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AN INTERMEDIATE NEWTON-KANTOROVICH METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract. We provide a semilocal convergence analysis for an easy to implement intermediate Newton-Kantorovich method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$P(x) = F(x) + G(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach Y, and $G\colon D \to Y$ is a continuous operator.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = T(x)$ for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative – when starting from one or several initial approximations, a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

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The Newton-type method (NTM)

$$(2)\qquad \begin{array}{rcl} x_{n+1} &=& x_n - \overline{A}(x_n)^{-1}P(x_n) \quad (n \geq 0), \quad (x_0 \in D), \\ P(x) &=& F(x) + G(x), \quad (x \in D), \end{array}$$

has been used by several authors to generate a sequence $\{x_n\}$ approximating $x^\star$, see [1–19]. In this case $\overline{A}(x)$ belongs to $L(X,Y)$, the space of bounded linear operators from X to Y, and it is an approximation to the Fréchet derivative $F'(x)$ of the operator $F(x)$, see [6, 7, 13, 15]. Note that at each step one operator evaluation is required, $P(x_n)$, and one inverse, $\overline{A}(x_n)^{-1}$.

In this paper, we consider the intermediate Newton-Kantorovich method (INKM)

$$(3)\qquad \begin{array}{rrcl} & x_{n+1} &=& x_n - \overline{A}(x_n)^{-1}P(x_n) \quad (x_0 \in D), \quad (n \geq 0), \\ \text{where} & A(x_n) &=& F'(x_n) + G'(x_0), \quad (n \geq 0). \end{array}$$

If $G = 0$, this scheme becomes the classical Newton-Kantorovich method for the equation $F(x) = 0$, and if $F = 0$, it becomes the modified Newton-Kantorovich method for the equation $G(x) = 0$.

Therefore, (INKM) provides an interesting unified setting for the study of both methods. Although convergence results for (INKM) can be immediately given from the studies mentioned above (see, in particular, [5, 6, 7, 19]), we decided to provide a direct convergence analysis in this study. (INKM) can also be interpreted as an intermediate scheme between the Newton-type method

$$(4)\qquad x_{n+1} = x_n - \left(F'(x_n) + G'(x_n)\right)^{-1} P(x_n) \quad (x_0 \in D), \quad (n \geq 0),$$

and the modified Newton-type method

$$(5)\qquad x_{n+1} = x_n - \left(F'(x_0) + G'(x_0)\right)^{-1} P(x_n) \quad (x_0 \in D), \quad (n \geq 0).$$

It is well-known that, although method (4) usually requires fewer iterations than the modified method (5) to achieve a desired level of accuracy, the latter is less expensive than the former to be implemented. This led several authors, e.g., [5, 6, 7, 19], to propose intermediate Newton methods which converge faster than (5) and are cheaper to implement than (4). (INKM) is such a scheme that is particularly useful in situations where the Fréchet-derivative $F'(x)$ is relatively easy to compute.

The above reasons are the justification for providing a convergence analysis for (INKM). The paper is organized as follows: Section 2 contains the semilocal convergence analysis for (INKM), and in Section 3 we present examples where our results can be applied to solve nonlinear equations, but earlier ones cannot be involved.

2. SEMILOCAL CONVERGENCE ANALYSIS OF (INKM)

We first need some lemmas on majorizing sequences for (INKM).

LEMMA 2.1. *Assume that there exist constants* $M_0 \geq 0$, $L_0 \geq 0$, $M \geq 0$, *and* $\eta \geq 0$ *such that*

$$L_0 < M. \tag{6}$$

Consider the polynomial f_1, *given by*

$$f_1(s) = 2M_0 s^2 - (2 - (M + L_0 + 2M_0)\eta)\, s + 2L_0\eta, \tag{7}$$

and denote by $\delta/2$ *its root which belongs to the interval* $(0,1)$. *For*

$$\delta_0 := \frac{(L_0 + M)\,\eta}{1 - M_0\eta} \tag{8}$$

and

$$\alpha := \begin{cases} \dfrac{-(L_0+M) + \sqrt{(L_0+M)^2 - 8M_0\,(L_0 - M)}}{4M_0}, & M_0 \neq 0 \\ \dfrac{M - L_0}{M + L_0}, & M_0 = 0. \end{cases} \tag{9}$$

the inequalities

$$\delta_0 \leq \delta \leq 2\alpha \tag{10}$$

hold true. Then the scalar sequence $\{t_n\}$ $(n \geq 0)$, *given by*

$$\begin{aligned} &t_0 = 0, \quad t_1 = \eta, \\ &t_{n+2} = t_{n+1} + \frac{M\,(t_{n+1} - t_n)^2 + L_0\,(t_{n+1} + t_n)}{2\,(1 - M_0 t_{n+1})}\,(t_{n+1} - t_n)\,, \end{aligned} \tag{11}$$

is increasing, bounded from above by

$$t^{\star\star} = \frac{2\eta}{2 - \delta}, \tag{12}$$

and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$. *Moreover, the following estimates hold for all* $n \geq 1$

$$t_{n+1} - t_n \leq \frac{\delta}{2}\,(t_n - t_{n-1}) \leq \left(\frac{\delta}{2}\right)^n \eta \tag{13}$$

and

$$t^\star - t_n \leq \frac{2\eta}{2 - \delta}\left(\frac{\delta}{2}\right)^n. \tag{14}$$

Proof. We show, using induction on the integer m, that

$$\begin{aligned} &0 < t_{m+2} - t_{m+1} \\ &= \frac{M\,(t_{m+1} - t_m)^2 + L_0\,(t_{m+1} + t_m)}{2\,(1 - M_0 t_{m+1})}\,(t_{m+1} - t_m) \leq \frac{\delta}{2}\,(t_{m+1} - t_m) \end{aligned} \tag{15}$$

and

$$M_0 t_{m+1} < 1. \tag{16}$$

If (15) and (16) hold, then so does (13), and, by (12),

$$
\begin{aligned}
t_{m+2} &\leq t_{m+1} + \frac{\delta}{2}\left(t_{m+1} - t_m\right) \\
&\leq t_m + \frac{\delta}{2}\left(t_m - t_{m-1}\right) + \frac{\delta}{2}\left(t_{m+1} - t_m\right) \\
&\leq \cdots \leq \eta + \left(\frac{\delta}{2}\right)\eta + \cdots + \left(\frac{\delta}{2}\right)^{m+1}\eta \\
&= \left[\frac{1 - \left(\frac{\delta}{2}\right)^{m+2}}{1 - \frac{\delta}{2}}\right]\eta < \frac{2\eta}{2-\eta} = t^{\star\star}.
\end{aligned}
\tag{17}
$$

The relations (15) and (16) are true for $m = 0$. Indeed, (15) and (16) become in this case, respectively

$$
0 < t_2 - t_1 = \frac{M\left(t_1 - t_0\right)^2 + L_0\left(t_1 + t_0\right)}{2\left(1 - M_0 t_1\right)}\left(t_1 - t_0\right) = \frac{\delta_0}{2}\left(t_1 - t_0\right) \leq \frac{\delta}{2}\left(t_1 - t_0\right)
$$

and

$$
M_0 t_1 < 1,
$$

which are true by the choices of δ_0, δ, by (10) and (11). Assume that (13), (15) and (16) hold true for all $m \leq n + 1$.

The estimate (15) can be re-written as

$$
M\left(t_{m+1} - t_m\right) + L_0\left(t_{m+1} - t_m\right) \leq \delta\left(1 - M_0 t_{m+1}\right).
$$

The inequalities (15) and (16) shall be true if

$$
M\left(t_{m+1} - t_m\right) + \left(L_0 + \delta M_0\right)t_{m+1} + L_0 t_m - \delta \leq 0 \quad \text{(by (13) and (17))} \tag{18}
$$

or

$$
\begin{aligned}
g_m\left(\frac{\delta}{2}\right) = M\left(\frac{\delta}{2}\right)^m \eta + \left(L_0 + \delta M_0\right)&\left[\frac{1 - \left(\frac{\delta}{2}\right)^{m+1}}{1 - \frac{\delta}{2}}\right]\eta \\
+ L_0&\left[\frac{1 - \left(\frac{\delta}{2}\right)^{m}}{1 - \frac{\delta}{2}}\right]\eta - \delta \leq 0.
\end{aligned}
\tag{19}
$$

The estimate (19) motivates us to replace $\delta/2$ by s, and to define the functions f_m $(m \geq 1)$ on $[0, +\infty)$ by

$$
\begin{aligned}
f_m(s) = M s^m \eta &+ \left(L_0 + 2sM_0\right)\left(1 + s + \cdots + s^m\right)\eta \\
&+ L_0\left(1 + s + \cdots + s^{m-1}\right)\eta - 2s.
\end{aligned}
\tag{20}
$$

We need to find a relationship between two consecutive terms f_m. For this we compute

$$
\begin{aligned}
f_{m+1}(s) &= Ms^{m+1}\eta + (L_0 + 2sM_0)\left(1 + s + \cdots + s^{m+1}\right)\eta \\
&\quad + L_0\left(1 + s + \cdots + s^{m}\right)\eta - 2s \\
&= Ms^{m+1}\eta + Ms^{m}\eta - Ms^{m}\eta + (L_0 + 2sM_0)\left(1 + s + \cdots + s^{m}\right)\eta \\
&\quad + (L_0 + 2sM_0)\, s^{m+1}\eta + L_0\left(1 + s + \cdots + s^{m-1}\right)\eta + L_0 s^{m}\eta - 2s.
\end{aligned}
$$

Hence

$$f_{m+1}(s) = f_m(s) + g(s)s^m\eta, \tag{21}$$

where

$$g(s) = 2M_0 s^2 + (L_0 + M)\, s + L_0 - M. \tag{22}$$

Note that α given by (9) belongs to $(0,1)$ and solves $g(s) = 0$. Moreover,

$$g(s) < 0, \quad s \in (0, \alpha). \tag{23}$$

The estimate (19) certainly holds if

$$f_m\left(\frac{\delta}{2}\right) \le 0 \quad (m \ge 1)\,. \tag{24}$$

Clearly (24) holds for $m = 1$ as equality (by (7)). We then get by (7), (21) and (23)

$$
\begin{aligned}
f_2\left(\frac{\delta}{2}\right) &= f_1\left(\frac{\delta}{2}\right) + g\left(\frac{\delta}{2}\right)\frac{\delta}{2}\eta \\
&= g\left(\frac{\delta}{2}\right)\frac{\delta}{2}\eta \le 0.
\end{aligned} \tag{25}
$$

Assume that (24) holds true for all $k \le m$. We then show (24) for m replaced by $m + 1$. Indeed, we have

$$f_{m+1}\left(\frac{\delta}{2}\right) = f_m\left(\frac{\delta}{2}\right) + g\left(\frac{\delta}{2}\right)\left(\frac{\delta}{2}\right)^m \eta \le 0, \tag{26}$$

which shows (24) for all m. Moreover, we obtain

$$f_\infty\left(\frac{\delta}{2}\right) := \lim_{m\to\infty} f_m\left(\frac{\delta}{2}\right) \le 0. \tag{27}$$

That completes the induction. Furthermore, the estimate (14) follows from (13), using standard majorization techniques (see [6, 7, 13, 15]). Finally note that the sequence $\{t_n\}$ is increasing, bounded from above by $t^{\star\star}$, and that it converges to its unique least upper bound $t^\star$. □

Remark 2.2. The hypotheses of Lemma 2.1 have been left as uncluttered as possible. Note that these hypotheses involve only computation at the initial point x_0 (see Theorem 2.7). In the next lemma we shall provide some simpler but stronger hypotheses.

LEMMA 2.3. *Let $M_0 \geq 0$, $L_0 \geq 0$, $M > 0$, and $\eta > 0$ be such that*

$$L_0 < M, \qquad 0 < h_{AH} = \sigma\,\eta \leq \frac{1}{2}, \tag{28}$$

where

$$\sigma := \frac{1}{4\alpha}\max\{M\alpha + (L_0 + 2\alpha\, M_0)\,(1+\alpha) + L_0, M + L_0 + 2\alpha\, M_0\}. \tag{29}$$

Then the following assertions hold:

- *f_1 has a positive root $\delta/2$,*
- $\max\{\delta_0, \delta\} \leq 2\alpha$,
- *the conclusions of Lemma 2.1 hold with α replacing $\delta/2$.*

Proof. It follows from (21) and (28) that

$$f_m(\alpha) = f_1(\alpha) \leq 0 \quad (m \geq 1), \tag{30}$$

which, together with $f_1(0) = 2L_0\eta > 0$, imply that there exists a positive root $\delta/2$ of the polynomial f_1, satisfying

$$\delta \leq 2\alpha. \tag{31}$$

It also follows from (8) and (28) that

$$\delta_0 \leq 2\alpha,$$

hence (24) holds with α replacing $\delta/2$ (by (30)). □

In order to cover the case $L_0 \geq M$, we provide the following alternative to the Lemmas 2.1 and 2.3.

LEMMA 2.4. *Let $M_0 > 0$, $L_0 > 0$, $M > 0$, and $\eta > 0$ be such that*

$$L_0 \geq M, \tag{32}$$

$$0 < h^1_{AH} = \phi\eta \leq \frac{1}{2}, \tag{33}$$

where

$$\phi := \frac{1}{2}\left[M_0 + 2L_0 + \sqrt{(M_0 + 2L_0)^2 - M_0^2}\right]. \tag{34}$$

Choose

$$\frac{\gamma}{2} \in [s_-, s_+] \tag{35}$$

and further assume that

$$\delta_0 \leq \gamma, \tag{36}$$

where s_- and s_+ (with $s_- \leq s_+$) are the roots of the equation

$$P(s) = s^2 - (1 - M_0\eta)s + L_0\eta. \tag{37}$$

Then the conclusions of Lemma 2.1 hold true for γ replacing δ.

Proof. It follows from (21) and (32) that

$$f_1(s) \leq f_2(s) \leq \cdots \leq f_m(s) \leq \cdots . \tag{38}$$

Thus (24) holds if

$$f_\infty(s) := \lim_{m\to\infty} q_m\left(\frac{\gamma}{2}\right) = \frac{L_0 + \gamma M_0}{1-\frac{\gamma}{2}}\eta + \frac{L_0\eta}{1-\frac{\gamma}{2}} - \gamma \leq 0 \tag{39}$$

or

$$\left(\frac{\gamma}{2}\right)^2 - (1 - M_0\eta)\left(\frac{\gamma}{2}\right) + L_0\eta \leq 0, \tag{40}$$

which is true by (33), (35) and (37). □

It turns out that hypotheses (6) and (32) can be dropped as follows:

Lemma 2.5. *Let $M_0 \geq 0$, $L_0 \geq 0$, $M > 0$, and $\eta > 0$ be such that* (33) *and the relations*

$$\alpha \in [s_-, s_+] \tag{41}$$

and

$$\delta_0 \leq 2\alpha \tag{42}$$

hold. Then the conclusions of Lemma 2.1 *hold with α replacing $\delta/2$.*

Proof. It follows from (21) that

$$f_{m+1}(\alpha) = f_m(\alpha) \quad (m \geq 1). \tag{43}$$

Thus (24) holds if

$$f_\infty(\alpha) := \lim_{m\to\infty} q_m(\alpha) \leq 0, \tag{44}$$

which is true by (37), (40) and (41). □

Remark 2.6. (Newton-Kantorovich method - the equation (3) with $G = 0$) If $L_0 = 0$, then we have, by Lemma 2.5, that

$$s_- = 0, \quad s_+ = 1 - M_0\eta, \tag{45}$$

$$\alpha = \frac{-M + \sqrt{M^2 + 8M_0M}}{4M_0}, \tag{46}$$

$$\delta_0 = \frac{M\eta}{1 - M_0\eta}. \tag{47}$$

Moreover, the conditions of Lemma 2.5 reduce to the ones given in [5, 6, 7]:

$$h^\star_{AH} = \overline{M}\eta \leq \frac{1}{2}, \tag{48}$$

where

$$\overline{M} = \frac{1}{8}\left[M + 4M_0 + \sqrt{M^2 + 8M_0M}\right]. \tag{49}$$

The Newton-Kantorovich hypothesis for solving nonlinear equations is famous for its simplicity and clarity, and it is given in [13, 15] by

$$h_k = M\eta \leq \frac{1}{2}. \tag{50}$$

Note that

$$M_0 \leq M \tag{51}$$

holds in general and that M/M_0 can be arbitrarily large (see [6, 7]). It follows from (48) and (50) that

$$h_k \leq \frac{1}{2} \quad \Longrightarrow \quad h^{\star}_{AH} \leq \frac{1}{2}, \tag{52}$$

but not necessarily vice versa, unless if $M_0 = M$. We also have that

$$\frac{h^{\star}_{AH}}{h_k} \to \frac{1}{4} \quad \text{as} \quad \frac{M_0}{M} \to 0. \tag{53}$$

Hence (48) at most quadruples the applicability of the Newton-Kantorovich method under the same computational cost, since in practice the computation of M requires that of M_0. The error bounds on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^{\star}\|$ $(n \geq 0)$ are also finer (see [6, 7]).

Now everything is prepared in order to show the following semilocal convergence theorem for (INKM).

THEOREM 2.7. *Let $F, G : D \subset X \to Y$ be Fréchet-differentiable operators. Assume that there exist $x_0 \in D$, a bounded inverse of $A(x_0) = F'(x_0) + G'(x_0)$, and constants $\eta > 0$, $M \geq 0$, $M_0 \geq 0$, $L_0 \geq 0$ such that for all $x, y \in D$*

$$\left\| A\left(x_0\right)^{-1} \left(F(x_0) + G(x_0)\right) \right\| \leq \eta, \tag{54}$$

$$\left\| A\left(x_0\right)^{-1} \left(F'(x) - F'(y)\right) \right\| \leq M \left\| x - y \right\|, \tag{55}$$

$$\left\| A\left(x_0\right)^{-1} \left(F'(x) - F'(x_0)\right) \right\| \leq M_0 \left\| x - x_0 \right\|, \tag{56}$$

$$\left\| A\left(x_0\right)^{-1} \left(G'(x) - G'(x_0)\right) \right\| \leq L_0 \left\| x - x_0 \right\|, \tag{57}$$

$$\overline{U}(x_0, t^{\star}) = \{x \in X, \|x - x_0\| \leq t^{\star}\} \subseteq D, \tag{58}$$

and such the hypotheses of one of the Lemmas 2.1, 2.3, 2.4 or 2.5 hold true. Then the sequence $\{x_n\}$ $(n \geq 0)$ generated by (INKM) is well-defined, remains in $\overline{U}(x_0, t^{\star})$ for all $n \geq 0$, and converges to a solution $x^{\star}$ of the equation $F(x) + G(x) = 0$ in $\overline{U}(x_0, t^{\star})$. Moreover, the following estimates hold for all $n \geq 0$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{59}$$

and

$$\|x_n - x^{\star}\| \leq t^{\star} - t_n, \tag{60}$$

where both the terms of the sequence $\{t_n\}$ $(n \geq 0)$ *and* $t^\star$ *are given in Lemma* 2.1. *Furthermore, the solution* $x^\star$ *of the equation* $F(x) + G(x) = 0$ *is unique in* $\overline{U}(x_0, T)$ *provided that*

$$T \geq t^\star, \tag{61}$$

$$\overline{U}(x_0, T) \subseteq D, \tag{62}$$

and

$$\frac{(M + L_0) + L_0 t^\star}{2\,(1 - M_0 t^\star)} < 1. \tag{63}$$

Proof. We show, using induction, that

$$\|x_{m+1} - x_m\| \leq t_{m+1} - t_m \tag{64}$$

and

$$\overline{U}\,(x_{m+1}, t^\star - t_{m+1}) \subseteq \overline{U}\,(x_m, t^\star - t_m)\,. \tag{65}$$

For every $z \in \overline{U}\,(x_1, t^\star - t_1)$ we have that

$$\begin{aligned} \|z - x_0\| &\leq \|z - x_1\| + \|x_1 - x_0\| \\ &\leq t^\star - t_1 + t_1 - t_0 = t^\star - t_0, \end{aligned}$$

so $z \in \overline{U}\,(x_0, t^\star - t_0)$. We also have

$$\|x_1 - x_0\| = \left\| A\,(x_0)^{-1}\,(F(x_0) + G(x_0)) \right\| \leq \eta = t_1 - t_0.$$

That is (64) and (65) are valid for $m = 0$. Assuming that they hold for $n \leq m$, we get

$$\begin{aligned} \|x_{m+1} - x_0\| &\leq \sum_{i=1}^{m+1} \|x_i - x_{i-1}\| \\ &\leq \sum_{i=1}^{m+1} (t_i - t_{i-1}) = t_{m+1} - t_0 = t_{m+1} \leq t^\star \end{aligned}$$

and, for all $t \in [0, 1]$,

$$\|x_m + t\,(x_{m+1} - x_m) - x_0\| \leq t_m + t\,(t_{m+1} - t_m) \leq t^\star.$$

Using (56), (16) and the induction hypotheses, we obtain

$$\begin{aligned} \left\| A\,(x_0)^{-1}\,[A\,(x_{m+1}) - A\,(x_0)] \right\| &\leq M_0\,\|x_{m+1-x_0}\| \\ &\leq M_0 t_{m+1} < 1. \end{aligned} \tag{66}$$

It follows from (66) and the Banach Lemma on invertible operators that $A\left(x_{m+1}\right)^{-1}$ exists and that

$$\begin{aligned} \left\|A\left(x_{m+1}\right)^{-1} A\left(x_0\right)\right\| &\leq \left(1-M_0\left\|x_{m+1}-x_0\right\|\right)^{-1} \\ &\leq \left(1-M_0 t_{n+1}\right)^{-1} \\ &\leq \left(1-M_0 t^{\star}\right)^{-1}. \end{aligned} \tag{67}$$

Using (INKM), we get the identity

$$\begin{aligned} F\left(x_{m+1}\right)+G\left(x_{m+1}\right) &= F\left(x_{m+1}\right)+G\left(x_{m+1}\right)-F\left(x_m\right)-G\left(x_m\right) \\ &\quad -\left(F'\left(x_m\right)+G'\left(x_0\right)\right)\left(x_{m+1}-x_m\right) \\ &= \left[F\left(x_{m+1}\right)-F\left(x_m\right)-F'\left(x_m\right)\left(x_{m+1}-x_m\right)\right] \\ &\quad +\left[G\left(x_{m+1}\right)-G\left(x_m\right)-G'\left(x_0\right)\left(x_{m+1}-x_m\right)\right] \\ &= \int_0^1 \left\{\left[F'\left(x_m+t(x_{m+1}-x_m)\right)-F'(x_m)\right]\right. \\ &\quad \left.+\left[G'\left(x_m+t(x_{m+1}-x_m)\right)-G'(x_0)\right]\right\}\left(x_{m+1}-x_m\right) \mathrm{d}t. \end{aligned} \tag{68}$$

Moreover, by (55), (57), (68), the induction hypotheses, and the triangle inequality, we obtain in turn

$$\begin{aligned} &\left\|A\left(x_0\right)^{-1}\left(F(x_{m+1})+G(x_{m+1})\right)\right\| \\ &\leq \left\|\int_0^1 A\left(x_0\right)^{-1}\left[F'\left(x_m+t(x_{m+1}-x_m)\right)-F'(x_m)\right]\right\| \left\|x_{m+1}-x_m\right\| \mathrm{d}t \\ &+\left\|\int_0^1 A\left(x_0\right)^{-1}\left[G'\left(x_m+t(x_{m+1}-x_m)\right)-G'(x_0)\right]\right\| \left\|x_{m+1}-x_m\right\| \mathrm{d}t \\ &\leq \frac{M}{2}\left\|x_{m+1}-x_m\right\|^2+\frac{L_0}{2}\left[\left\|x_{m+1}-x_0\right\|+\left\|x_m-x_0\right\|\right]\left\|x_{m+1}-x_m\right\| \\ &\leq \frac{1}{2}\left[M\left(t_{m+1}-t_m\right)+L_0\left(t_{m+1}+t_m\right)\right]\left(t_{m+1}-t_m\right). \end{aligned} \tag{69}$$

Then, by (INKM), (11), (67) and (69), we have

$$\begin{aligned} \left\|x_{m+2}-x_{m+1}\right\| &\leq \left\|A\left(x_{m+1}\right)^{-1} A\left(x_0\right)\right\| \left\|A\left(x_0\right)^{-1}\left(F(x_{m+1})+G(x_{m+1})\right)\right\| \\ &\leq \frac{1}{2\left(1-M_0 t_{m+1}\right)}\left[M\left(t_{m+1}-t_m\right)+L_0\left(t_{m+1}+t_m\right)\right]\left(t_{m+1}-t_m\right) \\ &= t_{m+2}-t_{m+1}, \end{aligned} \tag{70}$$

which shows (64) for all m.

Thus, for every $w \in \overline{U}(x_{m+2}, t^{\star}-t_{m+2})$, we have

$$\begin{aligned} \left\|w-x_{m+1}\right\| &\leq \left\|w-x_{m+2}\right\|+\left\|x_{m+2}-x_{m+1}\right\| \\ &\leq t^{\star}-t_{m+2}+t_{m+2}-t_{m+1}=t^{\star}-t_{m+1}, \end{aligned} \tag{71}$$

which shows (65) for all m. The previous lemmas imply that $\{t_n\}$ is a Cauchy sequence. It follows from (64) and (65) that $\{x_n\}$ is a Cauchy sequence in the Banach space X, hence it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $m \to \infty$ in (69), we get $F(x^\star) + G(x^\star) = 0$. The estimate (60) is obtained from (59) (i.e., (64)), using standard majorization techniques (see [6, 7, 13, 15]).

Finally, in order to show uniqueness, let $y^\star \in \overline{U}(x_0, T)$ be a solution of the equation $F(x) + G(x) = 0$. As in (68), using (INKM), we get the identity

$$\begin{aligned} x_{m+1} - y^\star = -A(x_m)^{-1} \bigg\{ \int_0^1 &\left[F'\left(y^\star + t(x_m - y^\star)\right) - F'(x_m)\right] \\ + \int_0^1 &\left[G'\left(y^\star + t(x_m - y^\star)\right) - G'(x_0)\right] \bigg\} (x_m - y^\star)\,\mathrm{d}t. \end{aligned} \tag{72}$$

Hence

$$\begin{aligned} \|x_{m+1} - y^\star\| &\le \frac{1}{2\left(1 - M_0 \|x_{m+1} - x_0\|\right)} [M \|x_m - y^\star\| \\ &\quad + L_0(\|x_m - x_0\| + \|x_0 - y^\star\|)] \|x_m - x^\star\| \\ &\le \frac{1}{2\left(1 - M_0 t^\star\right)} \left[MT + L_0\left(t^\star + T\right)\right] \|x_m - y^\star\| \\ &< \|x_m - y^\star\|, \end{aligned} \tag{73}$$

which implies $\lim\limits_{m\to\infty} x_m = y^\star$. On the other hand, we have seen above that $\lim\limits_{m\to\infty} x_m = x^\star$, so $x^\star = y^\star$. □

Note that the limit point $t^\star$ can be replaced by $t^{\star\star}$, given in (12), in the uniqueness hypotheses provided that $\overline{U}(x_0, t^{\star\star}) \subseteq D$, or in all hypotheses of Theorem 2.7.

3. A NUMERICAL EXAMPLE

We provide an example to show that Theorem 2.7 can be applied to solve a nonlinear equation. We point out that, in case of this example, earlier results cannot be applied.

Let $X = Y = \mathbb{R}$, $D = U(x_0, 1 - \theta)$, $x_0 = 1$, $\theta = 0.49$, and define the functions F, G on D, respectively, by

$$F(x) = x^3 - \theta,\ G(x) = \frac{1}{400} x^2.$$

Then we have

$$\eta = 0.170549085, \quad M_0 = \left\| \left(F'(x_0) + G'(x_0)\right)^{-1} 3(3-\theta) \right\| = 2.505823626,$$
$$M = \left\| 6\left(F'(x_0) + G'(x_0)\right)^{-1} (2-\theta) \right\| = 3.01497504,$$
$$L_0 = \frac{1}{200} \left\| \left(F'(x_0) + G'(x_0)\right)^{-1} \right\| = 0.001663894,$$
$$\delta_0 = 0.898453365, \quad \alpha = 0.53086525, \quad 2\alpha = 1.061613051,$$
$$s_- = 0.000495992, \quad s_+ = 0.572138081,$$
$$\phi = 1.319168225, \quad h_K = M_1\eta = 0.51448501 > 0.5.$$

Moreover,

$$\left\| \left(F'(x_0) + G'(x_0)\right)^{-1} \left(F'(x) + G'(x) - F'(y) - G'(y)\right) \right\| \le M_1 \|x - y\|$$

and

$$M_1 = \left\| \left(F'(x_0) + G'(x_0)\right)^{-1} \left[6(2-\theta) + \frac{1}{200} \right] \right\| = 3.016638934.$$

Hence there is no guarantee that the Newton-Kantorovich method converges to $x^\star$, since (50) is violated. However, the assumptions of Theorem 2.7 are satisfied, thus the Newton-Kantorovich method (5) starting at $x_0 = 1$ converges to $x^\star$. Other examples where $M_0 < M$ and $L_0 = 0$ (or not) can be found in [5, 6, 7].

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