MATHEMATICA, 66 (89), N° 2, 2024, pp. 278–288

FACE COUNTING FOR TOPOLOGICAL HYPERPLANE ARRANGEMENTS

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Abstract. Determining the number of pieces after cutting a cake is a classical problem. Roberts provided an exact solution by computing the number of chambers contained in a plane cut by lines. About 88 years later, Zaslavsky even computed the f-polynomial of a hyperplane arrangement, and consequently deduced the number of chambers of that latter. Recently, Forge and Zaslavsky introduced the more general structure of topological hyperplane arrangements. This article computes the f-polynomial of such arrangements when they are transsective, and therefore deduces their number of chambers.

MSC 2020. 05A15, 06A07.

Key words. Topological hyperplane, arrangement, Möbius polynomial.

1. INTRODUCTION

A classical basic problem was to determine the number of pieces obtained by cutting a cake d times. Deeper study of that problem has probably its origin in the article of Steiner [10] who computed the maximal number of chambers contained in a plane cut by several sets of parallel lines pointing in different directions. Roberts [9] fixed that problem by showing that

$$1 + d + \binom{d}{2} + \sum_{i=1}^{k} n_k \binom{k-1}{2} - \sum_{j=1}^{p} \binom{l_j}{2}$$

is the number of chambers contained in a plane cut by d lines, where n_k is the number of k-fold intersection points for $k \geq 3$, and p is the number of families of parallel lines containing respectively $l_1, \dots, l_p$ lines with $l_j \geq 2$. As mentioned in the book of Dimca [4] for instance, Schläfli extended that problem to the Euclidean space $\mathbb{R}^n$, and published in 1901 that the number of chambers in $\mathbb{R}^n$ partitioned by d hyperplanes is smaller that $\sum_{i=0}^{n} \binom{d}{i}$. That extended problem was, that time, solved by Zaslavsky [11]. He precisely expressed the f-polynomial of a hyperplane arrangement $\mathcal{A}$ by means of its Möbius polynomial, and deduced that its number of chambers is $\sum_{X \in L(\mathcal{A})} (-1)^{\operatorname{rank} X} \mu(\mathbb{R}^n, X)$, where $L(\mathcal{A})$ is the flat set of $\mathcal{A}$ and μ the Möbius

The author was supported by the Alexander von Humboldt Foundation.

DOI: 10.24193/mathcluj.2024.2.10

function. In an independent work, Alexanderson and Wetzel [1] obtained the f-polynomial of a plane arrangement in a space. More recently, Pakula [8] computed the number of chambers of pseudosphere arrangements. Note that pseudosphere arrangements are topologically equivalent to pseudohyperplane arrangements as one can read in the article of Deshpande [3] for example.

This article considers the more general case of topological hyperplane arrangements, or topoplane arrangements, introduced by Forge and Zaslavsky [7]. Transsective topoplane arrangements are even generalizations for pseudohyperplane arrangements that are known to be topological models for oriented matroids, like stated in the book of Björner et al. [2]. *This article determines the f-polynomial of a transsective topoplane arrangement $\mathscr{A}$ in a topological ball T, and deduces that $\sum_{X \in L(\mathscr{A})} (-1)^{\operatorname{rank} X} \mu(T, X)$ is its number of chambers, where $L(\mathscr{A})$ is the flat set of $\mathscr{A}$.*

In neighboring contexts, Dumitrescu and Mandal [5] established that the number of nonisomorphic simple arrangements of n pseudolines is bigger that $2^{cn^2 - O(n \ln n)}$ for some constant $c > 0.2083$, while Felsner and Scheucher [6] studied the circularizability of pseudocircle arrangements.

Recall that in the Euclidean space $\mathbb{R}^n$, an n-ball of radius r and center x is the set of all points of distance less than r from x, a topological n-ball is any subset which is homeomorphic to an n-ball, and an n-manifold is a subset with the property that each point has a neighborhood that is homeomorphic to an n-ball. Topological n-balls are important as building blocks of CW-complexes. However, they are not flexible enough to investigate topological properties of topoplane arrangements. More abstract objects, named deformed n-balls, must consequently be introduced in Section 2.

The study of topoplane arrangements really begins in Section 3. We namely fix the conjecture of Forge and Zaslavsky [7], mentioned in the introduction of their article, stating that solidity can be proved from the definition of a topoplane arrangement. Then, we prove that every chamber of a transsective topoplane arrangement is a deformed ball. These results allow us to compute the f-polynomial of a transsective topoplane arrangement in Section 4, and to deduce its number of chambers.

2. DEFORMED BALLS

This article uses the notations $[k] := \{1, 2, \dots, k\}$ for a positive integer k, and $\mathbb{N}_0$ for the set of nonnegative integers. Deformed balls, deformed ball complexes, as well as the Euler characteristic of a deformed ball complex are defined in this section.

Definition 2.1. Let n be a nonnegative integer. A *deformed n-ball* is a path connected n-manifold X in $\mathbb{R}^n$ such that the homotopy group $\pi_k(X, x_0)$ is trivial for each positive integer k and a distinguished point x_0 of X.

Definition 2.2. Let X be a deformed n-ball, and Y a deformed m-ball such that $n > m$ and $X \cap Y = \varnothing$. The sets X and Y can be glued together if the boundary ∂X of X contains Y. The set obtained from gluing Y onto X is the path connected space $X \sqcup Y$.

Recursive Construction of a System of Deformed Balls

We begin with a system $\big(X_1, \{X_1\}\big)$, where X_1 is a deformed n-ball.

- Let X_2 be a deformed m-ball such that X_2 can be glued onto X_1, if $n > m$, or X_1 can be glued onto X_2, if $n < m$. We get the extended system $\big(X_1 \sqcup X_2, \{X_1, X_2\}\big)$.
- Suppose that we have a positive integer k, and a system $\big(X, \{X_i\}_{i\in[k]}\big)$, where $X = \bigsqcup_{i\in[k]} X_i$ was obtained by gluing together the deformed balls $X_1, \dots, X_k$. This system can be extended with another deformed ball X_{k+1} if
  - $X \cap X_{k+1} = \varnothing$,
  - there exists $i \in [k]$ such that X_i and X_{k+1} can be glued together,
  - if I is the subset of $[k]$ such that X_i and X_{k+1} can be glued together for each $i \in I$, then $\bigsqcup_{i\in I} X_i$ is path connected.

  We obtain a new system $\big(X \sqcup X_{k+1}, \{X_i\}_{i\in[k+1]}\big)$ of deformed balls.

Definition 2.3. A topological space X is a *deformed ball complex* if there exist a positive integer k, and a set $\{X_i\}_{i\in[k]}$ of deformed balls such that $X = \bigsqcup_{i\in[k]} X_i$ and $\big(X, \{X_i\}_{i\in[k]}\big)$ is a system of deformed balls.

For a CW complex X, the Euler characteristic $\chi(X)$ is the alternating sum $\sum_{n\in\mathbb{N}_0}(-1)^n c_n$, where c_n is the number of topological n-balls of X. We need to generalize the definition of deformed ball complexes.

Definition 2.4. Let k be a positive integer, and $\big(X, \{X_i\}_{i\in[k]}\big)$ a system of deformed balls. The *Euler characteristic* of the deformed ball complex X is

$$\chi(X) := \sum_{n\in\mathbb{N}_0} (-1)^n c_n,$$

where c_n is the number of deformed n-balls in $\{X_i\}_{i\in[k]}$.

Example 2.5. In the left part of Figure 1, we have a deformed ball complex composed by the deformed 0-ball, 1-ball, and 3-ball represented in the right part of Figure 1. Its Euler characteristic is $(-1)^0 + (-1)^1 + (-1)^2 = 1$.

3. TOPOPLANE ARRANGEMENTS

This section is devoted to topoplane arrangements introduced by Forge and Zaslavsky [7]. Transsective topoplane arrangements are particularly of interest to us.

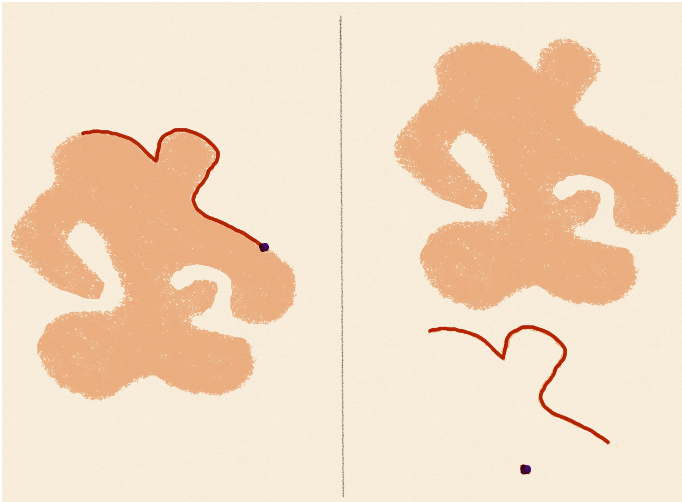

Fig. 1 – A complex formed by three deformed balls.

We fix in Proposition 3.10 the conjecture mentioned in the introduction of the article of Forge and Zaslavsky [7], stating that every restriction of a transsective topoplane arrangement is a transsective topoplane arrangement. Afterwards, we prove in Proposition 3.13 that every face of a transsective topoplane arrangement is a deformed ball.

Definition 3.1. Let n be a positive integer, and T a topological n-ball. A *topoplane* in T is a topological $(n-1)$-ball $H \subseteq T$ that divides T into two connected topological subspaces.

Definition 3.2. Let $\mathscr{A}$ be a finite set of topoplanes in a topological n-ball T. A *flat* of $\mathscr{A}$ is a nonempty intersection of topoplanes in $\mathscr{A}$. Denote by $L(\mathscr{A})$ the set composed by the flats of $\mathscr{A}$.

Example 3.3. The flat set generated by both topoplanes in the yellow open disk of Figure 2 is composed of the yellow disk, both topoplanes, and the four intersection points.

Definition 3.4. Let $\mathscr{A}$ be a finite set of topoplanes in a topological ball T. It is a *topoplane arrangement* if

(a) every flat in $L(\mathscr{A})$ is a topological ball,
(b) for every topoplane $H \in \mathscr{A}$ and each flat $X \in L(\mathscr{A})$, either $X \subseteq H$ or $H \cap X = \varnothing$ or $H \cap X$ is a topoplane in X.

Example 3.5. The flat set of the topoplane arrangement in Figure 3 is composed of $\mathbb{R}^3$, both topoplanes, and the intersection point.

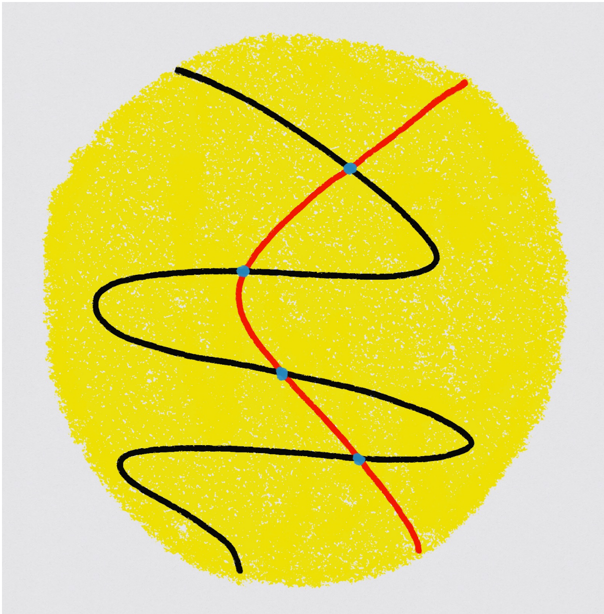

Fig. 2 – Two topoplanes in an open disk.

Proposition 3.6 ([7, Prop. 1]). *Let $\mathscr{A}$ be a topoplane arrangement in a topological ball T, and consider a flat $X \in L(\mathscr{A})$. The induced set of topological subspaces in X defined by*

$$\mathscr{A}^X := \{X \cap H \mid H \in \mathcal{A},\, X \nsubseteq H,\, X \cap H \neq \varnothing\}$$

is a topoplane arrangement in X.

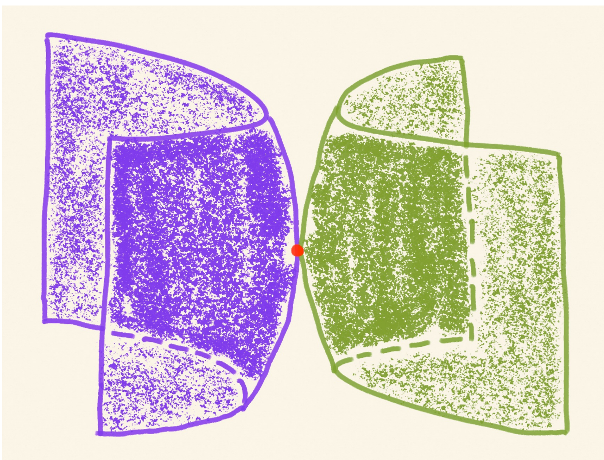

Fig. 3 – A topoplane arrangement in $\mathbb{R}^3$.

Definition 3.7. Let $\mathscr{A}$ be a topoplane arrangement in a topological ball T, and consider a flat $X \in L(\mathscr{A})$. The topoplane arrangement

$$\mathscr{A}^X := \{X \cap H \mid H \in \mathscr{A},\ X \nsubseteq H,\ X \cap H \neq \varnothing\}$$

in X is called the *restriction* of $\mathscr{A}$ on X.

Definition 3.8. Let $\mathscr{A}$ be a finite set of topoplanes in a topological ball T. A pair of distinct topoplanes $(H,\ K) \in \mathscr{A} \times \mathscr{A}$ forms a *transsection* if $H \setminus K$ is composed by two components which lie on opposite sides of K.

Definition 3.9. Let $\mathscr{A}$ be a topoplane arrangement in a topological ball T. It is said to be *transsective* if, for each pair of distinct topoplanes $(H,\ K) \in \mathscr{A} \times \mathscr{A}$, either $H \cap K = \varnothing$ or $(H,\ K)$ forms a transsection.

Proposition 3.10. *Let $\mathscr{A}$ be a topoplane arrangement in a topological ball T, and X a flat in $L(\mathscr{A})$. If $\mathcal{A}$ is transsective, then $\mathscr{A}^X$ is a transsective topoplane arrangement in X.*

Proof. Consider two distinct topoplanes in X, namely having the forms $X \cap H$ and $X \cap K$ with $H, K \in \mathscr{A}$. Suppose that $(X \cap H) \cap (X \cap K) \neq \varnothing$. Since $X \cap H \nsubseteq X \cap K$ and $\mathscr{A}^X$ is a topoplane arrangement as seen in Proposition 3.6, then $(X \cap H) \cap (X \cap K) = X \cap H \cap K$ is a topoplane in $X \cap H$. Hence $X \cap H \cap K$ divides $X \cap H$ into two connected topological subspaces $(X \cap H \cap K)^1$ and $(X \cap H \cap K)^{-1}$. Besides, the topoplane $X \cap K$ divides X into two connected topological subspaces $(X \cap K)^1$ and $(X \cap K)^{-1}$, and we have

- either $(X \cap H \cap K)^1 \subseteq (X \cap K)^1$ and $(X \cap H \cap K)^{-1} \subseteq (X \cap K)^{-1}$,
- or $(X \cap H \cap K)^{-1} \subseteq (X \cap K)^1$ and $(X \cap H \cap K)^1 \subseteq (X \cap K)^{-1}$.

In both cases, $(X \cap H) \setminus (X \cap K)$ is composed by two components in X which lie on opposite sides of $X \cap K$. The topoplane arrangement $\mathscr{A}^X$ is consequently transsective. ☐

Definition 3.11. Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball T. Denote by H^{-1} and H^1 both connected components obtained after division of T by a topoplane $H \in \mathscr{A}$. Moreover, set $H^0 = H$. The *sign map* of H is the function

$$\sigma_H : T \to \{-1,\ 0,\ 1\}, \quad v \mapsto \begin{cases} -1 & \text{if } v \in H^{-1}, \\ 0 & \text{if } v \in H^0, \\ 1 & \text{if } v \in H^1. \end{cases}$$

The sign map of $\mathscr{A}$ is the function $\sigma_{\mathscr{A}} : T \to \{-1,\ 0,\ 1\}^{\mathscr{A}},\ v \mapsto \big(\sigma_H(v)\big)_{H \in \mathscr{A}}$. And the *sign set* of $\mathscr{A}$ is the set

$$\sigma_{\mathscr{A}}(T) := \big\{\sigma_{\mathscr{A}}(v) \bigm| v \in T\big\}.$$

Definition 3.12. Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball T. A *face* of $\mathscr{A}$ is a subset F of T such that

$$\exists x \in \sigma_{\mathscr{A}}(T),\ F = \big\{v \in T \bigm| \sigma_{\mathscr{A}}(v) = x\big\}.$$

A *chamber* of $\mathscr{A}$ is a face F such that $\sigma_{\mathscr{A}}(F) \in \{-1, 1\}^{\mathscr{A}}$. Denote by $F(\mathscr{A})$ and $C(\mathscr{A})$ the sets composed by the faces and the chambers of $\mathscr{A}$, respectively.

PROPOSITION 3.13. *Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball T. Then, every face of $\mathscr{A}$ is a deformed ball.*

Proof. Assume T is a topological n-ball, and begin by considering a chamber $C \in C(\mathscr{A})$:

- Let $x \in C$, and $d = \min\big\{\mathrm{dist}(x, H) \parallel H \in \mathscr{A}\big\}$, where dist is a distance function on T. Then, the n-ball of radius $d/2$ and center x is included in C. The chamber C is consequently an n-manifold.
- Let $x, y \in C$. The fact that $\mathscr{A}$ is transsective and $\sigma_{\mathscr{A}}(x) = \sigma_{\mathscr{A}}(y)$ imply the path connectivity of x and y.
- The chamber C can naturally not contain holes, meaning that $\pi_k(C, x_0)$ is trivial for each positive integer k and distinguished point x_0 of C.

The chamber C is then a deformed ball. Consider a face $F \in F(\mathscr{A}) \setminus C(\mathscr{A})$, and the flat

$$X = \bigcap_{\substack{H \in \mathscr{A} \\ \sigma_H(F)=0}} H.$$

We know from Proposition 3.10 that $\mathscr{A}^X$ is a transsective topoplane arrangement in X. As F is a chamber of $\mathscr{A}^X$, it is therefore a deformed ball. □

PROPOSITION 3.14. *Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball T. Then,*

$$\sum_{F \in F(\mathscr{A})} \chi(F) = \chi(T).$$

Proof. On one side, if T is a topological 1-ball, then $\mathscr{A}$ is set of points dividing T into $\#\mathscr{A} + 1$ deformed 1-balls. Hence,

$$\sum_{F \in F(\mathscr{A})} \chi(F) = \#\mathscr{A}(-1)^0 + (\#\mathscr{A} + 1)(-1)^1 = -1 = \chi(T).$$

On the other side, if T is a topological n-ball, with $n \geq 2$, and $\#\mathscr{A} = 1$, then

$$\sum_{F \in F(\mathscr{A})} \chi(F) = (-1)^{n-1} + 2(-1)^n = (-1)^n = \chi(T).$$

Suppose now that T is a topological n-ball and $\#\mathscr{A} = m$, with $n \geq 2$ and $m \geq 2$. We proceed by induction, and assume that Proposition 3.14 is true for any transsective arrangement of r topoplanes in a topological s-ball if $s < n$, or $s = n$ and $r < m$. Let $H \in \mathscr{A}$, $\mathscr{A}' = \mathscr{A} \setminus \{H\}$, and consider the following subsets of $F(\mathscr{A}')$:

(1) $F^1 = \big\{F \in F(\mathscr{A}') \bigm| F \cap H \neq \varnothing,\, F \nsubseteq H\big\}$,
(2) $F^2 = \big\{F \in F(\mathscr{A}') \bigm| F \cap H = \varnothing\big\}$,
(3) and $F^3 = \big\{F \in F(\mathscr{A}') \bigm| F \subseteq H\big\}$.

The set $F(\mathscr{A}^H)$ is composed by the elements of F^3 and the faces F_H of $\mathscr{A}^H$ in one-to-one correspondence to the faces F in F^1 such that, if F is a deformed k-ball, F_H is a deformed $(k-1)$-ball dividing F into two deformed k-balls F_1 and F_2. We deduce

$$\begin{aligned}
&\sum_{F\in F(\mathscr{A})} \chi(F) \\
&= \sum_{F\in F^1(\mathscr{A}')} \big(\chi(F_1)+\chi(F_2)\big) + \sum_{F\in F^2(\mathscr{A}')} \chi(F) + \sum_{F\in F(\mathscr{A}^H)} \chi(F) \\
&= \sum_{F\in F^1(\mathscr{A}')} \big(\chi(F_1)+\chi(F_2)+\chi(F_H)\big) + \sum_{F\in F^2(\mathscr{A}')} \chi(F) + \sum_{F\in F^3(\mathscr{A}')} \chi(F) \\
&= \sum_{F\in F^1(\mathscr{A}')} \chi(F) + \sum_{F\in F^2(\mathscr{A}')} \chi(F) + \sum_{F\in F^3(\mathscr{A}')} \chi(F) \\
&= \sum_{F\in F(\mathscr{A}')} \chi(F) \\
&= \chi(T).
\end{aligned}$$

□

4. THE f-POLYNOMIAL OF A TOPOPLANE ARRANGEMENT

We finally get the f-polynomial of a transsective topoplane arrangement $\mathscr{A}$ in a topological ball T in Theorem 4.5 of this section. Besides, investigating the constant of that polynomial gives that $\sum_{X\in L(\mathscr{A})}(-1)^{\operatorname{rank} X}\mu(T,X)$ is the number of chambers of $\mathscr{A}$.

Definition 4.1. Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball. Define the *dimension* $\dim X$ of a flat X of $\mathscr{A}$ which is topological n-ball, as well as the dimension $\dim F$ of a face F of $\mathscr{A}$ which is a deformed n-ball, to be n. Call such flat and face of $\mathscr{A}$ n-flat and n-face, respectively.

Definition 4.2. Consider a transsective topoplane arrangement $\mathscr{A}$ in a topological n-ball. Let $f_i(\mathscr{A})$ be the number of i-faces of $\mathscr{A}$, and x a variable. The *f-polynomial* of $\mathscr{A}$ is

$$f_{\mathscr{A}}(x) := \sum_{i=0}^{n} f_i(\mathscr{A})\, x^{n-i}.$$

Definition 4.3. Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological n-ball. Define the *rank* of a flat $X\in L(\mathscr{A})$ to be $\operatorname{rank} X := n-\dim X$, and that of the topoplane arrangement $\mathscr{A}$ to be

$$\operatorname{rank}\mathscr{A} := \max\big\{\operatorname{rank} X\in\mathbb{N}_0 \;\big|\; X\in L(\mathscr{A})\big\}.$$

Recall that the Möbius function $\mu : L(\mathscr{A}) \times L(\mathscr{A}) \to \mathbb{Z}$ of a meet semilattice $L(\mathscr{A})$ is recursively defined, for $X, Y \in L(\mathscr{A})$, by

$$\mu(X,Y) := \begin{cases} 1 & \text{if } X = Y, \\ -\sum_{\substack{Z \in L(\mathscr{A}) \\ X \leq Z < Y}} \mu(X,Z) = -\sum_{\substack{Z \in L(\mathscr{A}) \\ X < Z \leq Y}} \mu(Z,Y) & \text{if } X < Y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.4. Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball, and x, y variables. The *Möbius polynomial* of $\mathscr{A}$ is

$$M_{\mathscr{A}}(x,y) := \sum_{X,Y \in L(\mathscr{A})} \mu(X,Y)\, x^{\operatorname{rank} X}\, y^{\operatorname{rank} \mathscr{A} - \operatorname{rank} Y}.$$

We can now state the main result of this article.

Theorem 4.5. *Let $\mathscr{A}$ be a transsective topoplane arrangement in a topological ball. The f-polynomial of $\mathscr{A}$ is*

$$f_{\mathscr{A}}(x) = (-1)^{\operatorname{rank} \mathscr{A}} M_{\mathscr{A}}(-x,-1).$$

Proof. We know from Proposition 3.10 and Proposition 3.13 that the pair $\big(X,\, F(\mathscr{A}^X)\big)$ forms a system of deformed balls. Thus, using Proposition 3.14,

$$\chi(X) = \sum_{i=0}^{\dim X} (-1)^i f_i(\mathscr{A}^X) = (-1)^{\dim X}.$$

Every i-face $F \in F(\mathscr{A}^X)$ is a chamber of a unique i-flat

$$\bigcap_{\substack{H \in \mathscr{A}^X \\ \sigma_H(F) = 0}} H \in L(\mathscr{A}^X).$$

Then

$$f_i(\mathscr{A}^X) = \sum_{\substack{Y \in L(\mathscr{A}^X) \\ \dim Y = i}} \#C\Big(\big(\mathscr{A}^X\big)^Y\Big),$$

and

$$\sum_{Y \in L(\mathscr{A}^X)} (-1)^{\dim Y} \#C\Big(\big(\mathscr{A}^X\big)^Y\Big) = (-1)^{\dim X}.$$

We have $L(\mathscr{A}^X) = \big\{Y \in L(\mathscr{A}) \bigm| Y \geq X\big\}$, and, for every $Y \in L(\mathscr{A}^X)$, also $C\Big(\big(\mathscr{A}^X\big)^Y\Big) = C(\mathscr{A}^Y)$. Hence,

$$\sum_{\substack{Y \in L(\mathscr{A}) \\ Y \geq X}} (-1)^{\dim Y} \#C(\mathscr{A}^Y) = (-1)^{\dim X}.$$

Using the Möbius inversion formula, we obtain

$$\sum_{\substack{Y \in L(\mathscr{A}) \\ Y \geq X}} (-1)^{\dim Y} \mu(X,Y) = (-1)^{\dim X} \# C(\mathscr{A}^X).$$

Besides,

$$(-1)^{\operatorname{rank} \mathscr{A}} M_{\mathscr{A}}(-x,-1) = \sum_{X,Y \in L(\mathscr{A})} (-1)^{\dim Y - \dim X} \mu(X,Y)\, x^{\operatorname{rank} X}.$$

Therefore, for every $0 \leq i \leq n$, the coefficient λ_{n-i} of x^{n-i} in the polynomial $(-1)^{\operatorname{rank} \mathscr{A}} M_{\mathscr{A}}(-x,-1)$ is

$$\lambda_{n-i} = \sum_{\substack{X \in L(\mathscr{A}) \\ \dim X = i}} \sum_{Y \in L(\mathscr{A}^X)} (-1)^{\dim Y - \dim X} \mu(X,Y) = \sum_{\substack{X \in L(\mathscr{A}) \\ \dim X = i}} \# C(\mathscr{A}^X) = f_i(\mathscr{A}).$$

□

EXAMPLE 4.6. Consider the arrangement $\mathscr{A}_{\text{ex}}$ formed by nine topoplanes in $\mathbb{R}^2$ represented in Figure 4. As its Möbius polynomial is $M_{\mathscr{A}_{\text{ex}}}(x,y) = 5x^2+y^2+9xy-11x-9y+6$, its f-polynomial is then $f_{\mathscr{A}_{ex}}(x) = 5x^2+20x+16$.

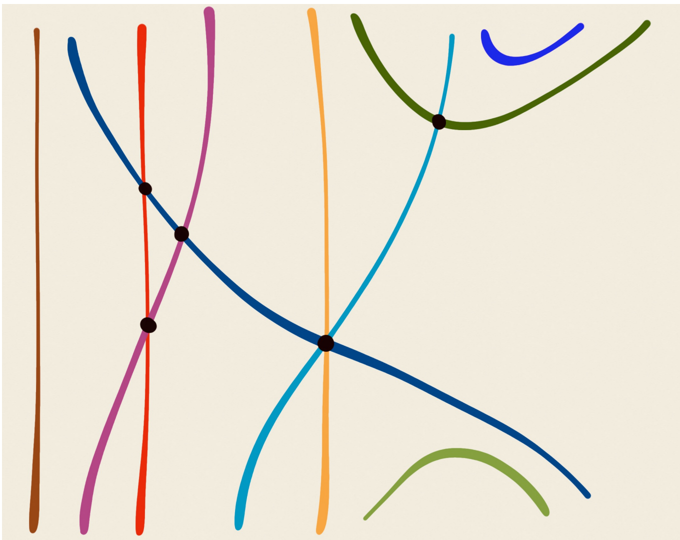

Fig. 4 – The topoplane arrangement $\mathscr{A}_{\text{ex}}$.

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Received July 2, 2023
Accepted August 25, 2024

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