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AN ALTERNATIVE SOLUTION TO THE PROBLEM OF THE REAL CUBIC MOMENT

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Abstract. In this article, we are interested in solving the real cubic truncated moment problem. We provide some results that make it possible to obtain a complete solution via a minimum representative measurement. Some numerical examples are also presented to emphasize the simplicity of our approach.

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1. INTRODUCTION

The theory of moment problems (MP) is an important area of applied mathematics, first formulated by T. J. Stieltjes in 1894. However, the study of moments and related topics dates back twenty years earlier. Indeed, according to Kreĭn [25], Chebychev (1874) and Markov (1884) used this object in their research on boundary values of integrals. Other research on this problem was carried out simultaneously, notably by P. Nevanlinna, M. Riesz and T. Carleman [4, 26, 31, 32].

Since then, numerous variants of this problem have emerged. In 1920 Hamburger extended the MP on $\mathbb{R}$ [22]. At the same time Hausdorff [23] explored the MP on an interval $[a, b] \subset \mathbb{R}$, which can easily be restricted to $[0, 1]$. The trigonometric MP deals with the case where the representing measure is supported on a torus [1–3, 35]. These one-dimensional problems have been studied widely. We refer the interested reader to [5, 18, 29, 30] for example.

The multidimensional moment problem concerns the case where the solution measure of the problem is supported on $\mathbb{K}^d$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $d \in \mathbb{N}$ [8, 10, 19, 20, 33].

The multidimensional $\mathcal{K}$-MP, answers the following question: given a multi-indexed sequence $\beta = (\beta_{\mathbf{i}})_{\mathbf{i}\in i\mathbb{N}^d}$ of $\mathbb{K}^d$, is there any positive Borel measure μ

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such that:

$$(1)\qquad \begin{cases} \mathbf{x}^{\mathbf{i}} \in L^1(\mu) \\ \beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} \mathrm{d}\mu \end{cases}, \text{ for all } \mathbf{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d \text{ and } \operatorname{supp} \mu \subseteq \mathcal{K}?$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{x}^{\mathbf{i}} = {x_1}^{i_1}.{x_2}^{i_2}.\ldots.{x_d}^{i_d}$.

A sequence that verifies the relation (1) is called a sequence of $\mathcal{K}$-moments, and the measure μ is a $\mathcal{K}$-representing measure for the sequence β. If $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, we use a sequence of moments and a representing measure of β in brief.

If in the problem (1), we take $|\mathbf{i}| = i_1 + i_2 + \ldots + i_d \leq m$ for a positive integer m, the problem is known as the $\mathcal{K}$-truncated moments problem, denoted in general by TMP [8, 27, 28].

For the bidimensional case, let us consider a doubly indexed finite sequence β of real numbers defined as follows

$$\beta \equiv \beta^{(m)} = \{\beta_{ij}\}_{i,j \in \mathbb{Z}_+, 0 \leq i+j \leq m} = \{\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{m0}, \ldots, \beta_{0m}\}$$

with $\beta_{00} > 0$. The truncated real moment problem (TRMP) associated to β consists in finding the existence of a Borel positive measure μ supported in $\mathbb{R}^2$ such that

$$(2)\qquad \begin{cases} x^i y^j \in L^1(\mu) \\ \beta_{ij} = \int x^i y^j \mathrm{d}\mu \end{cases}, \quad (i, j \in \mathbb{Z}_+, \quad 0 \leq i + j \leq m) \text{ and } \operatorname{supp} \mu \subset \mathbb{R}^2.$$

The truncated complex moment problem (TCMP) for a doubly indexed finite sequence

$$s \equiv s^{(m)} = (s_{ij})_{i,j \in \mathbb{Z}_+, 0 \leq i+j \leq m} = \{s_{00}, s_{01}, s_{10}, \ldots, s_{0m}, \ldots, s_{m0}\},$$

of complex numbers with $s_{00} > 0$ and $s_{ji} = \overline{s_{ij}}$ investigates the existence of a positive Borel measure σ supported on $\mathbb{C}$ such that,

$$\bar{z}^i z^j \in L^1(\sigma) \text{ and } s_{ij} = \int \bar{z}^i z^j \mathrm{d}\sigma, \quad (i, j \in \mathbb{Z}_+, \quad 0 \leq i + j \leq m)$$

and $\operatorname{supp} \sigma \subset \mathbb{C}^2$.

Curto and Fialkow proved in [7, Proposition 1.12] an equivalence between TRMP and TCMP. So, we simply speak about TMP. In [6, 7, 12, 21], the authors provide solutions for TMP when $m = 2$ (quadratic) and $m = 4$ (quartic). Their approaches are generally based on the positivity and flat extension of the associated moment matrix, as well as on the variety cone [20] for the explicit determination of the representing measure. For certain even values of m greater than 4, Curto and Fialkow [9, 19] employed in addition the notion of recursively generated and/or recursively determined moment matrices. The case $m = 6$ was studied by Curto et al in [11, 13], and by Yoo [36, 37] who investigated the case where the rank of the moment matrix is strictly less or equal than the cardinal of the cone of the associated variety.

For odd values of m, many studies have been undertaken. In [24], a complete solution is pointed out for the cubic TCMP ($m = 3$), based on the commutativity conditions of the matrices associated with the cubic moment sequence. In [14], Curto and Yoo presented an alternative solution of the nonsingular cubic TRMP, using the invariance under a degree-one transformation, positivity, flatness and recursively determined moment matrices. Recently, the authors [17] provided a solution for a class of quintic TRMP ($m = 5$).

If a sequence of moments $\beta \equiv \beta^{(m)} = \{\beta_{ij}\}_{0\leq i+j\leq m}$ admits one or more representing measures, Richter-Tchakaloff theorem [15] confirms the existence of a finite atomic representing measure.

Thus, in the case where (2) admits a solution, the sequence of moments $\beta^{(m)}$ has a finite atomic representing measure μ such that

$$\mu := \sum_{k=1}^{r} \rho_k \delta_{(x_k, y_k)},$$

where the positive numbers ρ_k, the d-tuples (x_k, y_k), $1 \leq k \leq r$ and $\delta_{(x_k,y_k)}$ are respectively called weights, atoms of the measure μ, and the Dirac measure at the point (x_k, y_k). The measure μ is then said to be r-atomic, and we have

$$\beta_{ij} = \rho_1 x_1^i y_1^j + \cdots + \rho_r x_r^i y_r^j = \int x^i y^j \,\mathrm{d}\mu, \quad 0 \leq i + j \leq m.$$

In this paper, we deal with the cubic TRMP. So, let $\beta \equiv \beta^{(3)} = \{\beta_{ij}\}_{0\leq i+j\leq 3}$ be a doubly indexed sequence with real values given with $\beta_{00} > 0$. As m is odd ($m = 3$), we gather the data of the sequence β in the following two matrices,

$$\mathcal{M}(1) := \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} \\ \beta_{10} & \beta_{20} & \beta_{11} \\ \beta_{01} & \beta_{11} & \beta_{02} \end{pmatrix} \text{ and } B(2) := \begin{pmatrix} \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{21} & \beta_{12} & \beta_{03} \end{pmatrix}. \tag{3}$$

First, we determine quartic moments $\beta_{40}, \beta_{31}, \beta_{22}, \beta_{31}, \beta_{22}, \beta_{13}$ and β_{04} to construct a positive semidefinite extension $\mathcal{M}(2)$ of the matrix $\mathcal{M}(1)$ as follows,

$$\mathcal{M}(2) := \left(\begin{array}{c|cc|ccc} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \hline \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{array}\right),$$

so that $\operatorname{rank} \mathcal{M}(2) = \operatorname{rank} \mathcal{M}(1)$. Otherwise, $\mathcal{M}(2)$ can be extended to a positive semidefinite matrix $\mathcal{M}(3)$ by calculating quintic moments (β_{50}, β_{41}, β_{32}, β_{23}, β_{14} and β_{05}), and sixtics (β_{60}, β_{51}, β_{42}, β_{33}, β_{24}, β_{15} and β_{06}),

$$\mathcal{M}(3) = \left(\begin{array}{c|cc|ccc|cccc}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\
\hline
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\
\hline
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{14} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{05} \\
\hline
\beta_{30} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{14} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{14} & \beta_{05} & \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06}
\end{array}\right),$$

such that $\operatorname{rank}\mathcal{M}(3) = \operatorname{rank}\mathcal{M}(2)$.

The main target in this work is to provide a simple and complete alternative solution to cubic TRMP, when the moment matrix $\mathcal{M}(1)$ is nonsingular, and to investigate the existence of a representing measure at most 4-atomic.

The remainder of this paper is organized as follows: Section 2 is devoted to stating some notations and some tools that will be used for solving the cubic TRMP, followed by Section 3, where we present our main results illustrated by numerical examples.

2. PRELIMINARIES

We denote by $\mathcal{P} = \mathbb{R}[x, y]$ the space polynomials with indeterminates and real coefficients, and for $k \geq 1$, $\mathcal{P}_k = \mathbb{R}_k[x, y]$ is the subspace of $\mathcal{P}$ consisting of polynomials with degree less than or equal to k. Recall that $\dim \mathcal{P}_k = \binom{2+k}{k}$. For $P(x, y) = \sum_{0\leq i+j\leq k} a_{ij}x^iy^j \in \mathcal{P}_k$, let $\widehat{P} \equiv (a_{ij})$ the column vector of the coefficients of P with respect to the base of $\mathcal{P}_k$ consisting of the monomials of $\mathcal{P}_k$ in lexicographic order in degrees. For example, for $k = 2$, these monomials are: 1, x, y, x^2, xy and y^2.

$M_{(p,q)}(\mathbb{R})$ denotes the set of $p \times q$ matrices with real coefficients. For a symmetric matrix A, we write $A \succeq 0$ if A is positive semidefinite and $A > 0$ if A is positive definite.

To a sequence of moments $\beta = \beta^{(2n)} \equiv \{\beta_{ij}\}_{0\leq i+j\leq 2n}$ where $\beta_{00} > 0$, we define a Riesz functional on $\mathcal{P}_k$ associated with β as follows

$$L_\beta : \mathcal{P}_n \longrightarrow \mathbb{R}$$
$$P(x, y) = \sum_{0\leq i+j\leq n} a_{ij}x^iy^j \longmapsto \sum_{0\leq i+j\leq n} a_{ij}\beta_{ij}.$$

To sequence β we associate the matrix moment $\mathcal{M}(n)$. The rows and columns of $\mathcal{M}(n)$ are successively denoted by the monomials X^iY^j where $0 \leq i+j \leq n$ in lexicographic degree order.

We define a bilinear form $\langle .,. \rangle_{\mathcal{M}(n)}$ on $\mathcal{P}_n$ by

$$\langle P, Q \rangle_{\mathcal{M}(n)} := L_\beta(PQ), \text{ for any } P, Q \in \mathcal{P}_n.$$

It is clear that the entry for $\mathcal{M}(n)$ in row X^iY^j and column X^pY^q, $0 \leq i+j \leq n$ and $0 \leq p+q \leq n$ is

$$L_\beta(X^iY^jX^pY^q) = \beta_{i+p,j+q}.$$

So, $\mathcal{M}(n)$ is a real symmetric matrix. Moreover, if μ is a representing measure for β then

$$\left\langle \mathcal{M}(n))\widehat{P}, \widehat{P} \right\rangle_{\mathcal{M}(n)} = L_\beta(P^2) = \int P^2 \mathrm{d}\mu \geq 0.$$

Since $\mathcal{M}(n)$ is symmetric, it follows that $\mathcal{M}(n) \succeq 0$. Therefore, the fact that being $\mathcal{M}(n)$ positive semidefinite is a necessary condition for the existence of a representative measure.

The matrix $\mathcal{M}(n)$ admits a decomposition by blocks

$$\mathcal{M}(n) = (B[i,j])_{0 \leq i,j \leq n},$$

$$\mathcal{M}(n) := \begin{pmatrix} B[0,0] & B[0,1] & \dots & B[0,n] \\ B[1,0] & M[1,1] & \dots & B[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ B[n,0] & B[n,1] & \dots & B[n,n] \end{pmatrix},$$

where,

$$B[i,j] = \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \dots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \dots & \beta_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{j,i} & \beta_{j-1,i+1} & \dots & \beta_{0,i+j} \end{pmatrix}, \quad 0 \leq i,j \leq n.$$

Thus, each block $B[i,j]$ has the Hankel property, i.e. it is constant on each cross diagonal. If we label the columns and rows of the moment matrix $\mathcal{M}(n)$ by considering the lexicographic order of the monomials in degree, 1, X, Y, X^2, XY, $Y^2, \dots$, X^n, $X^{n-1}Y, \dots$, XY^{n-1}, Y^n, then for, the matrix $\mathcal{M}(2)$ we have,

$$\mathcal{M}(2) = \begin{array}{c} \\ 1 \\ \\ X \\ Y \\ \\ X^2 \\ XY \\ Y^2 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & & X & Y & & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} \beta_{00} & | & \beta_{10} & \beta_{01} & | & \beta_{20} & \beta_{11} & \beta_{02} \\ -- & - & -- & -- & - & -- & -- & -- \\ \beta_{10} & | & \beta_{20} & \beta_{11} & | & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & | & \beta_{11} & \beta_{02} & | & \beta_{21} & \beta_{12} & \beta_{03} \\ -- & - & -- & -- & - & -- & -- & -- \\ \beta_{20} & | & \beta_{30} & \beta_{21} & | & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & | & \beta_{21} & \beta_{12} & | & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & | & \beta_{12} & \beta_{03} & | & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix} \end{array}.$$

In the following theorem, Shmul'yan [34] establishes a necessary and sufficient condition which ensures the positive extension and the flatness of a positive semidefinite matrix.

THEOREM 2.1. *Let $A \in M_{(n,n)}(\mathbb{R})$, $B \in M_{(n,p)}(\mathbb{R})$, and $C \in M_{(p,p)}(\mathbb{R})$ be matrices of real numbers. We have,*

$$\tilde{A} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0 \iff \begin{cases} A \succeq 0 \\ B = AW & (\textit{for some } W \in M_{(n,p)}(\mathbb{R})). \\ C \succeq W^T AW \end{cases}$$

Moreover

$$\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A) \iff C = W^T AW \textit{ for some } W \textit{ such that } AW = B.$$

When $\tilde{A}$ in Theorem 2.1 has the same rank as A, we say that $\tilde{A}$ is a flat extension of A. Moreover, if $A \succeq 0$ then each flat extension $\tilde{A}$ of A is positive semidefinite.

REMARK 2.2.

(a) According to the factorization lemma of Douglas [16], the condition $B = AW$ for a certain matrix W is equivalent to $\operatorname{Ran}(B) \subseteq \operatorname{Ran}(A)$.
(b) Since $A = A^T$, we obtain $W^T AW$ independent of W provided that $B = AW$.

According to Theorem 2.1, $\mathcal{M}(n) \succeq 0$ admits a flat positive semidefinite extension $\mathcal{M}(n)$ such that

$$\mathcal{M}(n+1) = \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1)^T & C(n+1) \end{pmatrix},$$

is equivalent to having the following two conditions,

(i) $B(n+1) = \mathcal{M}(n)W$ for a matrix W;
(ii) $C(n+1) = W^T \mathcal{M}(n)W$ is a Hankel matrix.

Let us notice also that we have

$$\begin{aligned} &\begin{pmatrix} I_p & 0 \\ -W^T & I_q \end{pmatrix} \mathcal{M}(n+1) \begin{pmatrix} I_p & -W \\ 0 & I_q \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}(n) & 0 \\ 0 & C(2) - W^T\mathcal{M}(n)W \end{pmatrix}, \end{aligned} \tag{4}$$

where I_p and I_q are the unit matrices of respective orders $p = n+2$ and $q = \frac{(n+1)(n+1)}{2}$. So from (4), we deduce that,

$$\operatorname{rank}\mathcal{M}(n+1) = \operatorname{rank}\mathcal{M}(n) + \operatorname{rank}\left(C(2) - W^T\mathcal{M}(n)W\right). \tag{5}$$

Let us set $\mathcal{C}_{\mathcal{M}(n)} := \operatorname{span}\left\{1, X, Y, X^2, XY, Y^2, \cdots, X^n, \cdots, Y^n\right\}$ the column space of the matrix $\mathcal{M}(n)$. The correspondence between $\mathcal{P}_n$ and $\mathcal{C}_{\mathcal{M}(n)}$, the column space of the matrix $\mathcal{M}(n)$, is given by $P(X,Y) = \mathcal{M}(n)\widehat{P}$ where $P = \sum_{0\le i+j\le 2n} a_{ij}x^iy^j$. That is, $P(X,Y)$ is a linear combination of $\mathcal{M}(n)$ columns.

We express the $\mathcal{M}(n)$ columns linear dependence by the following relations,

$$P_1(X,Y) = \mathbf{0}, \quad P_2(X,Y) = \mathbf{0}, \quad \ldots, \quad P_k(X,Y) = \mathbf{0},$$

for some polynomials $P_1, P_2, \ldots, P_k \in \mathcal{P}_n$, $k \in \mathbb{N}$ and $k \le \frac{(n+2)(n+1)}{2}$.

Let R_d be the set of linear dependencies between columns of $\mathcal{M}(n)$. Considering $\mathcal{Z}(P)$ the set of zeros of P, we define the algebraic variety of $\mathcal{M}(n)$ by

$$\mathcal{V} \equiv \mathcal{V}(\mathcal{M}(n)) := \bigcap_{P\in R_d} \mathcal{Z}(P).$$

The following two results will be useful to explicit the representing measure of $\beta = \beta^{(2n)}$ when it exists.

Proposition 2.3 ([6, Proposition 3.1]). *Suppose that μ is a representing measure of β. For $P \in \mathcal{P}_n$, we have*

$$\operatorname{supp}\mu \subseteq \mathcal{Z}(P) \Longleftrightarrow P(X,Y) = \mathbf{0}.$$

Using this proposition and by virtue of Corollary 3.7 in [6], we deduce

$$\operatorname{supp}\mu \subseteq \mathcal{V}(M(n)) \quad \text{and} \quad \operatorname{rank}\mathcal{M}(n) \le \operatorname{card}\operatorname{supp}\mu \le v := \operatorname{card}\mathcal{V}. \tag{6}$$

A representing measure μ for β is said to be minimal if $\operatorname{card}\operatorname{supp}\mu \le \operatorname{card}\operatorname{supp}\nu$ for any other representing measure for β and in this sense, the relation (6) shows that a rank-atomic measure is minimal.

Let us now recall a result that ensures the existence of a representing measure for a doubly indexed sequence $\beta^{(2n)}$.

Theorem 2.4 ([6, Theorem 5.13]). *The truncated moment sequence $\beta^{(2n)}$ has a* rank $\mathcal{M}$*-atomic representing measure if and only if $\mathcal{M}(n) \succeq 0$ and $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$.*

If $\mathcal{M}(n)$ admits a positive semidefinite extension $\mathcal{M}(n+1)$ such that $\mathcal{M}(n+1)$ is flat or has a flat extension $\mathcal{M}(n+2)$, then β admits a representing measure μ which is r-atomic where $r = \operatorname{rank} \mathcal{M}(n+1)$. By virtue of the flat extension Theorem 2.4, the algebraic variety $\mathcal{V}$ of $\mathcal{M}(n+1)$ consists of exactly r points.

Let us put $\mathcal{V} = \{(x_1, y_1), (x_2, y_2), \cdots, (x_r, y_r)\}$ and consider the Vandermonde matrix V given by

$$V = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{r-1} & x_r \\ y_1 & y_2 & y_3 & \dots & y_{r-1} & y_r \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_{r-1}^2 & x_r^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & \dots & x_{r-1}y_{r-1} & x_ry_r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n+1} & x_2^{n+1} & x_3^{n+1} & \dots & x_{r-1}^{n+1} & x_r^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{n+1} & y_2^{n+1} & y_3^{n+1} & \dots & y_{r-1}^{n+1} & y_r^{n+1} \end{pmatrix}.$$

Consider $\mathcal{B} = \{c_1, c_2, \cdots, c_r\}$ the basis of $\mathcal{C}_{\mathcal{M}(m)}$, the column space of $M(n+1)$, and $V_{|\mathcal{B}}$ the compression of V to the columns of $\mathcal{B}$. The weights ρ_k of the atoms $\{(x_k, y_k)\}$, $(1 \leq k \leq r)$ may be determined solving the following Vandermonde system,

$$(7) \qquad \left(V_{|\mathcal{B}}\right) \times (\rho_1 \quad \rho_2 \quad \cdots \quad \rho_r)^T = (L_\beta(c_1) \quad L_\beta(c_2) \quad \cdots \quad L_\beta(c_r))^T.$$

Hence, the representing measure of β is $\mu = \sum_{k=1}^{r} \rho_k \delta_{(x_k, y_k)}$.

We end this section with a reminder of recursively determined positive semidefinite moment matrices.

We recall that $\mathcal{M}(n)$ is recursively generated [10] if the following property is verified

$$P, Q, PQ \in \mathcal{P}_n, P(X, Y) = \mathbf{0} \Longrightarrow (PQ)(X, Y) = \mathbf{0}.$$

According to [19, Proposition 4.2], $\mathcal{M}(n)$ is recursively determined if it has the following column dependence relations,

$$(8) \qquad X^n = P(X, Y) = \sum_{i+j \leq n-1} a_{ij} X^i Y^j,$$

$$(9) \qquad Y^n = Q(X, Y) = \sum_{i+j \leq n, j \neq n} b_{ij} X^i Y^j,$$

or by similar relations with reversing the roles of P and Q.

The following lemma will be very useful in solving the cubic TRMP.

Lemma 2.5 ([14, Lemma 2.4]). *If $\mathcal{M}(2) \succeq 0$ and recursively determined (the relations* (8) *and* (9) *are verified with $n = 2$), then $\mathcal{M}(2)$ admits a flat extension $\mathcal{M}(3)$.*

Now, we are in a position to state our main results.

3. MAIN RESULTS

Let $\beta = \beta^{(3)} \equiv \{\beta_{ij}\}_{i+j\leq 3}$ be a real doubly indexed finite sequence with $\beta_{00} > 0$. As mentioned in Section 1, we can not group all the data of the sequence β in a single square matrix, so we have distributed the elements of the sequence over two matrices $\mathcal{M}(1)$ and $B(2)$ (see (2)). Thus, to solve the problem, we have to look first for a positive semidefinite extension $\mathcal{M}(2)$ in (3) of $\mathcal{M}(1)$, and then test its flatness.

This extension takes the form $\mathcal{M}(2) = \begin{pmatrix} \mathcal{M}(1) & B(2) \\ B(2)^T & C(2) \end{pmatrix}$ with $C(2)$ is a Hankel block containing the quartic moments,

$$C(2) = \begin{pmatrix} \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}. \tag{10}$$

and $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$, i.e. there exists a matrix W such that

$$\mathcal{M}(1)W = B(2)$$

according to the assertion (a) of Remark 2.2.

As $\mathcal{M}(1)$ is symmetric then $W^T\mathcal{M}(1)W$ is also.

So, we can write

$$W^T\mathcal{M}(1)W = \begin{pmatrix} x & a & b \\ a & y & t \\ b & t & z \end{pmatrix}, \tag{11}$$

where a, b, t, x, y and z are real numbers.

According to the Theorem 2.1, $\mathcal{M}(2) \succeq 0$, is equivalent to getting the following three conditions

$$\begin{cases} \text{(i) } \mathcal{M}(1) \succeq 0, \\ \text{(ii) } \mathcal{M}(1)W = B(2) \quad \text{and} \\ \text{(iii) } C(2) - W^T\mathcal{M}(1)W \succeq 0. \end{cases} \tag{12}$$

If the extension $\mathcal{M}(2)$ is flat, then there exists a representing measure; otherwise, we try to construct a flat extension $\mathcal{M}(3)$ of $\mathcal{M}(2)$.

In this context and before stating our main results we need the following two lemmas. Let $C(2)$ and $W^T\mathcal{M}(1)W$ be as defined in (10) and (11) respectively and which satisfy condition (iii) of (12).

Lemma 3.1. *We have the following equivalence,*

$$\operatorname{rank}(C(2) - W^T\mathcal{M}(1)W) = 0 \textit{ if and only if } y = b.$$

Proof. If $\operatorname{rank}(C(2) - W^T\mathcal{M}(1)W) = 0$ then $C(2) = W^T\mathcal{M}(1)W$.
Consequently, $\beta_{40} = x, \beta_{31} = a, \beta_{13} = c, \beta_{04} = z$ and $\beta_{22} = b = y$.
Conversely, if $y = b$ then $W^T\mathcal{M}(1)W$ is a Hankel matrix.

So, let us take $C(2) = W^T\mathcal{M}(1)W$. Therefore,

$$\operatorname{rank}(C(2) - W^T\mathcal{M}(1)W) = 0 \text{ and } C(2) - W^T\mathcal{M}(1)W \succeq 0.$$

□

From Lemma 3.1, we deduce the following Remark.

Remark 3.2. $\operatorname{rank}(C(2) - W^T\mathcal{M}(1)W) \geq 1$ if and only if $y \neq b$.

Now, we are in a position to state our first result.

Theorem 3.3. *Let $\beta = \beta^{(3)}$ be a real doubly indexed finite sequence, b and y are as defined in relation* (12)*.*

If $\mathcal{M}(1) > 0$, $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ and $b = y$, then β admits a unique representing measure 3*-atomic.*

Proof. If $b = y$ then by Lemma 3.1, we have $C(2) = W^T\mathcal{M}(1)W$.

Therefore, $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$. So, it is positive semidefinite and recursively determined.

By applying Lemma 2.5, $\mathcal{M}(2)$ admits a flat extension $\mathcal{M}(3)$.

Therefore, $\beta^{(4)}$, and particularly $\beta^{(3)}$, admits a unique representing measure $\operatorname{rank} \mathcal{M}(1)$-atomic. The uniqueness is deduced from the fact that $C(2)$ is unique. □

The following example illustrates the result in Theorem 3.3.

Example 3.4. Let $\beta = \beta^{(3)}$ be a doubly indexed real sequence with $\beta_{00} = 3$, $\beta_{10} = 2$, $\beta_{01} = 2$, $\beta_{20} = 2$, $\beta_{11} = -1$, $\beta_{02} = 2$, $\beta_{30} = 2$, $\beta_{21} = -1$, $\beta_{12} = 1$ and $\beta_{03} = 0$.

The two matrices associated with β are,

$$\mathcal{M}(1) = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B(2) = \begin{pmatrix} 2 & -1 & 2 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Calculations led to $\mathcal{M}(1)$ shows that $\mathcal{M}(1) > 0$.

Therefore, $\operatorname{rank} \mathcal{M}(1) = 3$ and

$$W = (\mathcal{M}(1))^{-1}B(2) = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

and

$$W^T\mathcal{M}W = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Since $b = y = 1$, $\mathcal{M}(1)$ admits a flat extension $\mathcal{M}(2)$ ($\operatorname{rank} M(2) = \operatorname{rank} \mathcal{M}(1)$).

From Theorem 3.3, we deduce that β admits a unique 3-atomic representing measure.

By choosing $C(2) = W^T \mathcal{M}(1) W$, $\mathcal{M}(2)$ is given by

$$\mathcal{M}(2) = \begin{pmatrix} 3 & 2 & 0 & 2 & -1 & 2 \\ 2 & 2 & -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 2 & 2 & -1 & 2 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ 2 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}.$$

The linear dependency relations between the columns of $\mathcal{M}(2)$ are,

$$X^2 = X =, \quad Y^2 = -2X - Y + 2 \quad \text{and} \quad XY = X + Y - 1.$$

Thus, the cone variety of $\mathcal{M}(2)$ is $\mathcal{V} = \{(0,1); (1,-1); (1,0)\}$. Solving the Vandermonde system (7), we find the weights $\rho_1 = \rho_2 = \rho_3 = 1$ associated with the three atoms.

Finally, the 3-atomic representative measure for β is,

$$\mu = \delta_{(0,1)} + \delta_{(1,-1)} + \delta_{(1.0)}.$$

Let us now state the last result concerning the case $\mathcal{M}(1) > 0$ and $y \neq b$.

THEOREM 3.5. *Let $\beta = \beta^{(3)}$ be a real doubly indexed finite sequence, b and y defined as in relation* (11)*. If $\mathcal{M}(1) > 0$, $\operatorname{Ran} B(2) \subseteq \operatorname{Ran} \mathcal{M}(1)$ and $b \neq y$, then β admits a representing measure* 4*-atomic.*

Proof. Since $b \neq y$ then for appropriate quartic moments (the entries of block $C(2)$), and according to Remark 3.2, we must have

$$\operatorname{rank}(C(2) - W^T \mathcal{M}(1) W) \;\geq\; 1.$$

If $b > y$, with the quartic moments given by

$$\beta_{40} = x,\; \beta_{31} = a,\; \beta_{22} = b,\; \beta_{13} = c \text{ and } \beta_{04} = z, \tag{13}$$

we have, $C(2) - W^T \mathcal{M}(1) W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b-y & 0 \\ 0 & 0 & 0 \end{pmatrix}$

If $b < y$, by taking

$$\beta_{40} = x+1,\; \beta_{31} = a,\; \beta_{22} = y,\; \beta_{13} = c \text{ and } \beta_{04} = (y-b)^2 - z, \tag{14}$$

we get, $C(2) - W^T \mathcal{M}(1) W = \begin{pmatrix} 1 & 0 & y-b \\ 0 & 0 & 0 \\ y-b & 0 & (y-b)^2 \end{pmatrix}$.

In both cases $C(2) - W^T \mathcal{M}(1) W$ is positive semidefinite with rank equal to 1.

So, according to Theorem 2.1, the extension matrix $\mathcal{M}(2)$ of $\mathcal{M}(1)$ is positive semidefinite. In addition, by the relation (5) we have $\operatorname{rank} \mathcal{M}(2) = 4$.

Hence, there exists a column in $\mathcal{M}(2)$ linearly independent with the columns $1, X$ and Y. This column is X^2 if $\beta_{22} = y$ or XY if $\beta_{40} = x$. In fact, we have

$$\beta_{40} = x \Rightarrow \det \begin{pmatrix} \mathcal{M}(1) & (X^2) \\ (X^2)^T & x \end{pmatrix} = 0,$$

and

$$\beta_{22} = y \Rightarrow \det \begin{pmatrix} \mathcal{M}(1) & (XY) \\ (XY)^T & y \end{pmatrix} = 0.$$

For the case $b > y$, XY is the column linearly independent with the columns $1, X$ and Y.

Hence, the columns X^2 and Y^2 are

$$X^2 = \alpha_1 XY + a_0 + a_1 X + a_2 Y \text{ and } Y^2 = \alpha_2 XY + b_0 + b_1 X + b_2 Y. \tag{15}$$

with

$$\begin{aligned}
\alpha_1 &= \frac{\det \begin{pmatrix} \mathcal{M}(1) & X^2 \\ (XY)^T & \beta_{31} \end{pmatrix}}{\det \begin{pmatrix} \mathcal{M}(1) & XY \\ (XY)^T & \beta_{22} \end{pmatrix}} \\
&= \frac{\det \begin{pmatrix} \mathcal{M}(1) & X^2 \\ (XY)^T & a \end{pmatrix} + (\beta_{31} - a) \det (\mathcal{M}(1))}{\det \begin{pmatrix} \mathcal{M}(1) & XY \\ (XY)^T & y \end{pmatrix} + (\beta_{22} - y) \det (\mathcal{M}(1))} \\
&= \frac{(\beta_{31} - a) \det (\mathcal{M}(1))}{(\beta_{22} - y) \det (\mathcal{M}(1))} \\
&= \frac{\beta_{31} - a}{\beta_{22} - y} = 0, \qquad (\beta_{31} = a \text{ and } \beta_{22} = b > y).
\end{aligned} \tag{16}$$

Similar calculations, as in (16), give

$$\alpha_2 = \frac{\beta_{13} - c}{\beta_{22} - y} = 0, \ (\beta_{13} = c \text{ and } \beta_{22} = b > y).$$

Finally, the relations (15) become,

$$X^2 = a_0 + a_1 X + a_2 Y \quad \text{ and } \quad Y^2 = b_0 + b_1 X + b_2 Y.$$

Consequently, $\mathcal{M}(2) \succeq 0$ and recursively determined.

Therefore, according to Lemma 2.5, $\mathcal{M}(2)$ admits a flat extension, and consequently $\beta^{(4)}$ and $\beta^{(3)}$ admits a representing measure 4-atomic.

For the case $b < y$, the column X^2, $1, X$ and Y are linearly independent in $\mathcal{M}(2)$.

Let us take $\beta_{22} = y$ and $\beta_{40} \neq x$ by employing relation (14).

So, the columns XY and Y^2 can be written as follows,

$$XY = \alpha_2 X^2 + c_0 + c_1 X + c_2 Y \text{ and } Y^2 = \alpha_3 X^2 + d_0 + d_1 X + d_2 Y. \tag{17}$$

By calculations, as in (16), we find

$$\alpha_2 = \frac{\beta_{31} - a}{\beta_{40} - x} = 0 \text{ and } \alpha_3 = \frac{\beta_{22} - b}{\beta_{40} - x} \neq 0, (\beta_{40} \neq x, \beta_{31} = a \text{ and } \beta_{22} = y \neq b).$$

So, the relations (17) become,

$$XY = c_0 + c_1X + c_2Y, \tag{18}$$

$$Y^2 = (y - b)X^2 + d_0 + d_1X + d_2Y. \tag{19}$$

Now, we focus on constructing the positive semidefinite extension $\mathcal{M}(3)$ of $\mathcal{M}(2)$.

As the condition of the recursivity generated must be respected, then from the relations (18) and (19) and by functional calculus, we obtain

$$X^2Y = c_0X + c_1X^2 + c_2XY, \tag{20}$$

$$XY^2 = c_0Y + c_1XY + c_2Y^2, \tag{21}$$

$$XY^2 = (y - b)X^3 + d_0X + d_1X^2 + d_2XY. \tag{22}$$

Using the relations (18)-(20), we get

$$\begin{aligned} Y^3 &= [c_0(c_2y - c_2b + d_1) + d_0d_2] \\ &\quad + [(c_0 + c_1c_2)(y - b) + c_1d_1 + d_1d_2]X \\ &\quad + [c_2(c_2y - c_2b + d_1) + d_0 + d_2^2]Y + (c_1 + d_2)(y - b)X^2. \end{aligned} \tag{23}$$

Noticing that the column XY^2 is defined by the relations (21) and (22), then by relation (9), these two relations must be similar.

Furthermore, since $y \neq b$ then

$$X^3 = -\left(\frac{d_0}{y-b}\right)X + \left(\frac{c_0}{y-b}\right)Y - \left(\frac{d_1}{y-b}\right)X^2 + \left(\frac{c_1 - d_2}{y-b}\right)XY + c_2Y^2. \tag{24}$$

Thus, using the definition of the columns X^3, X^2Y, XY^2 and Y^3, and by the relations (24), (20), (21) or (22) and (23) respectively, we complete the construction of the matrix $\mathcal{M}(3)$ as detailed in Remark 3.6.

Finally, Since these columns are written as linear combination of columns associated to monomials of degree at most 2, then $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$.

Whence, $\beta^{(3)}$ admits a finite measure 4-atomic. □

REMARK 3.6. In practice, the construction of the matrix $\mathcal{M}(3)$ can be performed as follows:

- If $\mathcal{M}(2)$ is recursively determined, to construct the matrix $\mathcal{M}(3)$, we begin by defining the columns X^3 and Y^3 by the functional calculation and the definitions of the columns X^2 and Y^2. Then, we compute the quintic moments in the columns X^3 and Y^3. This allows us to build the Hankel block $B[2, 3]$. Thus, the construction of the block $B(3)$ is completed.

By transposing the latter, one can construct the block $C(3)$ as previously. Thus the construction of $\mathcal{M}(3)$ is achieved.

- If $\mathcal{M}(2)$ is not recursively determined, then with the relations (20), (21) or (22), (23) and (24), we start calculating the quintic moments without conflict in order to complete the construction of the block $B(3)$. Then, we transpose $B(3)$ to calculate $C(3)$, which contains the sixth moments.

Now, we present two numerical examples illustrating both cases in Theorem 3.5. Namely, $b < y$ and $b > y$.

EXAMPLE 3.7. In this example, we take $b > y$.

Let $\beta^{(3)}$ be the be a real doubly indexed finite sequence defined by $\beta_{00} = 2$, $\beta_{10} = 1$, $\beta_{01} = 1$, $\beta_{20} = 2$, $\beta_{11} = 1$, $\beta_{02} = 2$, $\beta_{30} = 1$, $\beta_{21} = 2$, $\beta_{12} = 1$ and $\beta_{03} = 2$.

The two matrices associated to $\beta^{(3)}$ are,

$$\mathcal{M}(1) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } B(2) = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Calculations show that $\mathcal{M}(1) > 0$ and $\operatorname{rank} \mathcal{M}(1) = 3$.

So,

$$W = \mathcal{M}(1)^{-1} B(2) = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{-1}{4} & 1 & \frac{-1}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} \end{pmatrix} \text{ and } W^T \mathcal{M}(1) W = \begin{pmatrix} \frac{11}{4} & 1 & \frac{11}{4} \\ 1 & 2 & 1 \\ \frac{11}{4} & 1 & \frac{11}{4} \end{pmatrix}.$$

We have $b = \frac{11}{4} > y = 2$.

By the relation (13), we set $C(2) = \begin{pmatrix} \frac{11}{4} & 1 & \frac{11}{4} \\ 1 & \frac{11}{4} & 1 \\ \frac{11}{4} & 1 & \frac{11}{4} \end{pmatrix}$.

Then, the extension $\mathcal{M}(2)$ of $\mathcal{M}(1)$ is,

$$\mathcal{M}(2) = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & \frac{11}{4} & 1 & \frac{11}{4} \\ 1 & 2 & 1 & 1 & \frac{11}{4} & 1 \\ 2 & 1 & 2 & \frac{11}{4} & 1 & \frac{11}{4} \end{pmatrix}.$$

The computation of the nested determinants shows that $\mathcal{M}(2) \succeq 0$ and the dependency relations between the columns are,

$$X^2 = \frac{3}{4} - \frac{1}{4} X + \frac{3}{4} Y \quad \text{and} \quad Y^2 = \frac{3}{4} - \frac{1}{4} X + \frac{3}{4} Y.$$

Furthermore, the algebraic variety of $\mathcal{M}(2)$ is,

$$\mathcal{V}=\left\{\left(\frac{-3}{2},\frac{3}{2}\right);\left(\frac{1}{2},\frac{-1}{2}\right);\left(\frac{1-\sqrt{13}}{4},\frac{1-\sqrt{13}}{4}\right);\left(\frac{1+\sqrt{13}}{4},\frac{1+\sqrt{13}}{4}\right)\right\}.$$

With solving the Vandermonde system (8), we get the following

$$\rho_1=\frac{1}{6},\rho_2=\frac{1}{2},\rho_3=\frac{2}{39}\left(13-2\sqrt{13}\right)\text{ and }\rho_4=\frac{2}{39}\left(2\sqrt{13}+13\right),$$

related respectively to the following atoms

$$\left(\frac{-3}{2},\frac{3}{2}\right),\left(\frac{1}{2},\frac{-1}{2}\right),\left(\frac{1-\sqrt{13}}{4},\frac{1-\sqrt{13}}{4}\right)\text{ and }\left(\frac{1+\sqrt{13}}{4},\frac{1+\sqrt{13}}{4}\right).$$

Finally, the 4-atomic measure of $\beta^{(3)}$ is,

$$\begin{aligned}\mu=&\frac{1}{6}\delta_{\left(\frac{-3}{2},\frac{3}{2}\right)}+\frac{1}{2}\delta_{\left(\frac{1}{2},\frac{-1}{2}\right)}+\frac{26-4\sqrt{13}}{39}\delta_{\left(\frac{1-\sqrt{13}}{4},\frac{1-\sqrt{13}}{4}\right)}\\&+\frac{26+4\sqrt{13}}{39}\delta_{\left(\frac{1+\sqrt{13}}{4},\frac{1+\sqrt{13}}{4}\right)}.\end{aligned}$$

Using the steps described in Remark 3.6, we construct $\mathcal{M}(3)$ and we obtain

$$\mathcal{M}(3)=\begin{pmatrix}2&1&1&2&1&2&1&2&1&2\\1&2&1&1&2&1&\frac{11}{4}&1&\frac{11}{4}&1\\1&1&2&2&1&2&1&\frac{11}{4}&1&\frac{11}{4}\\2&1&2&\frac{11}{4}&1&\frac{11}{4}&\frac{13}{16}&\frac{53}{16}&\frac{13}{16}&\frac{53}{16}\\1&2&1&1&\frac{11}{4}&1&\frac{53}{16}&\frac{13}{16}&\frac{53}{16}&\frac{13}{16}\\2&1&2&\frac{11}{4}&1&\frac{11}{4}&\frac{13}{16}&\frac{53}{16}&\frac{13}{16}&\frac{53}{16}\\1&\frac{11}{4}&1&\frac{13}{16}&\frac{53}{16}&\frac{13}{16}&\frac{139}{32}&\frac{17}{32}&\frac{139}{32}&\frac{17}{32}\\2&1&\frac{11}{4}&\frac{53}{16}&\frac{13}{16}&\frac{53}{16}&\frac{17}{32}&\frac{139}{32}&\frac{17}{32}&\frac{139}{32}\\1&\frac{11}{4}&1&\frac{13}{16}&\frac{53}{16}&\frac{13}{16}&\frac{139}{32}&\frac{17}{32}&\frac{139}{32}&\frac{17}{32}\\2&1&\frac{11}{4}&\frac{53}{16}&\frac{13}{16}&\frac{53}{16}&\frac{17}{32}&\frac{139}{32}&\frac{17}{32}&\frac{139}{32}\end{pmatrix}.$$

Computations show that $\operatorname{rank}\mathcal{M}(3)=\operatorname{rank}\mathcal{M}(2)=4$. Consequently, $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$.

EXAMPLE 3.8. In this example, we choose $b<y$.

Let $\beta^{(3)}$ be the a real doubly indexed finite sequence defined by $\beta_{00}=9$, $\beta_{10}=2$, $\beta_{01}=1$, $\beta_{20}=2$, $\beta_{11}=-2$, $\beta_{02}=6$, $\beta_{30}=3$, $\beta_{21}=-2$, $\beta_{12}=2$ and $\beta_{03}=-3$. The two matrices associated to $\beta^{(3)}$ are,

$$\mathcal{M}(1)=\begin{pmatrix}9&2&1\\2&2&-2\\1&-2&6\end{pmatrix}\text{ and }B(2)=\begin{pmatrix}2&-2&6\\3&-2&2\\-2&2&-3\end{pmatrix}.$$

Calculations show that $\mathcal{M}(1)>0$ and $\operatorname{rank}\mathcal{M}(1)=3$.

So,

$$W = \mathcal{M}(1)^{-1}B(2) = \begin{pmatrix} -\frac{7}{19} & 0 & 1 \\ \frac{91}{38} & -1 & -1 \\ \frac{10}{19} & 0 & -1 \end{pmatrix}$$

and

$$W^T\mathcal{M}(1)W = \begin{pmatrix} \frac{205}{38} & -3 & 1 \\ -3 & 2 & -2 \\ 1 & -2 & 7 \end{pmatrix}.$$

We have, $b = 1 < y = 2$.

So, according to the relation (14), we set $C(2) = \begin{pmatrix} \frac{243}{38} & -3 & 2 \\ -3 & 2 & -2 \\ 2 & -2 & 8 \end{pmatrix}$.

Then, the extension $\mathcal{M}(2)$ of $\mathcal{M}(1)$ is,

$$\mathcal{M}(2) = \begin{pmatrix} 9 & 2 & 1 & 2 & -2 & 6 \\ 2 & 2 & -2 & 3 & -2 & 2 \\ 1 & -2 & 6 & -2 & 2 & -3 \\ 2 & 3 & -2 & \frac{243}{38} & -3 & 2 \\ -2 & -2 & 2 & -3 & 2 & -2 \\ 6 & 2 & -3 & 2 & -2 & 8 \end{pmatrix}.$$

One can easily check that $\mathcal{M}(2) \succeq 0$ and the dependency relations between the columns are,

$$XY = -X \quad \text{and} \quad Y^2 = \frac{26}{19} - \frac{129}{38}X - \frac{29}{19}Y + X^2. \tag{25}$$

The cone variety of $\mathcal{M}(2)$ is $\mathcal{V} = \{(x_i, y_i)\}_{i=1}^{i=4}$, where

$$(x_1, y_1) = \left(0; \frac{-29 - 3\sqrt{313}}{38}\right), \quad (x_2, y_2) = \left(0; \frac{-29 + 3\sqrt{313}}{38}\right),$$

$$(x_3, y_3) = \left(\frac{129 - 3\sqrt{633}}{76}; -1\right) \quad \text{and} \quad (x_4, y_4) = \left(\frac{129 + 3\sqrt{633}}{76}; -1\right).$$

Solving the Vandermonde system (8), we obtain the weights

$$\rho_1 = \frac{72929 - 3861\sqrt{313}}{22536}, \quad \rho_2 = \frac{72929 + 3861\sqrt{313}}{22536},$$

$$\rho_3 = \frac{57603 + 2089\sqrt{633}}{45576}, \quad \rho_4 = \frac{57603 - 2089\sqrt{633}}{45576}.$$

associated to the atoms $(x_i, y_i)_{1\le i\le 4}$ respectively.

Finally, the 4-atomic measure of $\beta^{(3)}$ is $\mu = \sum_{i=1}^{4} \rho_i \delta_{(x_i, y_i)}$.

The functional calculation on the dependency relations between the columns (25), define the columns X^3, X^2Y, XY^2 and Y^3 as linear dependency functions of the leftmost columns respectively,

$$\begin{aligned}
&X^3 = -\frac{36}{19}X + \frac{129}{38}X^2,\\
&X^2Y = -X^2,\\
&XY^2 = -XY,\\
&Y^3 = -\frac{754}{361} + \frac{3096}{361}X + \frac{1335}{361}Y - \frac{48}{19}X^2.
\end{aligned}$$

With these definitions, we construct the extension $\mathcal{M}(3)$ of $\mathcal{M}(2)$ as mentioned in Remark 3.6. We get,

$$\mathcal{M}(3) = \begin{pmatrix}
9 & 2 & 1 & 2 & -2 & 6 & 3 & -2 & 2 & -3\\
2 & 2 & -2 & 3 & -2 & 2 & \frac{243}{38} & -3 & 2 & -2\\
1 & -2 & 6 & -2 & 2 & -3 & -3 & 2 & -2 & 8\\
2 & 3 & -2 & \frac{243}{38} & -3 & 2 & \frac{23139}{1444} & -\frac{243}{38} & 3 & -2\\
-2 & -2 & 2 & -3 & 2 & -2 & -\frac{243}{38} & 3 & -2 & 2\\
6 & 2 & -3 & 2 & -2 & 8 & 3 & -2 & 2 & -\frac{219}{19}\\
3 & \frac{243}{38} & -3 & \frac{23139}{1444} & -\frac{243}{38} & 3 & \frac{2320083}{54872} & -\frac{23139}{1444} & \frac{243}{38} & -3\\
-2 & -3 & 2 & -\frac{243}{38} & 3 & -2 & -\frac{23139}{1444} & \frac{243}{38} & -3 & 2\\
2 & 2 & -2 & 3 & -2 & 2 & \frac{243}{38} & -3 & 2 & -2\\
-3 & -2 & 8 & -2 & 2 & -\frac{219}{19} & -3 & 2 & -2 & \frac{8574}{361}
\end{pmatrix}.$$

Calculations show that $\operatorname{rank}\mathcal{M}(3) = \operatorname{rank}\mathcal{M}(2) = 4$, i.e. $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$.

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