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ON SOME TRANSCENDENTAL CONTINUED FRACTIONS OVER A FINITE FIELD

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Abstract. The aim of the present paper is to present some families of transcendental continued fractions in positive characteristic with bounded and unbounded degrees through some specific irregularities of their partial quotients.

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1. INTRODUCTION

The question of how rationals can approximate algebraic numbers is a fundamental problem in Diophantine approximation theory. The behavior of the continued fraction expansion of these algebraic numbers is closely related to this subject. In particular, it is widely believed that the continued fraction expansion of any irrational algebraic number either is eventually periodic or contains arbitrarily large partial quotients. Khinchin [9] appears to have addressed this question initially, and a proof appears to be still far off. Examples of transcendental continued fractions are given as a first step towards solving this question. Liouville [12] created transcendental real numbers with a rapidly increasing sequence of partial quotients, which is where the first result of this kind originated. Other classes of transcendental continued fractions were subsequently constructed by different authors using deeper transcendence criteria from the Diophantine approximation. The work of Maillet [14], which has been continued by Baker [6], is particularly remarkable because he was the first to provide explicit examples of transcendental continued fractions with bounded partial quotients. More precisely, Baker proved that if $\alpha = [B_0, B_1, B_2, \ldots]$ where B_n is a block of k_n consecutive partial quotients such that

$$B_n = B_{n+1} = \ldots = B_{n+\lambda(n)-1},$$

for infinitely many positive integers n where $\lambda(n)$ is a sequence of integers verifying certain increasing properties, then α is transcendental. The proof of this result is based on combining the result of Liouville [12] and Roth's

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theorem. After this, several extensions and generalizations of Roth's theorem were then obtained. Schmidt has, in particular, proved in [19] a remarkable result, established in terms of simultaneous approximation of linear forms in logarithms with algebraic coefficients and known as the subspace theorem. It's a very profound result which has given rise to many interesting applications. Several new criteria of transcendence for continued fractions was given by Adamczewski and Bugeaud [1]. In 2004, Mkaouar [16] gave a similar result to Baker [6], concerning the transcendence of formal power series over a finite field. In [8], Hbaib et al. proved the following result which allows the construction of a family of transcendent continued fractions over $\mathbb{F}_q((T^{-1}))$ from an algebraic formal power series of degree more than 2.

THEOREM 1.1. *Let β be an algebraic formal power series such that* $\deg(\beta) > 0$ *and $\alpha = [B_1, B_2, \ldots]$ where B_i are finite blocks of partial quotients whose the first n_i-terms are those of the continued fraction expansion of β. Let d_i denote the sum of degrees of B_i and δ_i the sum of degrees of the first n_i-terms of B_i. If*

$$\liminf_{s \longrightarrow \infty} \frac{\sum_{j=1}^{s-1} d_j}{\delta_s} = 0,$$

then α is transcendental or quadratic.

Many explicit continued fractions are known for algebraic and non quadratic elements, with bounded and unbounded partial quotients, see for example [3, 4, 7, 20].

Recently, Ammous et al. [2] improved Theorem 1.1 by giving a new transcendental criterion depending on the length of the specific blocks appearing in the sequence of partial quotients. In the same way, the purpose behind this work is also to improve Theorem 1.1, by exposing new families of transcendental continued fraction over $\mathbb{F}_q(T)$, in an interesting way starting from the continued fraction expansion of some algebraic power series of degree greater than 2. However, we provide a slight refinement of their criteria.

This article is organized as follows: In Section 2, we define the field of formal series and the continued fraction expansions over this field. In Section 3, we recall some technical lemmas that allow us to prove our results. Section 4 is devoted to exposing our main results. Theorem 4.1 and Theorem 4.3 provide some transcendental continued fractions with unbounded degrees, whereas Theorem 4.4 illustrates a transcendental continued fraction with bounded degree. Section 5 contains the proofs of our main results.

2. FIELD OF FORMAL SERIES

Let p be a prime and q be a power of p. We denote by $\mathbb{F}_q$ the finite field of q elements, $\mathbb{F}_q[T]$ the ring of polynomials over $\mathbb{F}_q$, $\mathbb{F}_q(T)$ the field of rational function and $\mathbb{F}_q((T^{-1}))$ the field formal power series over $\mathbb{F}_q$.

For $\alpha \in \mathbb{F}_q((T^{-1}))$ and $\alpha \neq 0$ we have

$$\alpha = \sum_{n \geqslant n_0} a_n T^{-n},$$

where $n_0 \in \mathbb{Z}$, $a_n \in \mathbb{F}_q$ and $a_{n_0} \neq 0$. We define the degree of α, by $\deg \alpha = n_0$ and $\deg 0 = -\infty$. Then we define the absolute value on $\mathbb{F}_q((T^{-1}))$ by

$$|\alpha| = \begin{cases} q^{\deg \alpha} & \text{for } \alpha \neq 0, \\ 0 & \text{for } \alpha = 0. \end{cases}$$

Contrary to the usual absolute value on $\mathbb{Q}$, this non-Archimedean absolute value verifies

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$$

for every $\alpha, \beta \in \mathbb{F}_q((T^{-1}))$ and, in particular, $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$ as soon as $|\alpha| \neq |\beta|$.

Define the polynomial part of α, denoted by $[\alpha]$, as follows:

$$[\alpha] = \begin{cases} a_0 + a_{-1}T + \cdots + a_{n_0}T^{-n_0} & \text{if } n_0 \leq 0, \\ 0 & \text{else.} \end{cases}$$

The fractional part of α, denoted by $\{\alpha\}$, is defined as follows: $\{\alpha\} = \alpha - [\alpha]$. Then $\big|[\alpha]\big| \geq 1$ and $\big|\{\alpha\}\big| < 1$ for $\alpha \neq 0$.

As in classical continued fraction theory of real numbers, if $\alpha \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = [a_1, a_2, a_3, \ldots],$$

where $a_1 = [\alpha]$, $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) \geqslant 1$ for any $i \geqslant 1$. The sequence $(a_i)_{i \geqslant 1}$ is called the partial quotients of α and we denote by $\alpha_n = [a_n, a_{n+1}, \ldots]$ the n^{th} complete quotient of α. We define two sequences of polynomials P_n and Q_n by

$$P_1 = a_1, \qquad Q_1 = 1, \qquad P_2 = a_1 a_2 + 1, \qquad Q_2 = a_2$$

and

$$P_n = a_n P_{n-1} + P_{n-2}, \qquad Q_n = a_n Q_{n-1} + Q_{n-2}, \qquad \text{for any } n \geqslant 2.$$

We can write $\frac{P_n}{Q_n} = [a_1, a_2, \ldots, a_n]$ for $n \geqslant 1$ and we refer to $\left(\frac{P_n}{Q_n}\right)_{n \in \mathbb{N}}$ as the convergent sequence of α. It is easy to see that $\deg Q_{n+1} = \deg a_{n+1} + \deg Q_n$, thus $\deg Q_n = \sum_{j=2}^{n} \deg a_j$. Moreover we have for $n \geq 1$ the equality:

$$\alpha = [a_1, a_2, \ldots, a_n, \alpha_{n+1}] = \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}} \tag{1}$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$ is called a complete quotient of α.

We say that a formal power series has bounded partial quotients if the polynomial $(a_n)_{n\geq 1}$ are bounded in degrees.

We recall that α belonging to $\mathbb{F}_q((T^{-1}))$ is algebraic of degree d if there exists a polynomial $\Lambda \in \mathbb{F}_q[T][Y]$ irreducible of degree d such that $\Lambda(\alpha) = 0$. We say that α is quadratic if it is algebraic of degree $d = 2$. We recall that an element in $\mathbb{F}_q((T^{-1}))$ is quadratic if and only if its sequence of partial quotients is ultimately periodic, see [15].

In the classical theory of Diophantine approximation, we are concerned with how well an irrational number can be approximated by rational numbers. In order to measure the quality of approximation (the irrationality measure) of an irrational power series α, we define

$$\nu(\alpha) = -\limsup_{|Q|\to\infty} \log(|\alpha - P/Q|)/\log(|Q|).$$

We call $\nu(\alpha)$ the approximation exponent of α. By Roth's theorem, the irrationality measure of an irrational algebraic real number is 2. For power series over a finite field there is no analogue of Roth's theorem. However, according to Mahler's theorem [13], we have $\nu(\alpha) \in [2, n]$ if α is algebraic of degree $n > 1$ over $\mathbb{F}_q(T)$.

For a general account of continued fractions in power series fields and diophantine approximation, and also for more references on this matter, the reader can consult [10, 20, 21].

Now, let α be an algebraic formal power series of minimal polynomial $P(Y) = A_mY^m + A_{m-1}Y^{m-1} + \ldots + A_0$, where A_i are pairwise coprime over $\mathbb{F}_q[T]$. $H(\alpha)$ is defined as the maximal of absolute values of the coefficients of $P(Y)$.

Throughout the paper we are dealing with finite sequences (or words), consequently we recall the following notation on sequences in $\mathbb{F}_q[T]$. Let $B = a_1, a_2, \ldots, a_n$ be such a finite sequence, then we set $|B| = n$ for the length of the word B and $\varphi(B_n) = \max_{1\leqslant i\leqslant n}(\deg a_i)$. If we have two words B_1 and B_2, then B_1, B_2 denotes the word obtained by concatenation.

3. PRELIMINARY LEMMAS

We have to introduce some lemmas in order to prove the main results.

Lemma 3.1 ([17]). *Let α and β be two distinct numbers of degree, respectively, m and n over $\mathbb{F}_q(T)$. Then we have*

$$|\alpha - \beta| \geq H(\alpha)^{-n}H(\beta)^{-m}.$$

Lemma 3.2 ([18]). *Let $\alpha = [a_1, a_2, ...]$ and $\beta = [b_1, b_2, ...]$ be two formal series having the same first $n + 1$ partial quotients. Then*

$$|\alpha - \beta| \leqslant \frac{1}{|Q_n|^2}.$$

LEMMA 3.3 ([8]). *Let α be an algebraic formal power series of degree d such that $\alpha = [a_1, a_2, \ldots, a_t, \beta]$ where $a_1, \ldots, a_t \in \mathbb{F}_q[T]$, $\beta \in \mathbb{F}_q((T^{-1}))$. If $|\alpha| \geqslant 1$ and $|\beta| > 1$, then β is algebraic of degree d and*

$$H(\beta) \leqslant H(\alpha)\left|\prod_{i=1}^{t} a_i\right|^{d-2}.$$

LEMMA 3.4 ([3]). *Let $\beta \in \mathbb{F}_q((T^{-1}))$ be the irrational solution of strictly positive degree of the equation*

$$\beta^r - A\beta^{r-1} + 1 = 0,$$

where $r > 2$ is a power of p. Then

$$\beta = [b_1, \cdots, b_n, \cdots]$$

where $b_1 = A$ and for all $n \geq 1$:

$$b_{2n+1} = (-1)^n A,$$
$$b_{2n} = \begin{cases} -A^{-1}b_n^r & \text{if } n \text{ is odd;} \\ A^{-1}b_n^r & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, β is of degree r.

Define a finite sequence K_n whose terms consists of 4-tuples of elements in $\mathbb{F}_q[T]$ by

$$K_1 = b_1, b_2, b_3, b_4 = A, -A^{r-1}, -A, -A^{r^2-r-1},$$

and for $n \geq 1$

$$K_n = b_{4n+1}, b_{4n+2}, b_{4n+3}, b_{4n+4}.$$

Let K_∞ be the infinite sequence defined by

$$K_\infty = K_1, K_2, \ldots, K_n, \ldots.$$

Then we have

$$\beta = [K_\infty].$$

Set $u_m = \deg b_m$, $\lambda = \deg A$. So we have for $k \geq 1$

$$u_{2k+1} = \lambda$$

and

$$u_{2k} = \begin{cases} (r-1)\lambda & \text{if } k \text{ is odd;} \\ ru_k - \lambda & \text{if } k \text{ is even.} \end{cases}$$

So we have for even k, $u_{2k} = \lambda(r^l - r^{l-1} - \ldots - r - 1)$ when $2^{l-1} \| k$.

For a given n, $n = 2^t$ with $t > 1$. We have that

$$u_n = \lambda(r^t - r^{t-1} - \ldots - r - 1) = \lambda\left(\frac{(r-2)r^t + 1}{r-1}\right).$$

Indeed, as announced we have $\nu(\beta) = r$, then β is algebraic of degree r over $\mathbb{F}_q(T)$.

LEMMA 3.5. *Let $t > 0$ be an integer. Let $\beta \in \mathbb{F}_q((T^{-1}))$ satisfying*

$$\beta = [b_1, b_2, \ldots, b_t, b_1^r, b_2^r, \ldots, b_t^r, b_1^{r^2}, b_2^{r^2}, \ldots, b_t^{r^2}, \ldots]$$

such that $\gcd(\deg b_t, r) = 1$. *Then β is algebraic of degree $r+1$.*

Proof. An easy calculation ensures that β verifies the following

$$\beta = [b_1, \ldots, b_t, \beta^r].$$

Then, from (1), β satisfies

$$Q_t\beta^{r+1} - P_t\beta^r + Q_{t-1}\beta - P_{t-1} = 0,$$

where $\frac{P_t}{Q_t}$ is t^{th} convergent sequence of β. It remains to prove that β is algebraic of degree $r+1$. We denote

$$F(x) = x^{r+1} - \frac{P_t}{Q_t}x^r + \frac{Q_{t-1}}{Q_t}x - \frac{P_{t-1}}{Q_t}.$$

Then we have $F(\beta) = 0$. If there is a double root γ of $F(x)$, then $F(\gamma) = F'(\gamma) = 0$. Since $F'(x) = x^r + \frac{Q_{t-1}}{Q_t}$, we have the relation

$$-\gamma^r = \frac{Q_{t-1}}{Q_t}.$$

Applying to $F(\gamma) = 0$ we have:

$$\begin{aligned} 0 &= \gamma^r(\gamma - \frac{P_t}{Q_t}) + \frac{Q_{t-1}}{Q_t}\gamma - \frac{P_{t-1}}{Q_t} \\ &= -\frac{Q_{t-1}}{Q_t}(\gamma - \frac{P_t}{Q_t}) + \frac{Q_{t-1}}{Q_t}\gamma - \frac{P_{t-1}}{Q_t} \\ &= -\frac{Q_{t-1}}{Q_t}\frac{P_t}{Q_t} - \frac{P_{t-1}}{Q_t}. \end{aligned}$$

This implies that $P_tQ_{t-1} - P_{t-1}Q_t = (-1)^{t-1} = 0$, a contradiction. Hence $F(x)$ is separable over $\mathbb{F}_q((T^{-1}))$.

Now we consider the Newton polygon of $F(x)$, which is denoted by $\mathrm{N}(F)$. There are four points $(0, \deg b_t - \deg b_1)$, $(1, \deg b_t)$, $(r, -\deg b_1)$ and $(r+1, 0)$ on the xy-plane, and hence $\mathrm{N}(F)$ consists of two line segments. The first one has slope $\deg b_1$, which is a shift of the Newton polygon of $x - \beta$ and the second one has slope $-\frac{\deg b_t}{r}$. Suppose that $F(x) = (x-\beta)G(x)$ in $\mathbb{F}_q[T]$, then the second line segment is a shift of the Newton polygon of $G(x)$. Because $\gcd(\deg b_t, r) = 1$, by the theory of Newton polygons we see that $G(x)$ is irreducible over $\mathbb{F}_q$. Moreover $F(x)$ is irreducible over $\mathbb{F}_q$ since $\beta \notin \mathbb{F}_q$. So it follows that β is algebraic of degree $r+1$. □

LEMMA 3.6 ([11]). *Let $s, t \geq 1$ be integers and we put $q := p^s$ and $r := q^t$, where p is an odd prime number. Let λ be in $\mathbb{F}_q^*$ such that $\lambda \neq 2$ and put $\mu := 2 - \lambda$.*

Define a finite sequence H_n of elements of $\mathbb{F}_q[T]$, for $n \geq 1$ by

$$H_n = T, (\lambda T, \mu T)^{[(r^n-1)/2]}.$$

Let H_∞ be the infinite sequence defined by

$$H_\infty = H_1, H_2, \ldots, H_n, \ldots.$$

Let $\gamma \in \mathbb{F}_q((T^{-1}))$ such that

$$\gamma = [H_\infty].$$

Then γ satisfies the algebraic equation

$$(\lambda\mu)^{(r-1)/2} P_{r+1} \gamma^{r+1} - (\lambda\mu)^{(r-1)/2} Q_{r+1} \gamma^r + P_1 \gamma - Q_1 = 0, \tag{2}$$

where $(P_n/Q_n)_{n \geq 0}$ is the convergent sequence of γ.

It has been proved that $\deg \gamma = r + 1$ (see [5, Corollary 3.2]).

4. MAIN RESULTS

THEOREM 4.1. *Let $\beta = [b_1, \cdots, b_n, \cdots]$ be as defined above. Let $(t_i)_{i \geq 1}$ an increasing sequence of integers. Let $n_i = 2^{t_i - 1}$, for all $i \geq 1$. Let $\alpha \in \mathbb{F}_q((T^{-1}))$ such that $\alpha = [U_{n_1}, \ldots, U_{n_2}, \ldots] = [a_1, a_2, \ldots]$, where $(U_{n_i})_{i \geqslant 1}$ is a sequence of finite blocks of polynomials such that $U_{n_i} = b_1 b_2 \ldots b_{2n_i}$. If $\lim_{i \to +\infty} r^{t_i - t_{i-1}} = +\infty$, then α is transcendental.*

EXAMPLE 4.2. For $r = 3$, $A = T$ and $t_i = 2^i$ for all $i \geq 1$. Then the continued fraction

$$[\underbrace{T, 2T^2, 2T, 2T^5}_{U_{n_1}}, \underbrace{T, 2T^2, 2T, 2T^5, T, T^2, 2T, T^{14}, T, T^2, 2T, T^5, T, T^2, 2T, T^{41}}_{U_{n_2}},$$
$$T, 2T^2, 2T, 2T^5, T, \ldots]$$

is transcendental since $\lim_{i \to +\infty} 3^{2^i - 2^{i-1}} = +\infty$.

THEOREM 4.3. *Let $\alpha \in \mathbb{F}_q((T^{-1}))$ such that $\alpha = [U_1, V_1, \ldots, U_n, V_n, \ldots] = [a_1, a_2, \ldots]$, where $(U_n)_{n \geqslant 1}$ and $(V_n)_{n \geqslant 1}$ are two sequences of finite blocks of polynomials such that*

(i) *$U_i = u_{1,i}, u_{2,i}, \ldots, u_{t,i}, u_{1,i}^r, u_{2,i}^r, \ldots, u_{t,i}^r, \ldots, u_{1,i}^{r^{\eta_n - 1}}, \ldots, u_{t,i}^{r^{\eta_n - 1}}$, for any $i \geqslant 1$, with $u_{j,i} \in \mathbb{F}_q[T]$ of degree $\geqslant 1$ for all $1 \leq j \leq t$ and*

$$\gcd(\deg u_{t,i}, r) = 1.$$

(ii) *$(\eta_i)_{i \geqslant 0}$ is an increasing sequence of positive integers.*

(iii) *$(\deg u_{j,i})_{i \geqslant 0}$ is bounded for all $1 \leq j \leq n$.*

(iv) *$\varphi(V_n) \leqslant \varphi(U_n)$, for all $n \geqslant 1$.*

(v) *The sequence $(|V_n|/|U_n|)_{n \geqslant 1}$ is bounded.*

If α satisfies

$$\limsup_{n \to \infty} \frac{r^{\eta_n - \eta_{n-1}}}{(n-1)\eta_{n-1}} = +\infty,$$

then α is transcendental.

THEOREM 4.4. *Let $p > 2$, $q = p^s$ and $r = q^t$ where $s > 1$ and $t \geq 1$ are integers. Let $\alpha \in \mathbb{F}_q((T^{-1}))$ such that $\alpha = [A_1 B_1 ... A_n B_n ..] = [a_1, \ldots, a_n, \ldots]$, where $(A_n)_{n \geqslant 0}$ and $(B_n)_{n \geqslant 0}$ are two sequences of finite blocks of polynomials such that*

(i) $A_n = H_1 H_2 \ldots H_{n^2}$
(ii) $\varphi(B_n) = 1$
(iii) $|B_n| < |A_n|$ *for all* $n \geq 1$.

Then α is transcendental.

EXAMPLE 4.5. Let λ be in $\mathbb{F}_9^* \backslash \{1, 2\}$ and $\mu = 2 - \lambda$. Let $\alpha \in \mathbb{F}_9((T^{-1}))$ such that

$$\begin{aligned} \alpha &= [T, (\lambda T, \mu T)^{[(9-1)/2]}, T, T, (\lambda T, \mu T)^{[(9^2-1)/2]}, \\ &\quad T, T, T, (\lambda T, \mu T)^{[(9^3-1)/2]}, T, T, T, \ldots] \end{aligned}$$

Then α is transcendental.

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 4.1. Clearly α has unbounded partial quotients so it is not quadratic. We suppose that α is algebraic of degree $d > 2$. Let

$$\eta_{n_i} = \sum_{k=1}^{i-1} |U_{n_k}|.$$

Let $\alpha_{\eta_{n_i}} = [U_{n_i}, U_{n_{i+1}}, \ldots]$. Then by Lemma 3.1 we have

$$|\alpha_{\eta_{n_i}} - \beta| \quad \geqslant \quad H(\alpha_{\eta_{n_i}})^{-r} H(\beta)^{-d},$$

and Lemma 3.3 gives that

$$|\alpha_{\eta_{n_i}} - \beta| \quad \geqslant \quad H(\alpha)^{-r} \left| \prod_{k=1}^{\eta_{n_i}} a_k \right|^{(2-d)r} |A|^{-d}. \tag{3}$$

Furthermore, $\alpha_{\eta_{n_i}}$ and β have the same first partial quotients, hence by Lemma 3.2 we get

$$|\alpha_{\eta_{n_i}} - \beta| \leqslant |b_1 b_2 \ldots b_{2n_i}|^{-2}. \tag{4}$$

From the inequalities (3) and (4) we obtain

$$|b_1 b_2 \ldots b_{2n_i}|^2 \leqslant H(\alpha)^r \prod_{k=1}^{\eta_{n_i}} |a_k|^{(d-2)r} |A|^{-d}$$

whence

$$2\lambda \left(\frac{(r-2)r^{t_i} + 1}{r-1} \right) \leqslant r \deg(H(\alpha)) + r(d-2) \sum_{k=1}^{\eta_{n_i}} \deg a_k + d\lambda.$$

Hence

$$\frac{2\lambda(\frac{(r-2)r^{t_i}+1}{r-1})}{\sum_{k=1}^{\eta_{n_i}}\deg a_k} \leqslant \frac{r\deg(H(\alpha))+d\lambda}{\sum_{k=1}^{\eta_{n_i}}\deg a_k} + r(d-2). \tag{5}$$

As

$$\begin{aligned}\sum_{k=1}^{\eta_{n_i}}\deg a_k &= \sum_{j=1}^{i-1}|b_1b_2\dots b_{2n_j}| \\ &= \sum_{j=t_1}^{t_{i-1}}\lambda\frac{(r-2)r^j+1}{r-1}) \\ &= \lambda\frac{1}{(r-1)^2}((r-2)(r^{t_{i-1}+1}-r^{t_1})+(r-1)(t_{i-1}-t_1)),\end{aligned}$$

we have

$$\begin{aligned}\lim_{i\longrightarrow+\infty}\frac{2\lambda(\frac{(r-2)r^{t_i}+1}{r-1})}{\lambda\frac{1}{(r-1)^2}((r-2)(r^{t_{i-1}+1}-r^{t_1})+(r-1)(t_{i-1}-t_1))} &= \lim_{i\longrightarrow+\infty}r^{t_i-t_{i-1}} \\ &= +\infty.\end{aligned}$$

But the inequality (5) gives that the above limit is finite. This is a contradiction. □

Proof of Theorem 4.3. It is clear that α is not quadratic. Assume that α is algebraic of degree $d>2$. Set $\lambda_n=|U_n|$, $\gamma_n=|V_n|$ for all $n\geq 1$ and $k_n=\sum_{i=1}^{n-1}(\lambda_i+\gamma_i)$, for all $n\geq 1$. We denote

$$\beta_n=[u_{1,n},\dots,u_{t,n},u_{1,t}^r,u_{1,n}^r,\dots,u_{t,n}^r,u_{1,t}^{r^2},\dots,u_{t,n}^{r^2},\dots].$$

Then, from the previous lemma, β_n is algebraic of degree $r+1$. Now, Let $\alpha_{k_n}=[U_n,V_n,U_{n+1},V_{n+1}...]$ denote the k_n^{th} complete quotient of α. From Lemma 3.3, α_{k_n} is algebraic of degree $d>2$ and

$$H(\alpha_{k_n})\leqslant H(\alpha)\left|\prod_{i=1}^{k_n-1}a_i\right|^{d-2}$$

where $(a_i)_{i\geqslant 1}$ is the sequence of partial quotients of α.

On the other hand, according to Lemma 3.1 we obtain

$$\begin{aligned}|\alpha_{k_n}-\beta_n| &\geqslant H(\alpha_{k_n})^{-r-1}|\beta_n|^d \\ &\geqslant H(\alpha)^{-r-1}\left|\prod_{i=1}^{k_n}a_i\right|^{(2-d)(r+1)}|\beta_n|^d\end{aligned} \tag{6}$$

Furthermore, α_{k_n} and β_n have the same first λ_n partial quotients, hence by Lemma 3.2 we get

$$|\alpha_{k_n}-\beta_n|\leqslant|u_{1,n}\dots u_{t,n}u_{1,n}^r\dots u_{t,n}^r\dots u_{1,t}^{r^{\eta_n-1}}\dots u_{t,n}^{r^{\eta_n-1}}|^{-2}. \tag{7}$$

From the inequalities (6) and (7) we obtain

$$|u_{1,n}\dots u_{t,n}u_{1,n}^{r}\dots u_{t,n}^{r}\dots u_{1,t}^{r^{\eta_n-1}}\dots u_{t,n}^{r^{\eta_n-1}}|^2 \leqslant H(\alpha)^{r+1}\left|\prod_{i=1}^{k_n} a_i\right|^{(d-2)(r+1)} |\beta_n|^d$$

whence

$$\begin{aligned}&2\sum_{i=1}^{t}\deg u_{i,n}\left(\frac{r^{\eta_n}-1}{r-1}\right)\\ &\leqslant (r+1)\log_q(H(\alpha)) + (d-2)(r+1)\sum_{i=1}^{k_n}\deg a_i + d\sum_{i=1}^{t}\deg u_{i,n}.\end{aligned}$$

This gives that

$$\limsup_{n\to\infty}\frac{r^{\eta_n}}{\sum_{i=1}^{k_n}\deg a_i} \leqslant (d-2)(r^2-1).$$

Set $H = \sup_{1\leq j\leq t, i\geqslant 1}\deg(u_{j,i})$. As $\varphi(V_i) \leqslant \varphi(U_i)$ for all $i \geqslant 1$, we get $\deg(a_i) \leqslant r^{\eta_{n-1}-1}H$, for all $1 \leqslant i \leqslant k_n$. Therefore

$$\limsup_{n\to\infty}\frac{r^{\eta_n}}{r^{\eta_{n-1}-1}Hk_n} \leqslant (r^2-1)(d-2).$$

By hypothesis (v), there exists $c > 0$ such that $\gamma_i < c\lambda_i$ for all $i \geqslant 1$. Thus, $k_n < (c+1)(n-1)\lambda_{n-1} = (c+1)(n-1)t\eta_{n-1}$. Hence, we conclude that

$$\limsup_{n\to\infty}\frac{r^{\eta_n-\eta_{n-1}}}{(n-1)\eta_{n-1}} < \infty,$$

which contradicts the hypothesis. □

Proof of Theorem 4.4. It is clear that the length of the blocks A_n is increasing, so α is not quadratic. Assume that α is algebraic of degree $d > 2$. Set $\lambda_n = |A_n|$, $\gamma_n = |B_n|$ for all $n \geq 1$ and $k_n = \sum_{i=1}^{n-1}(\lambda_i+\gamma_i)$, for all $n \geq 1$. We denote

$$\gamma = [H_1, H_2, \dots, H_n, \dots].$$

Now, let $\alpha_{k_n} = [A_nB_nA_{n+1}B_{n+1}...]$ denote the k_n^{th} complete quotient of α. From Lemma 3.3, α_{k_n} is algebraic of degree $d > 2$ and

$$H(\alpha_{k_n}) \leqslant H(\alpha)\left|\prod_{i=1}^{k_n-1} a_i\right|^{d-2}$$

where $(a_i)_{i\geqslant 1}$ is the sequence of partial quotients of α.

On the other hand, according to Lemma 3.1 we obtain

$$|\alpha_{k_n}-\beta| \geqslant H(\alpha_{k_n})^{-r-1}H(\gamma)^{-d} \tag{8}$$

We have that γ satisfies the equation (2), then

$$H(\gamma) = |P_{r+1}| = |T|^{\deg P_{r+1}} = |T|^{\sum_{j=1}^{r+1}\deg a_j} = |T|^{r+1}.$$

So, the inequality (8) becomes

$$|\alpha_{k_n} - \gamma| \geqslant H(\alpha)^{-r-1} \left|\prod_{i=1}^{k_n-1} a_i\right|^{(2-d)(r+1)} |T|^{-d(r+1)} \tag{9}$$

Furthermore, α_{k_n} and γ have the same first λ_n partial quotients, hence by Lemma 3.2 we get

$$|\alpha_{k_n} - \beta| \leqslant |H_1 H_2 \dots H_{n^2}|^{-2}. \tag{10}$$

From the inequalities (9) and (10) we obtain

$$|H_1 H_2 \dots H_{n^2}|^2 \leqslant H(\alpha)^{r+1} \prod_{i=1}^{k_n-1} |a_i|^{(d-2)(r+1)} |T|^{(r+1)d}$$

whence

$$2\sum_{i=1}^{n^2} r^i = 2r\left(\frac{r^{n^2}-1}{r-1}\right) \leqslant (r+1)\deg(H(\alpha)) + (d-2)(r+1)\sum_{i=1}^{k_n-1} \deg a_i + (r+1)d.$$

Then

$$\limsup_{n\to\infty} \frac{r^{n^2}-1}{\sum_{i=1}^{k_n-1} \deg a_i} \leqslant \frac{(d-2)(r^2-1)}{r}$$

As $\varphi(A_i) = \varphi(B_i) = 1$ for all $i \geqslant 1$, we get $\deg a_i \leq 1$, for all $1 \leqslant i \leqslant k_n$. Therefore

$$\limsup_{n\to\infty} \frac{r^{n^2}-1}{k_n - 1} \leqslant \frac{(r^2-1)(d-2)}{r}. \tag{11}$$

As $|B_i| < |A_i|$ for $i \geq 1$, then $\gamma_i < \lambda_i$ for all $i \geqslant 1$. Thus, $k_n < 2\sum_{i=1}^{n-1} \lambda_i \leq 2(n-1)\lambda_{n-1}$. Hence, we have that

$$\limsup_{n\to\infty} \frac{r^{n^2}-1}{2(n-1)\lambda_{n-1} - 1} \leq \limsup_{n\to\infty} \frac{r^{n^2}-1}{k_n - 1}$$

On the other hand we have $\lambda_{n-1} = |A_{n-1}| = \sum_{i=1}^{(n-1)^2} r^i = r(\frac{r^{(n-1)^2}-1}{r-1})$. So

$$\limsup_{n\to\infty} \frac{r^{n^2}-1}{2(n-1)\lambda_{n-1} - 1} = \limsup_{n\to\infty} \frac{r^{n^2}-1}{2(n-1)(r^{(n-1)^2}-1)} = +\infty$$

which is a contradiction with (11). $\square$

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