tag{24}$$

for all $x,y\in\mathcal{G}$, where m is an integer with $m\neq 0,\pm1$ and $\alpha>0$. Then there exists a unique cubic mapping $\mathcal{C}:\mathcal{G}\longrightarrow\mathcal{X}$ such that

$$\|f(x)-\mathcal{C}(x)\|\leq\left|\frac{64}{\lambda}\right|\alpha\Gamma(\|x\|) \tag{25}$$

for all $x\in\mathcal{G}$.

Proof. Defining $\phi:\mathcal{G}\times\mathcal{G}\longrightarrow[0,\infty)$ via $\phi(x,y)=\alpha\left(\Gamma(\|x\|)+\Gamma(\|y\|)\right)$, we get

$$\lim_{n\to\infty}|8|^n\phi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\leq\lim_{n\to\infty}\left(\frac{\Gamma(|2|^{-1})}{|8|^{-1}}\right)^n\phi(x,y)=0$$

for all $x, y \in \mathcal{G}$. Moreover, we have that

$$\Phi(x) = \sup\left\{|8|^j \phi\left(\frac{x}{2^j}, 0\right) : 0 \le j \le n-1\right\} = \phi(x, 0) = \alpha\left(\Gamma(\|x\|)\right)$$

for all $x \in \mathcal{G}$. Also,

$$\lim_{l\to\infty} \sup\left\{|8|^j \phi\left(\frac{x}{2^j}, 0\right) : l \le j < n+l\right\} = 0 \quad (x \in \mathcal{G}).$$

The result follows now from Theorem 2. □

3. STABILITY OF (3) IN QUASI-BANACH SPACES

We recall first some basic facts concerning quasi-Banach space and some preliminary results which are taken from [4] and [28].

Definition 1. Let $\mathcal{X}$ be a real linear space. A *quasi-norm* is a real-valued function on $\mathcal{X}$ satisfying the following conditions:

(i) $\|x\| \ge 0$, for all $x \in \mathcal{X}$, and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|tx\| = |t|\|x\|$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$;
(iii) there is a constant $M \ge 1$ such that $\|x + y\| \le M(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

Note that condition (iii) implies that

$$\left\|\sum_{j=1}^{2n} x_j\right\| \le M^n \sum_{j=1}^{2n} \|x_j\| \quad \text{and} \quad \left\|\sum_{j=1}^{2n+1} x_j\right\| \le M^{n+1} \sum_{j=1}^{2n+1} \|x_j\|,$$

for all $n \ge 1$ and all $x_1, x_2, ..., x_{2n+1} \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest real M is called the *modulus of concavity of* $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \le 1)$ if $\|x+y\|^p \le \|x\|^p + \|y\|^p$, for all $x, y \in \mathcal{X}$. In this case, the quasi-Banach space is called a *p-Banach space*.

Given a p-norm, the function $d(x, y) := \|x - y\|^p$ gives rise to a translation invariant metric on $\mathcal{X}$. By the Aoki-Rolewicz Theorem [28] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, subsequently we will restrict our attention mainly to p-norms. Moreover, in [31], Tabor has investigated a version of the Hyers-Rassias-Gajda Theorem in quasi-Banach spaces.

Till the end of this section, let $\mathcal{X}$ be a real normed space with norm $\|\cdot\|_{\mathcal{X}}$ and let $\mathcal{Y}$ be a real p-Banach space with norm $\|\cdot\|_{\mathcal{Y}}$. In this section, by using an idea of Găvruţă [14], we prove the stability of (3) in the spirit of Hyers, Ulam, and Rassias.

THEOREM 3. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ such that*

$$\widetilde{\varphi}(x) := \sum_{k=0}^{\infty} \frac{1}{8^{kp}} \varphi^p(2^k x, 0) < \infty, \ \lim_{k\to\infty} \frac{1}{8^k} \varphi(2^k x, 2^k y) = 0 \tag{26}$$

and

$$\|\mathcal{D}_m f(x, y)\| \leq \varphi(x, y) \tag{27}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \neq 0, \pm 1$. Then there exists a unique cubic mapping $C : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{8} \left[\frac{\widetilde{\varphi}(x)}{\lambda}\right]^{\frac{1}{p}} \tag{28}$$

for all $x \in \mathcal{X}$.

Proof. Setting $y = 0$ in (27), we get

$$\left\|\frac{\lambda}{8} f(2x) - \lambda f(x)\right\| \leq \varphi(x, 0)$$

for all $x \in \mathcal{X}$. Hence

$$\left\|\frac{1}{8} f(2x) - f(x)\right\| \leq \frac{\varphi(x, 0)}{\lambda} \tag{29}$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ in (29) and dividing both sides of (29) by 8^n, we obtain

$$\left\|\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}\right\| \leq \frac{\varphi(2^n x, 0)}{\lambda 8^n} \tag{30}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Since $\mathcal{X}$ is a p-Banach space, we have

$$\begin{aligned}\left\|\sum_{j=k}^{n} \frac{f(2^{j+1}x)}{8^{j+1}} - \frac{f(2^j x)}{8^j}\right\|^p &\leq \sum_{j=k}^{n} \left\|\frac{f(2^{j+1}x)}{8^{j+1}} - \frac{f(2^j x)}{8^j}\right\|^p \\ &\leq \frac{1}{\lambda} \sum_{j=k}^{n} \frac{\varphi^p(2^j x, 0)}{8^{(j+1)p}}\end{aligned}$$

for all $x \in \mathcal{X}$ and all integers $n \geq k \geq 0$. Thus,

$$\left\|\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^k x)}{8^k}\right\|^p \leq \frac{1}{\lambda} \sum_{j=k}^{n} \frac{\varphi^p(2^j x, 0)}{8^{(j+1)p}} \tag{31}$$

for all $x \in \mathcal{X}$ and all integers $n \geq k \geq 0$. Due to the convergence of the series $\sum_{j\geq k} \frac{1}{8^j}$, the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is, by (26) and (31), a Cauchy sequence.

According to the completeness of $\mathcal{Y}$, there exists a map C so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{8^n} = C(x). \tag{32}$$

It follows from (26) and (32) that

$$\|\mathcal{D}_m C(x,y)\| \leq \lim_{n\to\infty} \frac{1}{8^n}\|\mathcal{D}_m f(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{8^n} = 0$$

for all $x, y \in \mathcal{X}$. Hence, by [8, Theorem 2.1], $C : \mathcal{X} \longrightarrow \mathcal{Y}$ is a cubic mapping. Putting $k = 0$ and letting n to infinity in (31), we see that (28) holds. In order to prove the uniqueness of C, assume that $C' : \mathcal{X} \longrightarrow \mathcal{Y}$ is another cubic mapping satisfying (28). Then

$$\begin{aligned} \|C(x) - C'(x)\|^p &= \frac{1}{8^{np}}\|f(2^n x) - C'(2^n x)\|^p \\ &\leq \lim_{n\to\infty} \frac{\widetilde{\varphi}(x)}{\lambda 8^{(n+1)p}} \\ &= \frac{1}{\lambda 8^p} \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{8^{kp}} \varphi^p(2^k x, 0) \end{aligned}$$

for all $x \in \mathcal{X}$. This completes the proof. $\square$

Corollary 3. *Let α be a non-negative real number and let $r, s \in (0,3)$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha(\|x\|^r + \|y\|^s) \tag{33}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \neq 0, \pm 1$, then there exists a unique cubic mapping $C : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\|f(x) - C(x)\| \leq \frac{\alpha\|x\|^r}{[\lambda(8^p - 2^{rp})]^{\frac{1}{p}}} \tag{34}$$

for all $x \in \mathcal{X}$. Furthermore, if, for each fixed $x \in \mathcal{X}$, the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $C(tx) = t^3 C(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Proof. Note that the inequality (33) implies that $f(0) = 0$. Putting $\varphi(x,y) = \alpha(\|x\|^r + \beta\|y\|^s)$ in Theorem 3, we obtain the inequality (34). Now, similar to the proof of [8, Theorem 4.3], we can show that $C(tx) = t^3 C(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$. $\square$

Theorem 4. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ such that*

$$\widetilde{\varphi}(x) := \sum_{k=1}^{\infty} 8^{kp} \varphi^p\left(\frac{x}{2^k}, 0\right) < \infty, \ \lim_{k\to\infty} 8^k \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \tag{35}$$

and

$$\|\mathcal{D}_m f(x,y)\| \leq \psi(x,y) \tag{36}$$

for all $x, y \in \mathcal{X}$*, where* m *is an integer with* $m \neq 0, \pm 1$*. Then there exists a unique cubic mapping* $C : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\|f(x) - C(x)\| \leq \left[\frac{8\widetilde{\varphi}(x)}{\lambda}\right]^{\frac{1}{p}} \tag{37}$$

for all $x \in \mathcal{X}$*.*

Proof. It follows from (35) that $\varphi(0,0) = 0$, so (36) implies that $f(0) = 0$. Putting $y = 0$ in (27), we get

$$\|f(2x) - 8f(x)\| \leq \frac{8\varphi(x,0)}{\lambda} \tag{38}$$

for all $x \in \mathcal{X}$. Replacing x by $\frac{x}{2^{n+1}}$ in (38) and multiplying both sides of (38) by 8^n, we deduce that

$$\left\|8^n f\left(\frac{x}{2^n}\right) - 8^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| \leq 8^{n+1} \frac{\varphi\left(\frac{x}{2^{n+1}}, 0\right)}{\lambda} \tag{39}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Since $\mathcal{Y}$ is a p-normed space, we have

$$\begin{aligned} \left\|\sum_{j=k}^{n}\left(8^j f\left(\frac{x}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right)\right\|^p &\leq \sum_{j=k}^{n}\left\|8^j f\left(\frac{x}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|^p \\ &\leq \frac{8}{\lambda}\sum_{j=k}^{n} 8^{(j+1)p}\varphi^p\left(\frac{x}{2^{j+1}}, 0\right) \end{aligned}$$

for all $x \in \mathcal{X}$ and all integers $n \geq k \geq 0$. Thus

$$\left\|8^k f\left(\frac{x}{2^k}\right) - 8^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\|^p \leq \frac{8}{\lambda}\sum_{j=k}^{n} 8^{(j+1)p}\varphi^p\left(\frac{x}{2^{j+1}}, 0\right) \tag{40}$$

for all $x \in \mathcal{X}$ and all integers $n \geq k \geq 0$. It follows from (35) and (40) that $\left\{8^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is a p-Banach space, there exists a map C so that

$$\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x). \tag{41}$$

Setting $k = 0$ and letting n to infinity in (40), we get that (37) holds. The rest of the proof is similar to the proof of Theorem 3. □

Corollary 4. *Let* α *be a non-negative real number and let* $r, s \in (3, \infty)$*. If* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha(\|x\|^r + \|y\|^s)$$

for all $x, y \in \mathcal{X}$*, where* m *is an integer with* $m \neq 0, \pm 1$*, then there exists a unique cubic mapping* $C : \mathcal{X} \longrightarrow \mathcal{Y}$ *satisfying*

$$\|f(x) - C(x)\| \leq 8\alpha\left(\frac{8}{[\lambda(2^{rp} - 8^p)]}\right)^{\frac{1}{p}} \|x\|^r \tag{42}$$

for all $x \in \mathcal{X}$. Furthermore, if, for each fixed $x \in \mathcal{X}$, the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $C(tx) = t^3 C(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Proof. The inequality (42) follows from Theorem 4, by taking $\varphi(x,y) := \alpha(\|x\|^r + \beta\|y\|^s)$. For the other statement we refer to the proof of Corollary 3. □

The cubic functional equation (3) can be superstable under some conditions, as it is shown by the next result.

Corollary 5. *Let r, s and α be a non-negative real numbers such that $r + s \neq 3$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha\|y\|^s \text{ (or } \alpha\|x\|^r\|y\|^s) \tag{43}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \neq 0, \pm 1$, then the mapping f is cubic. Furthermore, if, for each fixed $x \in \mathcal{X}$, the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is continuous, then $C(tx) = t^3 C(x)$ for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Proof. The inequality (43) shows that $f(0) = 0$. Putting $y = 0$ in (43), we get $f(2x) = 8f(x)\,(x \in \mathcal{X})$, and thus $f(x) = \frac{f(2^n x)}{8^n}$ for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. Letting $\varphi(x,y) = \alpha\|y\|^s$ (or $\varphi(x,y) = \alpha\|x\|^r\|y\|^s$) in Theorems 3 and 4, we get that $C = f$ is a cubic mapping. The rest of the proof follows from the proof of [8, Theorem 4.3]. □

Since the following corollaries are direct consequences of Theorems 3 and 4, respectively, we omit their proofs.

Corollary 6. *Let α be a non-negative real number and assume that $r+s \in (0,3)$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha(\|x\|^r + \|y\|^s + \|x\|^r\|y\|^s)$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \neq 0, \pm 1$, then there exists a unique cubic mapping $C : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\|f(x) - C(x)\| \leq \frac{\alpha\|x\|^r}{[\lambda(8^p - 2^{rp})]^{\frac{1}{p}}}$$

for all $x \in \mathcal{X}$.

Corollary 7. *Let α be a non-negative real number and assume that $r+s \in (3,\infty)$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha(\|x\|^r + \|y\|^s + \|x\|^r\|y\|^s)$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \neq 0, \pm 1$, then there exists a unique cubic mapping $C : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\|f(x) - C(x)\| \leq 8\alpha\left(\frac{8}{[\lambda(2^{rp} - 8^p)]}\right)^{\frac{1}{p}}\|x\|^r$$

for all $x \in \mathcal{X}$.

4. STABILITY OF (3) IN RANDOM NORMED SPACES

We first state the usual terminology, notations and conventions of the theory of random normed spaces, following [29] and [30]. The set of all probability distribution functions is denoted by

$$\Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1] |\, F \text{ is left-continuous and nondecreasing on } \mathbb{R}; \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1\}.$$

Let us define $D^+ := \{F \in \Delta^+ |\, l^- F(+\infty) = 1\}$, where $l^- F(x)$ denotes the left limit of the function f at the point x. The set Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element of Δ^+ with respect to this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2. ([29]) A mapping $\tau : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a *continuous triangular norm* (briefly, a continuous t-norm) if τ satisfies the following conditions:

(i) τ is commutative and associative;
(ii) τ is continuous;
(iii) $\tau(a,1) = a$ for all $a \in [0,1]$;
(iv) $\tau(a,b) \le \tau(c,d)$, whenever $a \le c$ and $c \le d$ for $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $\tau_P(a,b) = ab$, $\tau_M(a,b)$=min{a,b} and $\tau_L(a,b)$=max$\{a+b-1, 0\}$.

Definition 3. ([30]) A *random normed space* (briefly, RN-space) is a triple $(\mathcal{X}, \mu, \tau)$, where $\mathcal{X}$ is a vector space, τ is a continuous t-norm, and μ is a mapping from $\mathcal{X}$ into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \epsilon_0(t)$, for all $t > 0$, if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in \mathcal{X}$, all $\alpha \ne 0$ and all $t \ge 0$;
(RN3) $\mu_{x+y}(t+s) \ge \tau(\mu_x(t), \mu_x(s))$ for all $x, y \in \mathcal{X}$ and all $t, s \ge 0$.

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. Define the mapping $\mu : \mathcal{X} \longrightarrow D^+$ via $\mu_x(t) = \frac{t}{t+\|x\|}$ for all $x \in \mathcal{X}$ and all $t \ge 0$. Then $(\mathcal{X}, \mu, \tau_M)$ is a random normed space.

Definition 4. Let $(\mathcal{X}, \mu, \tau)$ be an RN-space.

(1) A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be *convergent* to a point $x \in \mathcal{X}$ if, for every $t > 0$ and every $\epsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \epsilon$ whenever $n \ge N$.
(2) A sequence $\{x_n\}$ in $\mathcal{X}$ is called a *Cauchy sequence* if, for every $t > 0$ and every $\epsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \epsilon$ whenever $n \ge m \ge N$.

(3) An RN-space $(\mathcal{X}, \mu, \tau)$ is said to be *complete* if every Cauchy sequence in $\mathcal{X}$ is convergent to a point in $\mathcal{X}$.

THEOREM 5. ([29]) *If $(\mathcal{X}, \mu, \tau)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Given a t-norm τ and a sequence $\{a_n\}$ in $[0, 1]$, we define $\tau_{j=1}^n a_j$ recursively by $\tau_{j=1}^1 a_j = a_1$ and $\tau_{j=1}^n a_j = \tau(\tau_{j=1}^{n-1} a_j, a_n)$, for all $n \geq 2$. We now establish the stability of the functional equation (3) in the setting of random normed spaces.

THEOREM 6. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Suppose that $\psi : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Z}$ is a mapping such that for some $0 < \alpha < 8$*

$$\Lambda_{\psi(2x,0)}(t) \geq \Lambda_{\alpha\psi(x,0)}(t) \qquad (x \in \mathcal{X},\ t > 0) \tag{44}$$

and

$$\lim_{n\to\infty} \Lambda_{\psi(2^n x, 2^n y)}(8^n t) = 1 \qquad (x, y \in \mathcal{X},\ t > 0). \tag{45}$$

If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping with $f(0) = 0$ and

$$\mu_{\mathcal{D}_m f(x,y)}(t) \geq \Lambda_{\psi(x,y)}(t) \tag{46}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, where m is an integer with $m \neq 0, \pm 1$, then there exists a unique cubic mapping $C : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\mu_{f(x)-\mathcal{C}(x)}(t) \geq \Lambda_{\psi(x,y)}\left(\frac{\lambda(8-\alpha)}{8}t\right) \tag{47}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Setting $y = 0$ in (46), we have

$$\mu_{\left(\frac{1}{8}f(2x)-f(x)\right)} \geq \Lambda_{\psi(x,0)}(\lambda t) \tag{48}$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ in (48) and using (44), we obtain

$$\begin{aligned} \mu_{\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}\right)} &\geq \Lambda_{\psi(2^n x,0)}(8^n \lambda t) \\ &\geq \Lambda_{\alpha^n \psi(x,0)}(8^n \lambda t) \\ &\geq \Lambda_{\psi(x,0)}\left(\left(\frac{8}{\alpha}\right)^n \lambda t\right) \end{aligned} \tag{49}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Applying the inequality (49), we get

$$\begin{aligned}\mu_{\left(\frac{f(2^n x)}{8^n}-f(x)\right)}\left(\frac{t}{\lambda}\sum_{j=0}^{n-1}\left(\frac{\alpha}{8}\right)^j\right) &= \mu_{\left(\sum_{j=0}^{n-1}\left(\frac{f(2^{j+1}x)}{8^{j+1}}-\frac{f(2^j x)}{8^j}\right)\right)}\left(\frac{t}{\lambda}\sum_{j=0}^{n-1}\left(\frac{\alpha}{8}\right)^j\right)\\ &\geq (\tau_M)_{j=0}^{n-1}(\Lambda_{\psi(x,0)}(t))\\ &= \Lambda_{\psi(x,0)}(t)\end{aligned}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Thus

$$\mu_{\left(\frac{f(2^n x)}{8^n}-f(x)\right)}(t) \geq \Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{\lambda}\sum_{j=0}^{n-1}\left(\frac{8}{\alpha}\right)^j}\right). \tag{50}$$

Substituting x into $2^l x$ in (50), we obtain

$$\mu_{\left(\frac{f(2^{n+l}x)}{8^{n+l}}-\frac{f(2^l x)}{8^l}\right)} \geq \Lambda_{\psi(x,0)}\left(\frac{t}{\left(\frac{1}{\lambda}\sum_{j=l}^{l+n}\left(\frac{\alpha}{8}\right)^j\right)}\right) \tag{51}$$

for all $x \in \mathcal{X}$ and all integers $n \geq l \geq 0$. Since the series $\sum_{j\geq l}\left(\frac{\alpha}{8}\right)^j$ is convergent, $\Lambda_{\psi(x,0)}\left(\frac{t}{\left(\frac{1}{\lambda}\sum_{j=l}^{l+n}\left(\frac{\alpha}{8}\right)^j\right)}\right)$ goes to 1 as l and n tend to infinity, and so $\left\{\frac{f(2^n x)}{8^n}\right\}$ is a Cauchy sequence in $(\mathcal{Y}, \mu, \tau_M)$. The completeness of $(\mathcal{Y}, \mu, \tau_M)$ as a RN-space implies that this sequence converges to some point $C(x) \in \mathcal{Y}$. It follows from (50) that, for each $\epsilon > 0$,

$$\begin{aligned}\mu_{(C(x)-f(x))}(t+\epsilon) &\geq \tau_M\left(\mu_{\left(C(x)-\frac{f(2^n x)}{8^n}\right)}(\epsilon), \mu_{\left(\frac{f(2^n x)}{8^n}-f(x)\right)}(t)\right)\\ &\geq \tau_M\left(\mu_{\left(C(x)-\frac{f(2^n x)}{8^n}\right)}(\epsilon), \Lambda_{\psi(x,0)}\left(\frac{t}{\left(\frac{1}{\lambda}\sum_{j=0}^{n-1}\left(\frac{\alpha}{8}\right)^j\right)}\right)\right)\end{aligned}$$

for all $x \in \mathcal{X}$. Letting n tend to infinity in the above inequality, we deduce that

$$\mu_{(C(x)-f(x))}(t+\epsilon) \geq \Lambda_{\psi(x,0)}\left(\frac{\lambda(8-\alpha)}{8}t\right). \tag{52}$$

Taking $\epsilon \to 0$ in (52), we get (47). Moreover, the inequality (46) implies that

$$\mu_{\frac{1}{8^n}\mathcal{D}_m f(2^n x, 2^n y)}(t) \geq \Lambda_{\psi(2^n x, 2^n y)}(8^n t) \tag{53}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Taking n to infinity in (53) and applying (45), we conclude that the mapping C is cubic. In order to prove the uniqueness of

C, assume that $C' : \mathcal{X} \longrightarrow \mathcal{Y}$ is another cubic mapping satisfying (47). Then

$$\begin{aligned}\mu_{\left(\frac{C(2^n x)}{8^n} - \frac{C'(2^n x)}{8^n}\right)}(t) &\geq \min\left\{\mu_{\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}\right)}\left(\frac{t}{2}\right), \mu_{\left(\frac{C'(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}\right)}\left(\frac{t}{2}\right)\right\} \\ &\geq \Lambda_{(\psi(2^n x,0))}\left(8^n \frac{\lambda(8-\alpha)}{16} t\right) \\ &\geq \Lambda_{(\psi(x,0))}\left(\left(\frac{8}{\alpha}\right)^n \frac{\lambda(8-\alpha)}{16} t\right)\end{aligned}$$

for all $x \in \mathcal{X}$. Therefore

$$\begin{aligned}\mu_{C(x)-C'(x)}(t) &= \lim_{n\to\infty} \mu_{\left(\frac{C(2^n x)}{8^n} - \frac{C'(2^n x)}{8^n}\right)}(t) \\ &\geq \lim_{n\to\infty} \Lambda_{(\psi(x,0))}\left(\left(\frac{8}{\alpha}\right)^n \frac{\lambda(8-\alpha)}{16} t\right) = 1.\end{aligned}$$

The above relations show that $C(x) = C'(x)$ for all $x \in \mathcal{X}$. This finishes the proof. $\square$

COROLLARY 8. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be real numbers such that $r, s \in [0, 3)$ and consider $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\mu_{\mathcal{D}_m f(x,y)}(t) \geq \Lambda_{\|x\|^r z_0}(t) \tag{54}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, where m is an integer with $m \neq 0, \pm 1$, then there exists a unique cubic mapping $\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfying

$$\mu_{f(x)-\mathcal{C}(x)}(t) \geq \Lambda_{\|x\|^r z_0}\left(\frac{\lambda(8-2^r)}{8}\right) \tag{55}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Setting $x = y = 0$ in (56), we see that $f(0) = 0$. Now, by defining $\psi(x, y) := (\|x\|^r + \|y\|^s)z_0$ and applying Theorem 6 for $\alpha = 2^r$, we get the desired result. $\square$

COROLLARY 9. *Let $\mathcal{X}$ be a linear space, $(\mathcal{Z}, \Lambda, \tau_M)$ be an RN-space and $(\mathcal{Y}, \mu, \tau_M)$ be a complete RN-space. Let r, s be non-negative real numbers such that $r + s \neq 3$ and consider $z_0 \in \mathcal{Z}$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\mu_{\mathcal{D}_m f(x,y)}(t) \geq \Lambda_{\|x\|^r \|y\|^s z_0}(t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$, where m is an integer with $m \neq 0, \pm 1$, then f is a cubic mapping.

Proof. Take $\psi(x, y) := \|x\|^r \|y\|^s z_0$. The assertion follows now from Theorem 6. $\square$

REFERENCES

[1] Aoki, T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan., **2** (1950), 64–66.
[2] Arunkumar, M., Rassias, M.J. and Zhang, Y., *Ulam-Hyers stability of a 2-variable AC-mixed type functional equation: direct and fixed point methods*, Journal of Modern Mathematics Frontier, **1** (2012), 10–26.
[3] Baktash, E., Cho, Y.J., Jalili, M., Saadati, R., and Vaezpour, S.M., *On the stability of cubic mappings and quadratic mappings in random normed spaces*, J. Inequal. Appl., (2008), Article ID 902187, DOI 10.1155/2008/902187, 11 pages.
[4] Benyamini,Y. and Lindenstrauss, J., *Geometric Nonlinear Functional Analysis*, Vol. 1, American Mathematical Society Colloquium Publications, **48**, American Mathematical Society, Providence, RI, 2000.
[5] Bodaghi, A., *Cubic derivations on Banach algebras*, Acta Math. Vietnam., **38** (2013), 517–528.
[6] Bodaghi, A. and Alias, I.A., *Approximate ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings*, Adv. Difference Equ., (2012), Article No. 11.
[7] Bodaghi, A., Alias, I.A. and Ghahramani, M.H., *Approximately cubic functional equations and cubic multipliers*, J. Inequal. Appl., (2011), Article No. 53.
[8] Bodaghi, A., Moosavi, S.M. and Rahimi, H., *The generalized cubic functional equation and the stability of cubic Jordan $*$-derivations*, Ann. Univ. Ferrara Sez. VII Sci. Mat, **59** (2013), 235–250.
[9] Czerwik, S., *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hambg., **62** (1992), 59–64.
[10] Eshaghi Gordji, M., Bodaghi, A. and Park, C., *A fixed point approach to the stability of double Jordan centralizers and Jordan multipliers on Banach algebras*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **73** (2011), 65–73.
[11] Eshaghi Gordji, M., Kaboli Gharetapeh, S., Savadkouhi, M.B., Aghaei, M. and Karimi, T., *On cubic derivations*, Int. J. Math. Anal., **4** (2010), 51, 2501–2514.
[12] Eskandani, G.Z. and Rassias, J.M., *Approximation of a general cubic functional equation in Felbin's type fuzzy normed linear spaces*, Results Math., 2014; DOI 10.1007/s00025-014-0367-5.
[13] Eskandani, G.Z., Rassias, J.M. and Găvruţă, P., *Generalized Hyers-Ulam stability for a general cubic functional equation in quasi-β-normed spaces*, Asian-Eur. J. Math., **4** (2011); DOI 10.1142/S1793557111000332.
[14] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.
[15] Hensel, K., *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math.-Verein., **6** (1897), 83–88.
[16] Hyers, D.H., *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.
[17] Hyers, D.H., Isac, G. and Rassias, Th.M., *Stability of Functional Equations in Several Variables,* Birkhäuser, Basel, 1998.
[18] Jun, K.W. and Kim, H.M., *The generalized Hyers-Ulam-Russias stability of a cubic functional equation*, J. Math. Anal. Appl., **274** (2002), 867–878.
[19] Jun, K.W. and Kim, H.M., *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl., **6** (2003), 289–302.
[20] Miheţ, D. and Radu, V., *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343** (2008), 567–572.
[21] Moslehian, M.S. and Rassias, Th.M., *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math., **1** (2007), 325–334.

[22] Najati, A., *Hyers-Ulam-Rassias stability of a cubic functional equation*, Bull. Korean Math. Soc., **44** (2007), 825–840.

[23] Najati, A., *The generalized Hyers-Ulam-Rassias stability stability of a cubic functional equation,* Turkish J. Math., **31** (2007), 395–408.

[24] Park, C. and Bodaghi, A., *On the stability of $*$-derivations on Banach $*$-algebras*, Adv. Difference Equ., (2012), Article No. 138.

[25] Rassias, J.M., *Solution of the Ulam stability problem for cubic mappings*, Glas. Mat. Ser. III, **36 (56)** (2001), 63–72.

[26] Rassias, Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.

[27] Ravi, K., Rassias, J.M. and Narasimman, P., *Stability of a cubic functional equation in fuzzy normed space*, Journal of Applied Analysis and Computation, **1** (2011), 411–425.

[28] Rolewicz, S., *Metric Linear Spaces*, second edition, PWN–Polish Scientific Publishers, Warsaw, D. Reidel Publishing Co., Dordrecht, 1984.

[29] Schweizer, B. and Sklar, A., *Probabilistic Metric Spaces*, Elsevier, North Holand, New York, 1983.

[30] Šerstnev, A.N., *On the motion of a random normed space*, Dokl. Akad. Nauk, **149** (1963), 280-283.

[31] Tabor, J., *stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math., **83** (2004), 243–255.

[32] Ulam, S.M., *Problems in Modern Mathematics,* Chapter VI, Science Ed., Wiley, New York, 1940.

[33] Xu, T.Z. and Rassias, J.M., *On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces*, Abstr. Appl. Anal., 2012, Article ID 926390, 23 pages.

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