A UNIVALENCE CRITERION FOR ANALYTIC FUNCTIONS
IN THE UNIT DISK

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Abstract. In this paper we obtain a univalence criterion involving the logarithmic derivative of \( z^2 f'(z)/f(z) \), where \( f(z) = z + a_2 z^2 + \ldots \) is an analytic function in the unit disk.

MSC 2000. 30C45.

Key words. Univalent functions, analytic extensions.

1. INTRODUCTION

Let \( U_r \) denote the disk \( \{ z \in \mathbb{C} : |z| < r \} \), \( r \in (0,1] \). We denote by \( A \) the class of functions \( A \) that are analytic in the unit disk \( U_1 = U = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( f(0) = 0, f'(0) = 1 \).

Before proving our results we need a brief summary of N.N. Pascu’s method of constructing univalence criteria \([4]\).

Definition 1. A function \( F : U_r \times \mathbb{C} \to \mathbb{C}, F = F(u,v) \) satisfies Pommerenke’s conditions in \( U_r, r \in (0,1] \) if:

i) the function \( L(z,t) = F(e^{-t}z, e^{t}z) \) is analytic in \( U_r \), for all \( t \in [0, \infty) \), locally absolutely continuous in \([0, \infty)\), locally uniform with respect to \( U_r \).

ii) the function \( G(e^{-t}z, e^{t}z), \) where \( G(u,v) = u v \cdot \frac{\partial F}{\partial u}(u,v)/\frac{\partial F}{\partial v}(u,v) \) is analytic in \( U_r \) for all \( t \in [0, \infty) \) and has an analytic extension in \( \mathbb{U} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) for all \( t > 0 \) and in \( U \) for \( t = 0 \). The analytic extension of the function \( G \) is denoted by \( H = H(e^{-t}z, e^{t}z) \) and is called the associate function of \( F(e^{-t}z, e^{t}z) \).

iii) \( \frac{\partial F}{\partial v}(0,0) \neq 0 \) and \( \frac{\partial F}{\partial u}(0,0)/\frac{\partial F}{\partial v}(0,0) \notin (-\infty, -1] \).

iv) the family of functions \( \left\{ F(e^{-t}z, e^{t}z)/\left[ e^{-t}\frac{\partial F}{\partial u}(0,0) + e^{t}\frac{\partial F}{\partial v}(0,0) \right] \right\}_{t \geq 0} \) forms a normal family in \( U_r \).

Theorem 1. \([4]\) Let \( F : U_r \times \mathbb{C} \to \mathbb{C}, F = F(u,v) \) be a function which satisfies Pommerenke’s conditions in \( U_r \) and let \( H = H(u,v) \) be the associate function of \( F \). If \( |H(z,z)| < 1, \) for all \( z \in U \)

and \( |H\left(z, \frac{1}{z}\right)| \leq 1, \) for all \( z \in U \setminus \{0\} \)
then the function $F(e^{-t}z, e^t z)$ has an analytic and univalent extension in $U$, for all $t \in [0, \infty)$.

2. SUFFICIENT CONDITIONS FOR UNIVALENCE

The following theorem is a direct application of N.N. Pascu’s method [4].

**Theorem 2.** Let $f \in A$ and let $\alpha$ be a complex number such that $\Re \alpha > \frac{1}{2}$.

If

$$
1 - \alpha \left[ 1 - (1 - |z|^2) \frac{zf'(z)}{f(z)} \right] + (1 - |z|^2) \frac{d}{dz} \left[ \log \frac{z^2 f'(z)}{f^2(z)} \right] \leq |z|^2
$$

for all $z \in U$, then the function $f$ is univalent in $U$.

**Proof.** We define

$$
F(u, v) = \left[ f(u) \right]^{1-\alpha} \left[ f(u) + \frac{(v - u) f'(u)}{1 - (v - u) \left( \frac{f(u)}{f(u) - \frac{1}{u}} \right)} \right]^\alpha.
$$

We shall prove that the function $F(u, v)$ satisfies the conditions of Theorem 1. Let

$$
L(z, t) = F(e^{-t}z, e^t z) = f(e^{-t}z) \left[ 1 + \frac{(e^{2t} - 1) \frac{e^{-t} f'(e^{-t}z)}{f(e^{-t}z)}}{1 - (e^{2t} - 1) \left( \frac{e^{-t} f'(e^{-t}z)}{f(e^{-t}z)} - 1 \right)} \right]^\alpha.
$$

Since $f(z) \neq 0$ for all $z \in U \setminus \{0\}$ the function

$$
f_1(z, t) = \frac{e^{-t} z f'(e^{-t}z)}{f(e^{-t}z)} = 1 + \ldots
$$

is analytic in $U$. The function

$$
f_2(z, t) = \frac{e^{-t} z f'(e^{-t}z)}{f(e^{-t}z)} - 1 = a_2 e^{-t} z + \ldots
$$

is also analytic in $U$. There exists $r \in (0, 1]$ such that the function

$$
f_3(z, t) = 1 + \frac{(e^{2t} - 1) f_1(z, t)}{1 - (e^{2t} - 1) f_3(z, t)} = e^{2t} + \ldots
$$

is analytic in $U_r$ and $f_3(z, t) \neq 0$ for all $z \in U_r$ and $t \in [0, \infty)$. Thus, we can choose an analytic branch in $U_r$ for the function

$$
f_4(z, t) = [f_3(z, t)]^\alpha = e^{2\alpha t} + \ldots.
$$

It follows that the function

$$
L(z, t) = f(e^{-t}z) f_4(z, t) = e^{(2\alpha - 1)t} z + \ldots,
$$
is analytic in $U_r$.

Further calculation shows that
\[
\frac{\partial L(z,t)}{\partial t} = -e^{-t} z \frac{\partial F}{\partial u} (e^{-t} z, e^t z) + e^t z \frac{\partial F}{\partial v} (e^{-t} z, e^t z) = a_1(t) z + \ldots.
\]

We obtain that $|\frac{\partial L(z,t)}{\partial t}|$ is bounded on $[0, T]$ for any fixed $T > 0$ and $z \in U_r$. Hence, the function $L(z,t)$ is locally absolutely continuous in $[0, \infty)$, locally uniform with respect to $U_r$.

We have
\[
a_1(t) = e^{-t} \frac{\partial F}{\partial u}(0,0) + e^t \frac{\partial F}{\partial v}(0,0) = e^{(2\alpha - 1)t}
\]
and hence $a_1(t) \neq 0$ and $\lim_{t \to \infty} |a_1(t)| = \lim_{t \to \infty} e^{\Re(2\alpha - 1)} = \infty$.

It is easy to check that there exists $K > 0$ such that $|F(e^{-t} z, e^t z)/a_1(t)| \leq K$, for all $z \in U_r$ and $t \in [0, \infty)$ and hence $\{F(e^{-t} z, e^t z)/a_1(t)\}_{t \geq 0}$ is a normal family in $U_r$.

From (2) we obtain
\[
G(u,v) = \frac{u}{v} \cdot \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v} = \frac{1 - \alpha}{\alpha} [u - (v - u) \frac{f'(u)}{f(u)}] + (v - u) \left[ \frac{1}{u} \frac{f''(u)}{f(u)} + \frac{f'(u)}{f(u)} \right].
\]

It follows that the function $G(e^{-t} z, e^t z)$ has an analytic extension
\[
H(e^{-t} z, e^t z) = \frac{1 - \alpha}{\alpha} \left[ e^{2t} - (e^{2t} - 1) \frac{e^{-t} z f'(e^{-t} z)}{f(e^{-t} z)} \right] + (e^{2t} - 1) \left[ 2 - 2 \frac{e^{-t} z f'(e^{-t} z)}{f(e^{-t} z)} + \frac{e^{-t} z f''(e^{-t} z)}{f'(e^{-t} z)} \right].
\]

We have
\[
|H(z, z)| = \left| \frac{1 - \alpha}{\alpha} \right| < 1,
\]
for all $z \in U$ and $\alpha \in \mathbb{C}$ with $\Re \alpha > \frac{1}{2}$, and
\[
\left| H \left( z, \frac{1}{z} \right) \right| = \left| \frac{1 - \alpha}{\alpha} \frac{1}{|z|^2} \left[ 1 - (1 - |z|^2) \frac{zf'(z)}{f(z)} \right] + \frac{1}{|z|^2} (1 - |z|^2) \frac{d}{dz} \left[ \log \frac{z^2 f'(z)}{f^2(z)} \right] \right| \leq 1,
\]
for all $z \in U \setminus \{0\}$.

The conditions of Theorem 1 being satisfied it follows that the function $F(e^{-t} z, e^t z)$ has an analytic and univalent extension $F_1(e^{-t} z, e^t z)$ in $U$ for all $t \in [0, \infty)$. In particular, the function $f(z) = F_1(z, z)$ is univalent in $U$. □
Remark 1. If in Theorem 2 the condition (1) is replaced by the condition
\[
\left| \frac{1 - \alpha}{\alpha} \left[ 1 - (1 - |z|^2) \frac{zf'(z)}{f(z)} \right] + (1 - |z|^2)z^2 \frac{d}{dz} \left[ \log \frac{z^2f'(z)}{f^2(z)} \right] \right| \leq q|z|^2, \quad z \in U,
\]
where \( q \in (0, 1) \), then, by Becker’s generalized \( q \)-chain theory [1], the function \( f \) is univalent in \( U \) and has a quasiconformal extension in \( \mathbb{C} \).

The following corollaries are specific applications of Theorem 2.

**Corollary 1.** [3] If \( f \in A \) and
\[
\left| \frac{z}{d} \left[ \log \frac{z^2f'(z)}{f^2(z)} \right] \right| \leq \frac{|z|^2}{1 - |z|^2}, \quad z \in U,
\]
then \( f \) is an univalent function in \( U \).

**Proof.** It follows from Theorem 2 with \( \alpha = 1 \). \( \square \)

**Corollary 2.** If \( f \in A \) and
\[
\left| (1 - |z|^2)z^2 \frac{d}{dz} \left[ \log \frac{z^2f'(z)}{f^2(z)} \right] + (1 - |z|^2)z^2 \frac{f''(z)}{f(z)} - 1 \right| \leq |z|^2, \quad z \in U,
\]
then the function \( f \) is univalent in \( U \).

**Proof.** It follows from Theorem 2 with \( \alpha \to \infty \). \( \square \)

**REFERENCES**


Received April 15, 2002

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