MATHEMATICA, 66 (89), N° 2, 2024, pp. 262–277

SOME REMARKS ON GENERALIZATIONS OF THE REVERSE ORDER LAW IN A $*$-RING

HAMIDEH RAHMANI and ALIREZA KAMEL MIRMOSTAFAEE

Abstract. We show that if $(1 - a^\dagger a)b$ is left *-cancelable, then the reverse order laws $(ab)^\dagger = b^\dagger a^\dagger$ and $(ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger$ are equivalent. By investigating the reverse order law $(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^\dagger$ in rings with involution, we will show that under certain circumstances the inclusion $(abb^\dagger a^\dagger a)\{5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}$ is always an equality.

MSC 2020. Primary 47A05; Secondary 15A09, 16U90.

Key words. Group inverse, Moore-Penrose inverse, left *-cancelable, EP element, idempotent element, reverse order law.

1. INTRODUCTION AND PRELIMINARIES

Let us start by recalling some definitions.

Definition 1.1 ([4]). By an involution on an unital ring R, we mean a function $a \mapsto a^*$ from R to itself such that

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^* \quad (a, b \in R).$$

If $a \in R$ satisfies $a^* = a$, then a is called self-adjoint (or Hermitian) and if $a \in R$ satisfies $a^* a = aa^*$, then a is called normal.

Throughout this paper, we will assume R is an associative ring with an involution. We also consider two kinds of generalized inverses in R, i.e. group inverses and Moore-Penrose inverses. Their formal definitions are given below.

Definition 1.2 ([1,2]). An element $a \in R$ is called

(a) *group invertible* if there exists $b \in R$ such that

$$aba = a,\ bab = b \text{ and } ab = ba.$$

This b is uniquely determined by the above identities and it is called the group inverse of a. The group inverse of a is denoted by $a^\sharp$. We also denote by $R^\sharp$ the set of all group invertible elements of R.

The authors would like to thank the referee for useful suggestions.
Corresponding author: Alireza Kamel Mirmostafaee.

DOI: 10.24193/mathcluj.2024.2.09

(b) *Moore-Penrose invertible* (or MP-invertible) if there is $b \in R$ such that following equations hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

If such an element b exists, then it is unique and it is denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of ring R is denoted by $R^\dagger$.

Note that if a is invertible, then $a^\sharp = a^\dagger = a^{-1}$, where a^{-1} is the ordinary inverse of a. Therefore, the above definition provides two generalizations of invertibility.

Notation 1.3. Consider the following equations

$$1)\ aba = a, \quad 2)\ bab = b, \quad 3)\ (ab)^* = ab, \quad 4)\ (ba)^* = ba, \quad 5)\ ab = ba.$$

If $A \subset \{1, 2, 3, 4, 5\}$ and a, b satisfy all equations of set A, then we say b is an A-inverse of a. The set of all A-inverses of a is denoted by $a\{A\}$. With this notation, $a\{1, 2, 5\} = \{a^\sharp\}$ and $a\{1, 2, 3, 4\} = \{a^\dagger\}$.

Definition 1.4 ([4]). Let $a \in R$. Then we say that an element $a \in R$ is left *-cancelable, if $a^*ax = a^*ay$ implies $ax = ay$ and we say that it is right *-cancelable if $xaa^* = yaa^*$ implies $xa = ya$. An element $a \in R$ is called *-cancelable if a is both left and right *-cancelable. The commutator of u and v is defined by $[u, v] = uv - vu$.

If a and b are invertible in R, then $(ab)^{-1} = b^{-1}a^{-1}$, which is known as the reverse order law. Note that this rule cannot be extended to other generalized inverses [7–9]. Some mathematicians have tried to obtain conditions under which the reverse order law holds for generalized inverses. In particular, we have the following:

Theorem 1.5 ([7, Theorem 3]). *Let $a, b \in R^\dagger$ and $(1 - a^\dagger a)b$ be left *-cancelable. Then the following conditions are equivalent:*

(i) *ab is Moore-Penrose invertible and $(ab)^\dagger = b^\dagger a^\dagger$;*

(ii) *$[a^\dagger a, bb^*] = 0$ and $[bb^\dagger, a^*a] = 0$.*

D. Mosić and D. S. Djordjević proved the following equivalent statements for the reverse order law $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

Theorem 1.6 ([7, Theorem 1.1]). *Let $a, b \in R^\dagger$ and $(1 - a^\dagger a)b$ be left *-cancelable. Then the following statements are equivalent:*

(i) *$abb^\dagger a^\dagger ab = ab$;*

(ii) *$b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;*

(iii) *$a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;*

(iv) *$a^\dagger abb^\dagger$ is an idempotent;*

(v) *$bb^\dagger a^\dagger a$ is an idempotent;*

(vi) *$a^\dagger abb^\dagger \in R^\dagger$ and $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;*

(vii) *$a^\dagger abb^\dagger \in R^\dagger$ and $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.*

The reverse order law $(ab)^\sharp = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution were studied in [7] and the authors obtained the following.

Theorem 1.7 ([7, Theorem 2.1]). *Let $a, b, a^\dagger abb^\dagger \in R^\dagger$ and $ab \in R^\sharp$. Then the following statements are equivalent:*

(i) $(ab)^\sharp = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;
(ii) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in ab\{5\}$;
(iii) $(a^\dagger abb^\dagger)^\dagger = b(ab)^\sharp a$ *and* $abaa^\dagger = ab = b^\dagger bab$.

Inspired by the papers [4,6,7], we show that in a ring with involution R if $a, b, ab \in R^\dagger$ and $a^\dagger a = bb^\dagger$, then $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $[a^\dagger a, {b^*}^\dagger b^\dagger] = 0$ and $[bb^\dagger, a^\dagger {a^*}^\dagger] = 0$. We also obtain the necessary and sufficient conditions for establishing the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$. Moreover, we show that if $(1 - a^\dagger a)b$ is Moore-Penrose invertible, then $[(1 - a^\dagger a)b]^\dagger = b^\dagger(1 - a^\dagger a)$. Using left *- cancelability of $(1 - a^\dagger a)b$, we prove some necessary and sufficient conditions for the hybrid reverse order law $(ab)^\sharp = (abb^\dagger a^\dagger ab)^\dagger$ in rings with involution. Finally, we investigate necessary and sufficient conditions for the equations $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. Our results can be considered as an application of Theorem 1.6. Applying Theorem 1.7 will obtain several equivalent conditions for new reverse order law $(ab)^\sharp = (abb^\dagger a^\dagger ab)^\dagger$.

D. Mosić and D. S. Djordjević, in [6] showed that under certain conditions the inclusion $(ab)\{1, 5\} \subseteq b\{1, 3, 4\}a\{1, 3, 4\}$ becomes an equality:

Theorem 1.8 ([6]). *Let R be a ring with involution, let a and $b \in R^\dagger$,and let $(1 - a^\dagger a)b$ be left *-cancelable. If $ab \in R^\sharp$, then the inclusion $(ab)\{1, 5\} \subseteq b\{1, 3, 4\}a\{1, 3, 4\}$ is always an equality.*

D. Mosić and D. S. Djordjević, in [7] presented several equivalent conditions for the reverse order law $(ab)^\sharp = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$. Also, they [7] obtained conditions which guarantee the equality of the following inclusions

$$(ab)\{5\} \subseteq b\{1, 3, 4\}a^\dagger abb^\dagger\{1, 3, 4\}a\{1, 3, 4\}$$

and

$$abb^\dagger\{5\} \subseteq a^\dagger abb^\dagger\{1, 3, 4\}a\{1, 3, 4\}.$$

Theorem 1.9 ([7]). *Let R be a ring with involution, let a, b and $a^\dagger abb^\dagger \in R^\dagger$. If $ab \in R^\sharp$, then the inclusion $(ab)\{5\} \subseteq b\{1, 3, 4\}.a^\dagger abb^\dagger\{1, 3, 4\}a\{1, 3, 4\}$ is an equality.*

Theorem 1.10 ([7]). *Let $a, b, a^\dagger abb^\dagger \in R^\dagger$, and $abb^\dagger \in R^\sharp$. Then following statements are equivalent:*

(i) $abb^\dagger\{5\} \subseteq a^\dagger abb^\dagger\{1, 3, 4\}a\{1, 3, 4\}$;
(ii) $abb^\dagger\{5\} = a^\dagger abb^\dagger\{1, 3, 4\}a\{1, 3, 4\}$.

In this paper, for $b(abb^\dagger a^\dagger ab)^\dagger \in (abb^\dagger a^\dagger a)\{1,5\}$, we obtain an equivalent condition. Also, we obtain conditions which guarantee the equality

$$(abb^\dagger a^\dagger a)\{5\} = b(abb^\dagger a^\dagger ab)^{(1,3,4)}$$

in rings with involution. Moreover, we will show that under certain circumstances the following inclusion is always an equality

$$(abb^\dagger a^\dagger a)\{5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}.$$

By using the Moore-Penrose invertibility property, we will find the inverse of some special elements of a ring with involution.

2. RESULTS

We start this section with the following theorem which will be frequently used furthermore.

THEOREM 2.1 ([1]). *For any $a \in R^\dagger$, the following are satisfied:*

(i) $(a^\dagger)^\dagger = a$;
(ii) $(a^*)^\dagger = (a^\dagger)^*$;
(iii) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;
(iv) $(aa^*)^\dagger = (a^\dagger)^* a^\dagger$;
(v) $a^* = a^\dagger a a^* = a^* a a^\dagger$;
(vi) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$;
(vii) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

We also need the following results.

LEMMA 2.2 ([7]). *If $a \in R^\dagger$, then*

(i) $a.a\{1,3\} = \{aa^\dagger\}$;
(ii) $a\{1,4\}.a = \{a^\dagger a\}$.

LEMMA 2.3 ([7]). *Let a, b, and $a^\dagger abb^\dagger \in R^\dagger$. Then the following conditions are satisfied:*

(i) $(a^\dagger abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger a$;
(ii) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger(a^\dagger abb^\dagger)^\dagger$.

The following lemma gives two equalities that we will be used all over the paper.

LEMMA 2.4. *Let $a, b, abb^\dagger a^\dagger ab \in R^\dagger$. Then the following conditions are satisfied:*

(i) $(abb^\dagger a^\dagger ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger aa^\dagger$;
(ii) $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger b(abb^\dagger a^\dagger ab)^\dagger$.

Proof. (i) follows from the following:

$$\begin{aligned}(abb^\dagger a^\dagger ab)^\dagger aa^\dagger &= (abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger aa^\dagger \\ &= (abb^\dagger a^\dagger ab)^\dagger (aa^\dagger (abb^\dagger a^\dagger ab)\ (abb^\dagger a^\dagger ab)^\dagger)^* \\ &= (abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger \\ &= (abb^\dagger a^\dagger ab)^\dagger.\end{aligned}$$

Since

$$\begin{aligned}b^\dagger b(abb^\dagger a^\dagger ab)^\dagger &= b^\dagger b(abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger \\ &= ((abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)b^\dagger b)^* (abb^\dagger a^\dagger ab)^\dagger \\ &= (abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger \\ &= (abb^\dagger a^\dagger ab)^\dagger,\end{aligned}$$

(ii) follows immediately. □

Remark 2.5. Let R be a ring with involution and a and $b \in R^\dagger$. If $a^\dagger a = bb^\dagger$, then $(1 - a^\dagger a)b = 0$. So $(1 - a^\dagger a)b$ is left *-cancelable.

Therefore if $a, b \in R^\dagger$ and $a^\dagger a = bb^\dagger$, the equivalent conditions (i) and (ii) of Theorem 1.5 hold.

The following theorem gives another condition, equivalent to conditions (i) and (ii) of Theorem 1.5.

Theorem 2.6. *Let R be a ring with involution, $a, b, ab \in R^\dagger$ and $a^\dagger a = bb^\dagger$. Then $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $[a^\dagger a, b^{*\dagger} b^\dagger] = 0$ and $[bb^\dagger, a^\dagger a^{*\dagger}] = 0$.*

Proof. Let $(ab)^\dagger = b^\dagger a^\dagger$. Then by our assumption and Theorem 2.1, we have:

$$\begin{array}{llllll} & bb^\dagger a^\dagger a^{*\dagger} & = & b(b^\dagger a^\dagger abb^\dagger a^\dagger)a^{*\dagger} & = a^\dagger abb^\dagger bb^\dagger a^\dagger a^{*\dagger} \\ = & a^\dagger abb^\dagger a^\dagger a^{*\dagger} & = & a^\dagger abb^\dagger a^\dagger a^{*\dagger} a^\dagger a & = a^\dagger (abb^\dagger a^\dagger)^* a^{*\dagger} a^\dagger a \\ = & a^\dagger a^{*\dagger} bb^\dagger a^* a^{*\dagger} a^\dagger a & = & a^\dagger a^{*\dagger} bb^\dagger a^\dagger a & = a^\dagger a^{*\dagger} a^\dagger abb^\dagger \\ = & a^\dagger a^{*\dagger} bb^\dagger. \end{array}$$

Therefore $[bb^\dagger, a^\dagger a^{*\dagger}] = 0$. On the other hand

$$\begin{array}{llllll} & b^{*\dagger} b^\dagger a^\dagger a & = & b^{*\dagger} b^\dagger a^\dagger aa^\dagger a & = b^{*\dagger} b^\dagger a^\dagger abb^\dagger \\ = & b^{*\dagger} (b^\dagger a^\dagger ab)^* b^\dagger & = & b^{*\dagger} b^* a^\dagger ab^{*\dagger} b^\dagger & = (bb^\dagger)^* a^\dagger ab^{*\dagger} b^\dagger \\ = & bb^\dagger a^\dagger ab^{*\dagger} b^\dagger & = & a^\dagger abb^\dagger b^{*\dagger} b^\dagger & = a^\dagger ab^{*\dagger} b^\dagger. \end{array}$$

Hence $[a^\dagger a, b^{*\dagger} b^\dagger] = 0$.

Let $[a^\dagger a, b^{*\dagger} b^\dagger] = 0$ and $[bb^\dagger, a^\dagger a^{*\dagger}] = 0$. We prove four conditions for Moore-Penrose invertibility.

1) $abb^\dagger a^\dagger ab = aa^\dagger abb^\dagger b = ab$, (by assumption $a^\dagger a = bb^\dagger$.)
2) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger bb^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger$, (by assumption $a^\dagger a = bb^\dagger$.)

3) $(abb^\dagger a^\dagger)^* = a^{*\dagger}bb^\dagger a^* = aa^\dagger a^{*\dagger}bb^\dagger a^* = abb^\dagger a^\dagger a^{*\dagger}a^* = abb^\dagger a^\dagger$, (by assumption $[bb^\dagger, a^\dagger a^{*\dagger}] = 0$ and Theorem 2.1.)
4) $(b^\dagger a^\dagger ab)^* = b^* a^\dagger ab^{*\dagger} = b^* a^\dagger ab^{*\dagger}b^\dagger b = b^* b^{*\dagger}b^\dagger a^\dagger ab = b^\dagger a^\dagger ab$, (by assumption $[a^\dagger a, b^{*\dagger}b^\dagger] = 0$ and Theorem 2.1.)

By 1), 2), 3) and 4) we have $(ab)^\dagger = b^\dagger a^\dagger$. □

Corollary 2.7. *Let R be a ring with involution, let $a, b, ab \in R^\dagger$ and $a^\dagger a = bb^\dagger$. Then the following conditions are equivalent:*

(i) $[a^\dagger a, b^{*\dagger}b^\dagger] = 0$ *and* $[bb^\dagger, a^\dagger a^{*\dagger}] = 0$;
(ii) $[a^\dagger a, bb^*] = 0$ *and* $[bb^\dagger, a^* a] = 0$.
(iii) $(ab)^\dagger = b^\dagger a^\dagger$;

The next result shows that under certain conditions $(1 - a^\dagger a)b$ is Moore-Penrose invertible.

Theorem 2.8. *Let R be a ring with involution, let a and $b \in R^\dagger$. If $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$, $b^\dagger = b^*$ and $a^\dagger = a^*$, then $(1 - a^\dagger a)b$ is Moore-Penrose invertible with Moore-Penrose inverse $b^\dagger(1 - a^\dagger a)$.*

Proof. We prove the four conditions for Moore-Penrose invertibility.

Condition 1

$$\begin{aligned}(1 - a^\dagger a)b[b^\dagger(1 - a^\dagger a)](1 - a^\dagger a)b &= (1 - a^\dagger a)bb^\dagger(1 - a^\dagger a - a^\dagger a + a^\dagger aa^\dagger a)b\\ &= (1 - a^\dagger a)bb^\dagger(1 - a^\dagger a)b\\ &= (1 - a^\dagger a)(bb^\dagger b - bb^\dagger a^\dagger ab)\\ &= (1 - a^\dagger a)(b - bb^\dagger a^\dagger ab)\\ &= (1 - a^\dagger a)(b - a^\dagger abb^\dagger b)\\ &= (1 - a^\dagger a)(b - a^\dagger ab)\\ &= b - a^\dagger ab - a^\dagger ab + a^\dagger aa^\dagger ab\\ &= (1 - a^\dagger a)b.\end{aligned}$$

Condition 2

$$\begin{aligned}b^\dagger(1 - a^\dagger a)[(1 - a^\dagger a)b]b^\dagger(1 - a^\dagger a) &= b^\dagger(1 - a^\dagger a)bb^\dagger(1 - a^\dagger a)\\ &= b^\dagger(1 - a^\dagger a)(bb^\dagger - bb^\dagger a^\dagger a)\\ &= (b^\dagger - b^\dagger a^\dagger a)(bb^\dagger - bb^\dagger a^\dagger a)\\ &= b^\dagger - b^\dagger a^\dagger a - b^\dagger a^\dagger abb^\dagger + b^\dagger a^\dagger abb^\dagger a^\dagger a\\ &= b^\dagger - b^\dagger a^\dagger a - b^\dagger bb^\dagger a^\dagger a + b^\dagger bb^\dagger a^\dagger aa^\dagger a\\ &= b^\dagger - b^\dagger a^\dagger a - b^\dagger a^\dagger a + b^\dagger a^\dagger a = b^\dagger(1 - a^\dagger a).\end{aligned}$$

Condition 3

$$\begin{aligned}[(1-a^\dagger a)bb^\dagger(1-a^\dagger a)]^* &= [bb^\dagger(1-a^\dagger a)]^*(1-a^\dagger a)^*\\ &= (1-a^\dagger a)^*(bb^\dagger)^*(1-a^\dagger a)^*\\ &= (1-a^\dagger a)(bb^\dagger)(1-a^\dagger a).\end{aligned}$$

Condition 4

$$\begin{aligned}[b^\dagger(1-a^\dagger a)(1-a^\dagger a)b]^* &= [b^\dagger(1-a^\dagger a)b]^*\\ &= (b^\dagger b - b^\dagger a^\dagger ab)^*\\ &= (b^\dagger b)^* - (b^\dagger a^\dagger ab)^*\\ &= (b^\dagger b)^* - b^*(a^\dagger a)^*(b^\dagger)^*\\ &= b^\dagger b - b^\dagger a^\dagger ab\\ &= b^\dagger(1-a^\dagger a)(1-a^\dagger a)b.\end{aligned}$$ □

In order to state the next main result of this paper, we need the following result.

Theorem 2.9 ([1]). *Let R be a ring with involution and let $a \in R$. Then the following conditions are equivalent:*

(i) *a is Moore-Penrose invertible;*
(ii) *a is left *-cancelable and a^*a is group invertible;*
(iii) *a is right *-cancelable and aa^* is group invertible;*
(iv) *a is *-cancelable and both a^*a and aa^* are group invertible.*

By Theorems 2.8 and 2.9, we have the following.

Corollary 2.10. *Let R be a ring with involution, let a and $b \in R^\dagger$. If $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$, $b^\dagger = b^*$ and $a^\dagger = a^*$, then $(1-a^\dagger a)b$ is left *-cancelable.*

The following theorem provides other equivalent conditions to the conditions presented in Theorem 1.6.

Theorem 2.11. *Let R be a ring with involution, let $ab, a, b \in R^\dagger, b^\dagger = b^*$, $a^\dagger = a^*$ and $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. Then the following statements are equivalent:*

(i) $abb^\dagger a^\dagger ab = ab$;
(ii) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;
(iii) $a^\dagger abb^\dagger$ *is an idempotent;*
(iv) $bb^\dagger a^\dagger a$ *is an idempotent;*
(v) $a^\dagger abb^\dagger \in R^\dagger$ *and* $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;
(vi) $a^\dagger abb^\dagger \in R^\dagger$ *and* $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$;
(vii) $b(abb^\dagger a^\dagger ab)^\dagger a = bb^\dagger a^\dagger a$;
(viii) $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$.

Proof. By Theorem 1.6 and Corollary 2.10, (i)-(vi) are equivalent.

(vii)⇒(viii) Let $b(abb^\dagger a^\dagger ab)^\dagger a = bb^\dagger a^\dagger a$. Multiplying by $b^\dagger$ from the left side and multiplying by $a^\dagger$ from the right side we get $b^\dagger b(abb^\dagger a^\dagger ab)^\dagger aa^\dagger = b^\dagger a^\dagger$. By applying Lemma 2.4, we get $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$.

(viii)⇒(vii) Let $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. Multiplying by b from the left side and multiplying by a from the right side, we get $b(abb^\dagger a^\dagger ab)^\dagger a = bb^\dagger a^\dagger a$.

(viii)⇒(i) Let (viii) hold. By our assumption we have $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$ and therefore by (viii) we have $(ab)^\dagger = b^\dagger a^\dagger$, hence $abb^\dagger a^\dagger ab = ab$.

(i)⇒(viii) Let (i) hold. Then the equivalent statements (i)-(vi) are satisfied. We prove four conditions for Moore-Penrose invertibility:

1) $(abb^\dagger a^\dagger ab)b^\dagger a^\dagger(abb^\dagger a^\dagger ab) = (abb^\dagger a^\dagger ab)$, (by (iv)).
2) $b^\dagger a^\dagger(abb^\dagger a^\dagger ab)b^\dagger a^\dagger = b^\dagger a^\dagger abb^\dagger a^\dagger$, (by (iii)).
3) $((abb^\dagger a^\dagger ab)b^\dagger a^\dagger)^* = a^{\dagger *}bb^\dagger a^\dagger abb^\dagger a^* = (abb^\dagger a^\dagger ab)b^\dagger a^\dagger$.
4) $(b^\dagger a^\dagger(abb^\dagger a^\dagger ab))^* = b^* a^\dagger abb^\dagger a^\dagger ab^{\dagger *} = b^\dagger a^\dagger(abb^\dagger a^\dagger ab)$.

So $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. Hence (viii) holds. □

The following corollary shows that the equivalent conditions of the previous theorem are equivalent with the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$.

Corollary 2.12. *Let R be a ring with involution, let $ab, a, b, abb^\dagger a^\dagger ab \in R^\dagger$ and let $b^\dagger = b^*, a^\dagger = a^*$ and $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. Then $(ab)^\dagger = b^\dagger a^\dagger$.*

Proof. By assumption, the equivalent statements of Theorem 2.11 are satisfied. By (i) we have $(abb^\dagger a^\dagger ab)^\dagger = (ab)^\dagger$. On the other hand by (viii) we have $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. So $(ab)^\dagger = b^\dagger a^\dagger$. □

The following theorem shows that conditions of the previous theorem can be replaced by $a^\dagger a = bb^\dagger$.

Theorem 2.13. *Let R be a ring with involution, let $a, b \in R^\dagger$, and $a^\dagger a = bb^\dagger$. Then the conditions* (i)-(viii) *of the previous theorem hold.*

Proof. (i)-(vi) are equivalent by Theorem 1.6 and Remark 2.5.

(vii)⇒(viii): Let $b(abb^\dagger a^\dagger ab)^\dagger a = b^\dagger a^\dagger$. Multiplying by $b^\dagger$ from the left side and multiplying by $a^\dagger$ from the right side, we get $b^\dagger b(abb^\dagger a^\dagger ab)^\dagger aa^\dagger = b^\dagger a^\dagger$. By applying Lemma 2.4 we get $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$.

(viii)⇒(vii): Let $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. Multiplying by b from the left side and multiplying by a from the right side, we get $b(abb^\dagger a^\dagger ab)^\dagger a = bb^\dagger a^\dagger a$.

(viii)⇒(ii): Let (viii) hold, then $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. On the other hand, by our assumption, we have $a^\dagger a = bb^\dagger$. Therefore $(ab)^\dagger = b^\dagger a^\dagger$. So $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$.

(ii)⇒(viii): Let (ii) hold, then (i)-(vi) are equivalent by Theorem 1.6 and Remark 2.5. We show that $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. To achieve this goal, we investigate the four conditions for Moore-Penrose invertibility.

1) $(abb^\dagger a^\dagger ab)b^\dagger a^\dagger(abb^\dagger a^\dagger ab) = (abb^\dagger a^\dagger ab)$, (by (i)).
2) $b^\dagger a^\dagger(abb^\dagger a^\dagger ab)b^\dagger a^\dagger = b^\dagger a^\dagger abb^\dagger a^\dagger$, (by (ii)).
3)

$$\begin{array}{rlll} & ((abb^\dagger a^\dagger ab)b^\dagger a^\dagger)^* & = ((abb^\dagger bb^\dagger b)b^\dagger a^\dagger)^* & = (abb^\dagger a^\dagger)^* \\ = & (a^\dagger a)^* & = a^\dagger a & = a^\dagger aa^\dagger a \\ = & abb^\dagger a^\dagger & = (abb^\dagger a^\dagger ab)b^\dagger a^\dagger, & \end{array}$$

by our assumptions and (i).

4)

$$\begin{array}{rlll} & (b^\dagger a^\dagger(abb^\dagger a^\dagger ab))^* & = (b^\dagger a^\dagger(aa^\dagger aa^\dagger ab))^* & = (b^\dagger a^\dagger ab)^* \\ = & (b^\dagger b)^* & = b^\dagger b & = b^\dagger bb^\dagger b \\ = & b^\dagger a^\dagger ab & = (b^\dagger a^\dagger abb^\dagger a^\dagger)ab, & \end{array}$$

by our assumptions and (ii). □

Corollary 2.14. *Let R be a ring with involution, let $ab, a, b, abb^\dagger a^\dagger ab \in R^\dagger$. If $a^\dagger a = bb^\dagger$, then $(ab)^\dagger = b^\dagger a^\dagger$.*

The following theorem and Corollary 2.10 show that if $(1 - a^\dagger a)b$ is left *-cancelable, then two reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ and $(ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger$ are equivalent.

Theorem 2.15. *Let $a, b, ab, abb^\dagger a^\dagger ab \in R^\dagger$ and $(1-a^\dagger a)b$ be left *-cancelable. Then $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $(ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger$.*

Proof. Let $(ab)^\dagger = b^\dagger a^\dagger$, then $ab = abb^\dagger a^\dagger ab$. So that $(ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger$. Conversely, let $(ab)^\dagger = (abb^\dagger a^\dagger ab)^\dagger$, then $ab = abb^\dagger a^\dagger ab$. Therefore all of the equivalent statements of Theorem 2.11 are satisfied. By (viii) we have $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger a^\dagger$. So by our assumption $(ab)^\dagger = b^\dagger a^\dagger$. □

In Theorem 1.7, the reverse order law $(ab)^\sharp = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ is studied. In the following theorem, we investigate the new reverse order law $(ab)^\sharp = (abb^\dagger a^\dagger ab)^\dagger$ in rings with involution.

Theorem 2.16. *Let $a, b, abb^\dagger a^\dagger ab \in R^\dagger, ab \in R^\sharp, (1 - a^\dagger a)b$ be left *-cancelable. Then following statements are equivalent:*

(i) $(ab)^\sharp = (abb^\dagger a^\dagger ab)^\dagger$;
(ii) $(abb^\dagger a^\dagger ab)^\dagger \in ab\{1, 5\}$;
(iii) $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger b(ab)^\sharp aa^\dagger$ *and* $abaa^\dagger = ab = b^\dagger bab$.

Proof. (i)⇒(ii) It is clear.

(ii)⇒(iii) Let $(abb^\dagger a^\dagger ab)^\dagger \in ab\{1,5\}$, by our assumptions and Lemma 2.4 we have

$$\begin{aligned} abaa^\dagger &= ab(abb^\dagger a^\dagger ab)^\dagger abaa^\dagger \\ &= abab(abb^\dagger a^\dagger ab)^\dagger aa^\dagger \\ &= abab(abb^\dagger a^\dagger ab)^\dagger \\ &= ab(abb^\dagger a^\dagger ab)^\dagger ab \\ &= ab. \end{aligned}$$

Moreover,

$$\begin{aligned} b^\dagger bab &= b^\dagger bab(abb^\dagger a^\dagger ab)^\dagger ab \\ &= b^\dagger b(abb^\dagger a^\dagger ab)^\dagger abab \\ &= (abb^\dagger a^\dagger ab)^\dagger abab \\ &= ab(abb^\dagger a^\dagger ab)^\dagger ab \\ &= ab. \end{aligned}$$

By (ii), the equivalent statements of Theorem 1.6 are satisfied. Therefore $(abb^\dagger a^\dagger ab)^\dagger \in ab\{1,2,5\}$, Hence $(ab)^\sharp = (abb^\dagger a^\dagger ab)^\dagger$ and by Lemma 2.4, we have

$$(abb^\dagger a^\dagger ab)^\dagger = b^\dagger b(abb^\dagger a^\dagger ab)^\dagger aa^\dagger = b^\dagger b(ab)^\sharp aa^\dagger.$$

(iii)⇒(i) Let $(abb^\dagger a^\dagger ab)^\dagger = b^\dagger b(ab)^\sharp aa^\dagger$, then

$$\begin{aligned} (abb^\dagger a^\dagger ab)^\dagger &= b^\dagger b(ab)^\sharp aa^\dagger \\ &= b^\dagger bab{(ab)^\sharp}^3 abaa^\dagger \\ &= ab{(ab)^\sharp}^3 ab \\ &= (ab)^\sharp ab(ab)^\sharp. \\ &= (ab)^\sharp, \end{aligned}$$

which proves the result. □

Example 2.17. Consider 2×2 block matrices $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbb{C} \setminus \{0\}$. It is clear that

$$A^\dagger = \begin{bmatrix} 0 & 1/a \\ 1/a & 0 \end{bmatrix}, \qquad B^\dagger = \begin{bmatrix} 1/b & 0 \\ 0 & 1/b \end{bmatrix}, \qquad AB = \begin{bmatrix} 0 & ab \\ ab & 0 \end{bmatrix}.$$

Since the statements of Corollary 2.12 are satisfied, we obtain

$$(AB)^\dagger = B^\dagger A^\dagger = \begin{bmatrix} 0 & 1/ab \\ 1/ab & 0 \end{bmatrix}.$$

In [10], the elements whose Moore-Penrose inverses are idempotent in rings with involution are investigated:

Lemma 2.18 ([10]). *Let $a \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\dagger}$ *and* $a^{\dagger}$ *is idempotent;*
(ii) $a \in R^{\dagger}$ *and* $a^2 = aa^*a$*;*
(iii) $a \in R^{\sharp}$ *and* $a^2 = aa^*a$*.*

Lemma 2.19. *Let $a, b, a^{\dagger}$ and $b^{\dagger}$ be idempotent and $a, b, abb^*a^*ab \in R^{\dagger}$. Then the following conditions are satisfied:*

(i) $(abb^*a^*ab)^{\dagger} = (abb^*a^*ab)^{\dagger}aa^*$;
(ii) $(abb^*a^*ab)^{\dagger} = b^*b(abb^*a^*ab)^{\dagger}$.

Proof. (i):

$$\begin{aligned}(abb^*a^*ab)^{\dagger}aa^* &= (abb^*a^*ab)^{\dagger}(abb^*a^*ab)(abb^*a^*ab)^{\dagger}aa^* \\ &= (abb^*a^*ab)^{\dagger}(aa^*(abb^*a^*ab)\ (abb^*a^*ab)^{\dagger})^* \\ &= (abb^*a^*ab)^{\dagger}((abb^*a^*ab)\ (abb^*a^*ab)^{\dagger})^* \\ &= (abb^*a^*ab)^{\dagger}(abb^*a^*ab)(abb^*a^*ab)^{\dagger} \\ &= (abb^*a^*ab)^{\dagger}.\end{aligned}$$

(ii):

$$\begin{aligned}b^*b(abb^*a^*ab)^{\dagger} &= b^*b(abb^*a^*ab)^{\dagger}(abb^*a^*ab)(abb^*a^*ab)^{\dagger} \\ &= ((abb^*a^*ab)^{\dagger}(abb^*a^*ab)b^*b)^*(abb^*a^*ab)^{\dagger} \\ &= ((abb^*a^*ab)^{\dagger}(abb^*a^*ab))^*(abb^*a^*ab)^{\dagger} \\ &= (abb^*a^*ab)^{\dagger}(abb^*a^*ab)(abb^*a^*ab)^{\dagger} \\ &= (abb^*a^*ab)^{\dagger}.\end{aligned}$$

□

Theorem 2.20. *Let $a, b, abb^{\dagger}a^{\dagger}ab \in R^{\dagger}$, $a, b, a^{\dagger}$ and $b^{\dagger}$ be idempotent and $(ab) \in R^{\sharp}$. Then following statements are equivalent:*

(i) $(ab)^{\sharp} = b(abb^*a^*ab)^{\dagger}a$*;*
(ii) $(abb^*a^*ab)^{\dagger} = b^*(ab)^{\sharp}a^*$ *and* $aba^*a = ab = bb^*ab$*.*

Proof. (i)⇒(ii) We have

$$\begin{aligned} & aba^*a && = (ab)(ab)^{\sharp}(ab)a^*a && = (ab)^2(ab)^{\sharp}a^*a \\ = \ & (ab)^2b(abb^*a^*ab)^{\dagger}aa^*a && = (ab)^2b(abb^*a^*ab)^{\dagger}a && = (ab)^2(ab)^{\sharp} \\ = \ & (ab)(ab)^{\sharp}(ab) && = ab. && \end{aligned}$$

Moreover,

$$\begin{array}{llll} & bb^*ab & = bb^*(ab)(ab)^\sharp(ab) & = bb^*(ab)^\sharp(ab)^2 \\ = & bb^*b(abb^*a^*ab)^\dagger a(ab)^2 & = b(abb^*a^*ab)^\dagger a(ab)^2 & = (ab)^\sharp(ab)^2 \\ = & (ab)(ab)^\sharp(ab) & = ab. & \end{array}$$

By Lemma 2.18,

$$b^*(ab)^\sharp a^* = b^*b(abb^*a^*ab)^\dagger aa^* = (abb^*a^*ab)^\dagger.$$

(ii)⇒(i) We have

$$\begin{aligned} b(abb^*a^*ab)^\dagger a &= bb^*(ab)^\sharp a^*a \\ &= bb^*ab((ab)^\sharp)^3(aba^*a) \\ &= (ab)((ab)^\sharp)^3(ab) \\ &= (ab)^\sharp, \end{aligned}$$

which proves the result. □

Now, we give an equivalent condition for $b(abb^\dagger a^\dagger ab)^\dagger \in (abb^\dagger a^\dagger a)\{1,5\}$.

THEOREM 2.21. *If $a, b, abb^\dagger a^\dagger ab \in R^\dagger$, then following statements are equivalent:*

(i) $b(abb^\dagger a^\dagger ab)^\dagger \in (abb^\dagger a^\dagger a)\{1,5\}$;
(ii) $b(abb^\dagger a^\dagger ab)\{1,3,4\} \subseteq (abb^\dagger a^\dagger a)\{1,5\}$.

Proof. (i)⇒(ii) Suppose that $b(abb^\dagger a^\dagger ab)^\dagger \in (abb^\dagger a^\dagger a)\{1,5\}$. For

$$(abb^\dagger a^\dagger ab)^{\{1,3,4\}} \in (abb^\dagger a^\dagger ab)\{1,3,4\},$$

by Lemma 2.2 and (i) we obtain:

$$\begin{aligned} b(abb^\dagger a^\dagger ab)^{\{1,3,4\}}(abb^\dagger a^\dagger a) &= b(abb^\dagger a^\dagger ab)^{\{1,3,4\}}(abb^\dagger a^\dagger a)[b(abb^\dagger a^\dagger ab)^\dagger](abb^\dagger a^\dagger a) \\ &= b(abb^\dagger a^\dagger ab)^\dagger(abb^\dagger a^\dagger a)[b(abb^\dagger a^\dagger ab)^\dagger](abb^\dagger a^\dagger a) \\ &= b(abb^\dagger a^\dagger ab)^\dagger(abb^\dagger a^\dagger a)(abb^\dagger a^\dagger a)[b(abb^\dagger a^\dagger ab)^\dagger] \\ &= (abb^\dagger a^\dagger a)b(abb^\dagger a^\dagger ab)^\dagger(abb^\dagger a^\dagger a)[b(abb^\dagger a^\dagger ab)^\dagger] \\ &= (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger \\ &= (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^{\{1,3,4\}} \\ &= (abb^\dagger a^\dagger a)b(abb^\dagger a^\dagger ab)^{\{1,3,4\}}. \end{aligned}$$

Moreover, by Lemma 2.2 and (i) we have

$$\begin{aligned} abb^\dagger a^\dagger a &= (abb^\dagger a^\dagger a)[b(abb^\dagger a^\dagger ab)^\dagger](abb^\dagger a^\dagger a) \\ &= (abb^\dagger a^\dagger a)b(abb^\dagger a^\dagger ab)^{\{1,3,4\}}(abb^\dagger a^\dagger a). \end{aligned}$$

(ii)⇒(i) is clear. □

Corollary 2.22. *If* $a, b, abb^\dagger a^\dagger ab \in R^\dagger, abb^\dagger a^\dagger a \in R^\sharp$ *and*

$$(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^\dagger,$$

then $b(abb^\dagger a^\dagger ab)\{1,3,4\} \subseteq (abb^\dagger a^\dagger a)\{1,5\}$.

Proof. Suppose that $(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^\dagger$. Then

$$b(abb^\dagger a^\dagger ab)^\dagger \in (abb^\dagger a^\dagger a)\{1,5\}.$$

Then, by Theorem 2.21, we have $b(abb^\dagger a^\dagger ab)\{1,3,4\} \subseteq (abb^\dagger a^\dagger a)\{1,5\}$. □

Lemma 2.23. *Let* $a, b, abb^\dagger a^\dagger ab \in R^\dagger$*, and* $abb^\dagger a^\dagger a \in R^\sharp$*. If*

$$(abb^\dagger a^\dagger a)\{1,5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\},$$

then we have $(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^\dagger$.

Proof. Let $(abb^\dagger a^\dagger a)\{1,5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}$. Then there is

$$(abb^\dagger a^\dagger ab)^{\{1,3,4\}} \in (abb^\dagger a^\dagger ab)\{1,3,4\}$$

such that $(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^{(1,3,4)}$. By Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} b(abb^\dagger a^\dagger ab)^\dagger &= b(abb^\dagger a^\dagger ab)^\dagger (abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^\dagger \\ &= b(abb^\dagger a^\dagger ab)^{(1,3,4)}(abb^\dagger a^\dagger ab)(abb^\dagger a^\dagger ab)^{(1,3,4)} \\ &= (abb^\dagger a^\dagger a)^\sharp (abb^\dagger a^\dagger a)(abb^\dagger a^\dagger a)^\sharp \\ &= (abb^\dagger a^\dagger a)^\sharp, \end{aligned}$$

which proves the result. □

Now, we prove that $(abb^\dagger a^\dagger a)\{5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}$ is equivalent to

$$(abb^\dagger a^\dagger a)\{5\} = b(abb^\dagger a^\dagger ab)\{1,3,4\}.$$

Theorem 2.24. *Let* $a, b, abb^\dagger a^\dagger ab \in R^\dagger$*, and* $abb^\dagger a^\dagger a \in R^\sharp$*. Then following statements are equivalent:*

(i) $(abb^\dagger a^\dagger a)\{5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}$.

(ii) $(abb^\dagger a^\dagger a)\{5\} = b(abb^\dagger a^\dagger ab)\{1,3,4\}$.

Proof. (i)⇒(ii) Suppose that $(abb^\dagger a^\dagger a)\{5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}$, then

$$(abb^\dagger a^\dagger a)\{1,5\} \subseteq b(abb^\dagger a^\dagger ab)\{1,3,4\}.$$

By the previous lemma, $(abb^\dagger a^\dagger a)^\sharp = b(abb^\dagger a^\dagger ab)^\dagger$. Therefore, by Corollary 2.22, we have

$$b(abb^\dagger a^\dagger ab)\{1,3,4\} \subseteq (abb^\dagger a^\dagger a)\{1,5\}.$$

Hence $b(abb^\dagger a^\dagger ab)\{1,3,4\} \subseteq (abb^\dagger a^\dagger a)\{5\}$. It follows that

$$(abb^\dagger a^\dagger a)\{5\} = b(abb^\dagger a^\dagger ab)\{1,3,4\}.$$

Therefore (ii) holds.

(ii)⇒(i) is clear. □

In [3], M. M. Karizaki et al. investigated the invertibility of Moore-Penrose invertible elements of operators on a Hilbert C^*-module. We are going to find the inverse of some special elements via Moore-Penrose inverses. In order to achieve this goal, we need the following.

Definition 2.25. Let R be a ring with involution. An element $a \in R$ is said to be *EP* if $a^\dagger = a^\sharp$ and $a \in R^\dagger \cap R^\sharp$.

Lemma 2.26 ([5]). *Let R be a ring with involution and let $a \in R^\dagger$. Then a is an EP if and only if $aa^\dagger = a^\dagger a$.*

Lemma 2.27. *Let R be a ring with involution and $a \in R^\dagger$. If a is a normal element, then a is an EP element.*

Proof. Let a be normal. By Theorem 2.1 we have:

$$\begin{array}{lllll} & aa^\dagger & = & aa^*(aa^*)^\dagger & = a^*a(a^*a)^\dagger \\ = & a^*aa^\dagger(a^*)^\dagger & = & a^*(a^*)^\dagger & = a^*(aa^*)^\dagger a \\ = & a^*(a^*)^\dagger a^\dagger a & = & a^*(aa^*)^\dagger a & = a^\dagger a. \end{array}$$

□

Theorem 2.28. *Let a be an EP element. Then $1-aa^\dagger-a^\dagger$ and $1-a-aa^\dagger$, are invertible.*

Proof. We have

$$\begin{aligned} &(1-aa^\dagger-a^\dagger)(1-a-aa^\dagger)\\ &= 1-a-aa^\dagger-aa^\dagger+aa^\dagger a+aa^\dagger aa^\dagger-a^\dagger+a^\dagger a+a^\dagger aa^\dagger\\ &= 1-a-aa^\dagger-aa^\dagger+aa^\dagger a+aa^\dagger-a^\dagger+a^\dagger a+a^\dagger=1. \end{aligned}$$

Moreover,

$$\begin{aligned} &(1-a-aa^\dagger)(1-aa^\dagger-a^\dagger)\\ &= 1-aa^\dagger-a^\dagger-a+aaa^\dagger+aa^\dagger-aa^\dagger+aa^\dagger aa^\dagger+aa^\dagger a^\dagger\\ &= 1-aa^\dagger-a^\dagger-a+a+aa^\dagger-aa^\dagger+aa^\dagger+a^\dagger=1. \end{aligned}$$

Therefore $1-aa^\dagger-a^\dagger$ and $1-a-aa^\dagger$ are invertible. □

Corollary 2.29. *Let a be an EP element. Then*

$$1-aa^\dagger-a^{*\dagger}a^\dagger \quad \text{and} \quad 1-aa^*-aa^\dagger$$

are invertible.

Proof. Since a^*a is normal, then a^*a is EP. Now we replace a by aa^*, and hence, by the previous theorem, we have

$$\begin{aligned} 1-aa^\dagger-a^\dagger &= 1-(aa^*)(aa^*)^\dagger-(aa^*)^\dagger\\ &= 1-aa^*(a^*)^\dagger a^\dagger-(a^*)^\dagger a^\dagger\\ &= 1-aa^\dagger-a^{*\dagger}a^\dagger. \end{aligned}$$

We also have

$$\begin{aligned} 1-a-aa^{\dagger} &= 1-(aa^*)-(aa^*)(aa^*)^{\dagger} \\ &= 1-aa^*-aa^*(a^*)^{\dagger}a^{\dagger} \\ &= 1-aa^*-aa^{\dagger}. \end{aligned}$$

Furthermore the result follows from Theorem 2.28. □

Finally, we give an example to illustrate our results.

EXAMPLE 2.30. Consider 2×2 block matrices

$$A=\begin{bmatrix}0 & a\\ a & 0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}b & 0\\ 0 & b\end{bmatrix},$$

where $a,b\in\mathbb{C}\setminus\{0\}$.

It is clear that

$$A^{\dagger}=\begin{bmatrix}0 & 1/a\\ 1/a & 0\end{bmatrix},\quad B^{\dagger}=\begin{bmatrix}1/b & 0\\ 0 & 1/b\end{bmatrix},\quad AB=\begin{bmatrix}0 & ab\\ ab & 0\end{bmatrix}.$$

A, B and AB are EP elements. It is clear that statements of previous theorem are satisfied. Therefore

$$\begin{aligned} (1-AA^{\dagger}-A^{*\dagger}A^{\dagger})^{-1} &= 1-AA^*-AA^{\dagger}, \\ (1-BB^{\dagger}-B^{*\dagger}B^{\dagger})^{-1} &= 1-BB^*-BB^{\dagger} \end{aligned}$$

and

$$(1-(AB)(AB)^{\dagger}-(AB)^{*\dagger}(AB)^{\dagger})^{-1}=1-(AB)(AB)^*-(AB)(AB)^{\dagger}.$$

REFERENCES

[1] K. P. S. Bhaskara Rao, *The theory of generalized inverses over commutative rings*, Algebra, Logic and Applications, Vol. 17, Taylor and Francis, London, 2002.

[2] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev., **8** (1966), 518–521.

[3] M. M. Karizaki, M. Hassani and M. Amyari, *Moore-Penrose inverse of product operators in Hilbert C^*-modules*, Filomat, **30** (2016), 3397–3402.

[4] J.J. Koliha, D.S. Djordjević and D. Cvetković, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl., **426** (2007), 371–381.

[5] J. J. Koliha and P. Patricio, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc., **137** (2002), 137–152.

[6] D. Mosić and D. S. Djordjević, *Some results on the reverse order law in rings with involution*, Aequationes Math., **83** (2012), 271–282.

[7] D. Mosić and D. S. Djordjević, *The reverse order law $(ab)^{\sharp}=b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$ in rings with involution*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **109** (2015), 257–265.

[8] Y. Tian, *On mixed-type reverse-order laws for Moore-Penrose inverse of a matrix product*, Int. J. Math. Math. Sci., **2004** (2004), 3103–3116.

[9] Y. Tian, *The reverse-order law $(AB)^{\dagger}=B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ and its equivalent equalities*, J. Math. Kyoto Univ., **45** (2005), 841–850.

[10] H. Zhu, J. Chen, Y. Zhou, *On elements whose Moore-Penrose inverse is idempotent in a *-ring*, Turkish J. Math., **45** (2021), 878–889.

Received September 4, 2023
Accepted February 28, 2024

University of Mashhad
Department of Pure Mathematics, Ferdowsi
P.O. Box 1159, Mashhad, 91775, Mashhad, Iran
E-mail: `hrahmani2013@yahoo.com`

University of Mashhad
Center of Excellence in Analysis on Algebraic Structures
Department of Pure Mathematics, Ferdowsi
P.O. Box 1159, Mashhad, 91775, Mashhad, Iran
E-mail: `mirmostafaei@ferdowsi.um.ac.ir`