MATHEMATICA, 66 (89), N° 2, 2024, pp. 168–187

# $S$-PRIME PROPERTY IN LATTICES

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**Abstract.** Let $\pounds$ be a bounded distributive lattice and $S$ a join closed subset of $\pounds$. Following the concept of $S$-prime ideals (resp. weakly $S$-prime ideals), we define $S$-prime filters (resp. weakly $S$-prime filters) of $\pounds$. We will make an extensive investigation of the basic properties and possible structures of these filters.

**MSC 2020.** Primary 97H50, 06A11; Secondary 16G30, 06D25.

**Key words.** Lattice, filter, $S$-prime filter, weakly $S$-prime filter.

## 1. INTRODUCTION

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are definitely a natural choice of generalizations of rings, and it is appropriate to ask which properties of rings can be extended to lattices. The lack of subtraction in lattices shows that many results in rings have no counterparts in lattices, hence, it ought to be in the literature.

The main aim of this article is that of extending some results obtained for ring theory to the theory of lattices.

The notion of prime ideals has a significant place in the theory of rings, and it is used to characterize certain classes of rings. For years, there have been many studies and generalizations on this issue. See, for example, [1, 3, 5, 8, 9, 13–15].

Anderson and Smith generalized the concept of prime ideals in [3]. We recall from [3] that a nonzero proper ideal $I$ of a commutative ring $R$ is said to be a weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$, then either $a \in I$ or $b \in I$ (also see [8]).

In 2019, Hamed and Malek [13] introduced the notion of an $S$-prime ideal, i.e. let $S \subseteq R$ be a multiplicative set and $I$ an ideal of $R$ disjoint from $S$. We say that $I$ is $S$-prime if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have $sa \in I$ or $sb \in I$.

The author would like to thank the referee for a number of comments which have improved the paper.

DOI: 10.24193/mathcluj.2024.2.02

Almahdi et. al. [1] introduced the notion of a weakly $S$-prime ideal as follows: We say that $I$ is a weakly $S$-prime ideal of $R$ if there is an element $s \in S$ such that for all $x, y \in R$, if $0 \neq xy \in I$, then $xs \in I$ or $ys \in I$. Let $\pounds$ be a bounded distributive lattice.

Our objective in this paper is to extend the notion of $S$-primeness from commutative rings to $S$-primeness in lattices, and to investigate the relations between $S$-prime filters, weakly $S$-prime filters, weakly prime filters and prime filters. We say that a subset $S \subseteq \pounds$ is join closed if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (if $\mathbf{p}$ is a prime filter of $\pounds$, then $\pounds \setminus \mathbf{p}$ is a join closed subset of $\pounds$). Among many results in this paper, the first, introduction section contains elementary observations needed later on.

In Section 2, we give the basic properties of $S$-prime filters. At first, we give the definition of $S$-prime filters (Definition 2.1) and we give an example (Example 2.2) of an $S$-prime filter of $\pounds$ that is not a prime filter. It is shown (Theorem 2.4) that $\mathbf{p}$ is an $S$-prime filter of $\pounds$ if and only if there exists $s \in S$ such that for all $\mathbf{q}, \mathbf{r}$ filters of $\pounds$, if $\mathbf{q} \vee \mathbf{r} \subseteq \mathbf{p}$, then $s \vee \mathbf{q} \subseteq \mathbf{p}$ or $s \vee \mathbf{r} \subseteq \mathbf{p}$. It is proved (Theorem 2.8) that if $\mathbf{q}$ is a filter of $\pounds$, $\mathbf{p_1}, \dots, \mathbf{p_n}$ are $S$-prime filters of $\pounds$ and $\mathbf{q} \subseteq \cup_{i=1}^{n} \mathbf{p_i}$, then there exist $s \in S$ and $i \in \{1, \dots, n\}$ such that $s \vee \mathbf{q} \subseteq \mathbf{p_i}$. It is shown (Theorem 2.13) that if $S$ is a strongly join closed subset of $\pounds$, then each filter of $\pounds$ disjoint with $S$ is contained in a minimal $S$-prime filter of $\pounds$. Also, we show that every $S$-maximal filter of $\pounds$ is an $S$-prime filter (Proposition 2.15). In the rest of this section, we investigate the properties of $S$-prime filters similar to prime filters. In particular, we investigate the behavior of $S$-prime filters under homomorphism, in factor lattices, $S$-Noetherian lattices, and in Cartesian products of lattices (see Proposition 2.17, Proposition 2.18, Theorem 2.21, Theorem 2.22, Theorem 2.25, Corollary 2.28, Proposition 2.29, Theorem 2.30).

Section 3 is dedicated to the investigation of the basic properties of weakly $S$-prime filters. At first, we give the definitions of weakly $S$-prime filters and weakly prime filters (Definition 3.1) and we give an example (Example 3.2) of a weakly $S$-prime filter of $\pounds$ that is not a $S$-prime filter (so it is not a prime filter of $\pounds$). It is proved (Theorem 3.4) that if $S$ is a join closed subset of $\pounds$ and $\mathbf{p}$ is a weakly $S$-prime filter of $\pounds$ that is not $S$-prime, then $\mathbf{p} = \{1\}$. Theorem 3.6 proves that a filter $\mathbf{p}$ is weakly $S$-prime if and only if there exists $s \in S$ such that for each $x \notin (\mathbf{p} :_{\pounds} s)$ we have either $(\mathbf{p} :_{\pounds} x) \subseteq (\mathbf{p} :_{\pounds} s)$ or $(\mathbf{p} :_{\pounds} x) = (1 :_{\pounds} x)$. Also, we show that if $S$ is a join closed subset of $\pounds$, then every weakly $S$-prime filter of $\pounds$ is prime if and only if $\pounds$ is a $\pounds$-domain and every $S$-prime filter of $\pounds$ is prime (Proposition 3.9). In the rest of this section, we investigate the properties of weakly $S$-prime filters similarly to weakly prime filters. In particular, we investigate the behavior of weakly $S$-prime filters under homomorphism, in factor lattices, $S$-Noetherian lattices, and in Cartesian products of lattices (see Theorem 3.10, Theorem 3.14, Corollary 3.15, Theorem 3.16, Corollary 3.17).

Let us recall some notions and notations.

By a lattice we mean a poset $(\pounds, \leq)$ in which every couple of elements $x, y$ has a greatest lower bound (g.l.b. - called the meet of $x$ and $y$, and written $x \wedge y$) and a least upper bound (l.u.b. - called the join of $x$ and $y$, and written $x \vee y$).

A lattice $\pounds$ is complete when each of its subsets $X$ has a l.u.b. and a g.l.b. in $\pounds$. Setting $X = \pounds$, we see that any nonvoid complete lattice contains a least element 0 and a greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1).

A lattice $\pounds$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c$ in $\pounds$ (equivalently, $\pounds$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c$ in $\pounds$).

A non-empty subset $F$ of a lattice $\pounds$ is called a filter, if for $a \in F$, $b \in \pounds$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$).

A proper filter $F$ of $\pounds$ is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$.

A proper filter $F$ of $\pounds$ is said to be maximal if $G$ is a filter in $\pounds$ with $F \subsetneqq G$, then $G = \pounds$ [10, 11].

Let $A$ be subset of a lattice $\pounds$. Then the filter generated by $A$, denoted by $T(A)$, is the intersection of all filters that contain $A$.

A filter $F$ is called finitely generated if there is a finite subset $A$ of $F$ such that $F = T(A)$.

A lattice $\pounds$ with 1 is called a $\pounds$-domain if whenever $a \vee b = 1$ $(a, b \in \pounds)$, then $a = 1$ or $b = 1$ (so $\pounds$ is $\pounds$-domain if and only if $\{1\}$ is a prime filter of $\pounds$).

First we need the following lemma proved in [4–7].

LEMMA 1.1. *Let $\pounds$ be a lattice.*

(i) *A non-empty subset $F$ of $\pounds$ is a filter of $\pounds$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \pounds$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \pounds$.*

(ii) *If $F_1, \cdots, F_n$ are filters of $\pounds$ and $a \in \pounds$, then*

$$\bigvee_{i=1}^{n} F_i = \left\{ \bigvee_{i=1}^{n} a_i : a_i \in F_i \right\} \text{ and } a \vee F_i = \{a \vee a_i : a_i \in F_i\}$$

*are filters of $\pounds$ and $\bigvee_{i=1}^{n} F_i = \bigcap_{i=1}^{n} F_i$.*

(iii) *Let $A$ be an arbitrary non-empty subset of $\pounds$. Then*

$$T(A) = \{x \in \pounds : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}.$$

*Moreover, if $F$ is a filter and $A$ is a subset of $\pounds$ with $A \subseteq F$, then*

$$T(A) \subseteq F, \quad T(F) = F \quad \text{and} \quad T(T(A)) = T(A).$$

(iv) *If $\pounds$ is distributive, $F, G$ are filters of $\pounds$ and $y \in \pounds$, then*

$$(G :_{\pounds} F) = \{x \in \pounds : x \vee F \subseteq G\},$$
$$(F :_{\pounds} T(\{y\})) = (F :_L y) = \{a \in \pounds : a \vee y \in F\} \text{ and}$$
$$(1 :_{\pounds} y) = \{x \in \pounds : x \vee y = 1\}$$

*are filters of $\pounds$.*

(v) *If $\{F_i\}_{i\in\Delta}$ is a chain of filters of $\pounds$, then $\bigcup_{i\in\Delta} F_i$ is a filter of $\pounds$.*

(vi) *If $\pounds$ is distributive, $G, F_1, \cdots, F_n$ are filters of $\pounds$, then*

$$G \vee \left(\bigwedge_{i=1}^{n} F_i\right) = \bigwedge_{i=1}^{n} (G \vee F_i).$$

(vii) *If $\pounds$ is distributive and $F_1, \cdots, F_n$ are filters of $\pounds$, then*

$$\bigwedge_{i=1}^{n} F_i = \{\wedge_{i=1}^{n} a_i : a_i \in F_i\}$$

*is a filter of $\pounds$ and $F_i \subseteq \bigwedge_{i=1}^{n} F_i$ for each $i$.*

## 2. CHARACTERIZATION OF $S$-PRIME FILTERS

In this section, we collect some basic properties concerning $S$-prime filters. We remind the reader the following definition.

Definition 2.1. Assume that $\mathbf{p}$ is a filter of $\pounds$ and let $S$ be a join closed subset of $\pounds$ disjoint with $S$. We say that $\mathbf{p}$ is an $S$-prime filter of $\pounds$ if there is an element $s \in S$ such that for all $x, y \in \pounds$ if $x \vee y \in \mathbf{p}$, then $x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$.

Example 2.2. (a) If $S = \{0\}$, then the prime and the $S$-prime filters of $\pounds$ are the same.

(b) If $\mathbf{p}$ is a prime filter of $\pounds$ disjoint with $S$, then $\mathbf{p}$ is an $S$-prime filter.

(c) Assume that $\pounds = \{0, a, b, c, 1\}$ is a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$ and let $S = \{0, a\}$. Then $S$ is a join closed subset of $\pounds$ and $\mathbf{p} = \{1, c\}$ is an $S$-prime filter of $\pounds$. Note that $S \cap \mathbf{p} = \emptyset$. Also, $\mathbf{p}$ is not a prime filter of $\pounds$ because $a \vee b = c \in \mathbf{p}$ and $a, b \notin \mathbf{p}$. Thus an $S$-prime filter need not be a prime filter.

Proposition 2.3. *Assume that $\mathbf{p}$ is a filter of $\pounds$ and let $S$ be a join closed subset of $\pounds$ disjoint with $\mathbf{p}$. The following assertions are equivalent:*

(i) *$\mathbf{p}$ is an $S$-prime filter of $\pounds$;*

(ii) *$(P :_{\pounds} s)$ is a prime filter of $\pounds$ for some $s \in S$.*

*Proof.* (i)$\Rightarrow$(ii) Since $\mathbf{p}$ is an $S$-prime filter, we conclude that there is an element $s \in S$ such that for all $x, y \in \pounds$ with $x \vee y \in \mathbf{p}$ we have $x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$.

Now, we show that $(\mathbf{p} :_{\pounds} s)$ is a prime filter of $\pounds$. Let $x, y \in \pounds$ such that $x \vee y \in (P :_{\pounds} s)$. Then $x \vee y \vee s \in \mathbf{p}$ gives $x \vee s \vee s = x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$ which means that $x \in (P :_{\pounds} s)$ or $y \in (P :_{\pounds} s)$. Thus $(P :_{\pounds} s)$ is a prime filter of $\pounds$.

The implication (ii)⇒(i) is obvious. □

THEOREM 2.4. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *The following assertions are equivalent:*

(i) $\mathbf{p}$ *is an* $S$*-prime filter of* $\pounds$;

(ii) *There exists* $s \in S$ *such that for any two* $\mathbf{q}, \mathbf{r}$ *filters of* $\pounds$, *if* $\mathbf{q} \vee \mathbf{r} \subseteq \mathbf{p}$, *then* $s \vee \mathbf{q} \subseteq \mathbf{p}$ *or* $s \vee \mathbf{r} \subseteq \mathbf{p}$.

*Proof.* (i)⇒(ii) Since $\mathbf{p}$ is an $S$-prime filter, we conclude that there is an element $s \in S$ such that for all $x, y \in \pounds$, if $x \vee y \in \mathbf{p}$, then $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. On the contrary, assume that for all $u \in S$, there are $\mathbf{q}_u, \mathbf{r}_u$ two filters of $\pounds$ with $\mathbf{q}_u \vee \mathbf{r}_u \subseteq \mathbf{p}$, but $u \vee \mathbf{q}_u \nsubseteq \mathbf{p}$ and $u \vee \mathbf{r}_u \nsubseteq \mathbf{p}$. So there exist $\mathbf{q}_s, \mathbf{r}_s$ two filters of $\pounds$ with $\mathbf{q}_s \vee \mathbf{r}_s \subseteq \mathbf{p}$, but $s \vee \mathbf{q}_s \nsubseteq \mathbf{p}$ and $s \vee \mathbf{r}_s \nsubseteq \mathbf{p}$, as $s \in S$. This shows that there exist $x_s \in \mathbf{q}_s$ and $y_s \in \mathbf{r}_s$ such that $s \vee x_s \notin \mathbf{p}$ and $s \vee y_s \notin \mathbf{p}$ which is impossible, as $\mathbf{p}$ is an $S$-prime filter, i.e. (ii) holds.

(ii)⇒(i) Let $x, y \in \pounds$ such that $x \vee y \in \mathbf{p}$. Set $\mathbf{q} = T(\{x\})$ and $\mathbf{r} = T(\{y\})$. Then $\mathbf{q} \vee \mathbf{r} \subseteq \mathbf{p}$ gives that there exits $s \in S$ such that $s \vee x \in s \vee \mathbf{q} \subseteq \mathbf{p}$ or $s \vee y \in s \vee \mathbf{r} \subseteq \mathbf{p}$ by (ii), i.e. (i) holds. □

COROLLARY 2.5. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *Then* $\mathbf{p}$ *is an* $S$*-prime filter if and only if there exists* $s \in S$ *such that for all* $\mathbf{q_1}, \cdots, \mathbf{q_n}$ *filters of* $\pounds$, *if* $\mathbf{q_1} \vee \cdots \vee \mathbf{q_n} \subseteq \mathbf{p}$, *then* $s \vee \mathbf{q_i} \subseteq \mathbf{p}$ *for some* $i \in \{1, \cdots, n\}$.

*Proof.* Let $\mathbf{p}$ be an $S$-prime filter of $\pounds$. Then there is an element $s \in S$ such that for all $x, y \in \pounds$, if $x \vee y \in \mathbf{p}$, then $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. We use induction on $n$. We can take $n = 2$ as a base case by Theorem 2.4. Let $n \geq 3$, assume that the property holds up to the order $n-1$ and let $\mathbf{q_1}, \cdots, \mathbf{q_n}$ filters of $\pounds$ such that $\mathbf{q_1} \vee \cdots \vee \mathbf{q_n} = (\mathbf{q_1} \vee \cdots \vee \mathbf{q_{n-1}}) \vee \mathbf{q_n} \subseteq \mathbf{p}$. Then by Theorem 2.4, $s \vee \mathbf{q_n} \subseteq \mathbf{p}$ or $(s \vee \mathbf{q_1}) \vee \mathbf{q_2} \vee \cdots \vee \mathbf{q_{n-1}} \subseteq \mathbf{p}$. Therefore $s \vee \mathbf{q_n} \subseteq \mathbf{p}$ or $(s \vee s \vee \mathbf{q_1} = s \vee \mathbf{q_1} \subseteq \mathbf{p}$ or $s \vee \mathbf{q_i} \subseteq \mathbf{p}$ for some $i \in \{2, \cdots, n-1\})$. In the same way we prove that $s \vee \mathbf{q_i} \subseteq \mathbf{p}$ for some $i \in \{1, 2, \cdots, n\}$. □

COROLLARY 2.6. *Let* $\mathbf{p}$ *be a proper filter of* $\pounds$. *Then* $\mathbf{p}$ *is a prime filter if and only if for all* $\mathbf{q_1}, \cdots, \mathbf{q_n}$ *filters of* $\pounds$, *if* $\mathbf{q_1} \vee \cdots \vee \mathbf{q_n} \subseteq \mathbf{p}$, *then* $\mathbf{q_i} \subseteq \mathbf{p}$ *for some* $i \in \{1, \cdots, n\}$.

*Proof.* Take $S = \{0\}$ in Corollary 2.5. □

COROLLARY 2.7. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *Then* $\mathbf{p}$ *is an* $S$*-prime filter if and only if there exists* $s \in S$ *such that for all* $x_1, x_2, \cdots, x_n \in \pounds$, *if* $x_1 \vee \cdots \vee x_n \in \mathbf{p}$, *then* $s \vee x_i \in \mathbf{p}$ *for some* $i \in \{1, \cdots, n\}$.

*Proof.* Assume that $\mathbf{p}$ is an $S$-prime filter of $\pounds$ and let $x_1, \cdots, x_n \in \pounds$ such that $x_1 \vee \cdots \vee x_n \in \mathbf{p}$. So $T(\{x_1\}) \vee \cdots \vee T(\{x_n\}) \subseteq \mathbf{p}$. Then by Corollary 2.5, there exists $s \in S$ such that $s \vee x_i \in s \vee T(\{x_i\}) \subseteq \mathbf{p}$ for some $i \in \{1, \cdots, n\}$. For the converse, take $n = 2$. □

Compare the next theorem with Theorem 2 in [13].

THEOREM 2.8. *Let $S$ be a join closed subset of $\pounds$. Let $\mathbf{q}$ be a filter of $\pounds$ and $\mathbf{p_1}, \cdots, \mathbf{p_n}$ be $S$-prime filters of $\pounds$. If $\mathbf{q} \subseteq \cup_{i=1}^{n} \mathbf{p_i}$, then there exist $s \in S$ and $i \in \{1, \cdots, n\}$ such that $s \vee \mathbf{q} \subseteq \mathbf{p_i}$.*

*Proof.* By Proposition 2.3, for each $i \in \{1, \cdots, n\}$, there exists $s_i \in S$ such that $(\mathbf{p_i} :_{\pounds} s_i)$ is a prime filter of $\pounds$. Then $\mathbf{q} \subseteq \cup_{i=1}^{n} \mathbf{p_i} \subseteq \cup_{i=1}^{n} (\mathbf{p_i} :_{\pounds} s_i)$ gives that there exists $i \in \{1, \cdots, n\}$ such that $\mathbf{q} \subseteq (\mathbf{p_i} :_{\pounds} s_i)$ by [6, Theorem 2.1 (ii)]; this implies that $s_i \vee \mathbf{q} \subseteq \mathbf{p_i}$. □

DEFINITION 2.9. Let $S$ be a join closed subset of $\pounds$. We say that $S$ is a *strongly join closed subset* if for each family $\{s_i\}_{i \in \Lambda}$ of elements of $S$ we have

$$(\cap_{i \in \Lambda} T(\{s_i\})) \cap S \neq \emptyset.$$

EXAMPLE 2.10. Assume that $S = \{s_1, \cdots, s_k\}$ is a join closed subset of $\pounds$ and let $\{s_{i_1}, \cdots, s_{i_t}\} \subseteq S$. Set $s = s_{i_1} \vee \cdots \vee s_{i_t}$ (so $s \in S$). Then for each $j \in \{i_1, \cdots, i_t\} = S'$, $s \in T(s_{i_j})$; hence $s \in (\cap_{j \in S'} T(\{s_{i_j}\})) \cap S$. Thus every finite join closed subset of $\pounds$ is a strongly join closed subset.

THEOREM 2.11. *Assume that $S$ is a strongly join closed subset of $\pounds$ and let $\{\mathbf{p_i}\}_{i \in \Lambda}$ be a chain of $S$-prime filters of $\pounds$. Then $\mathbf{p} = \cap_{i \in \Lambda} \mathbf{p_i}$ is an $S$-prime filter of $\pounds$.*

*Proof.* For each $i \in \Lambda$, there is an element $s_i \in S$ such that for all $x, y \in \pounds$ with $x \vee y \in \mathbf{p_i}$ we have $s_i \vee x \in \mathbf{p_i}$ or $s_i \vee y \in \mathbf{p_i}$. Since $S$ is a strongly join closed subset, we conclude that $(\cap_{i \in \Lambda} T(\{s_i\})) \cap S \neq \emptyset$. Consider $t \in (\cap_{i \in \Lambda} T(\{s_i\})) \cap S$. Then for each $i \in \Lambda$, $t = s_i \vee a_i$, where $a_i \in \pounds$.

Now we will show that for all $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$ we have $t \vee a \in \mathbf{p}$ or $t \vee b \in \mathbf{p}$, i.e. $\mathbf{p}$ is $S$-prime. Let $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$ and suppose that $t \vee a \notin \mathbf{p}$. Then $t \vee a \notin \mathbf{p_j}$ for some $j \in \Lambda$. Let $k \in \Lambda$. Then $\mathbf{p_k} \subseteq \mathbf{p_j}$ or $\mathbf{p_j} \subseteq \mathbf{p_k}$. We split the proof into two cases.

**Case 1.** $\mathbf{p_k} \subseteq \mathbf{p_j}$. Since $t \vee a \notin \mathbf{p_j}$, we conclude that $t \vee a = s_k \vee a_k \vee a \notin \mathbf{p_k}$; so $s_k \vee a \notin \mathbf{p_k}$. This shows that $s_k \vee b \in \mathbf{p_k}$; hence $s_k \vee a_k \vee b = t \vee b \in \mathbf{p_k}$. Thus $t \vee b \in \mathbf{p}$.

**Case 2.** $\mathbf{p_j} \subseteq \mathbf{p_k}$. Since $t \vee a = s_j \vee a_j \vee a \notin \mathbf{p_j}$, we get that $s_j \vee a \notin \mathbf{p_j}$; so $s_j \vee b \in \mathbf{p_j} \subseteq \mathbf{p_k}$ which gives $t \vee b = s_j \vee a_j \vee b \in \mathbf{p_k}$, and so $t \vee b \in \mathbf{p}$. □

Assume that $S$ is a join closed subset of $\pounds$ and let $F$ be a filter of $\pounds$ disjoint with $S$. Let $\mathbf{p}$ be an $S$-prime filter of $\pounds$ such that $F \subseteq \mathbf{p}$. We say that $\mathbf{p}$ is a minimal $S$-prime filter over $F$ if $\mathbf{p}$ is minimal in the set of the $S$-prime filters containing $F$. Let $\mathbb{F}(\pounds)$ be the set of all filters of $\pounds$.

Lemma 2.12. *Suppose that $F$ is a filter of $\pounds$ and let $S$ be a join closed subset of $\pounds$ such that $S \cap F = \emptyset$. Then there exists an $S$-prime filter $\mathbf{p}$ of $\pounds$ such that $F \subseteq \mathbf{p}$.*

*Proof.* Set $\Omega = \{G \in \mathbb{F}(\pounds) : F \subseteq G \text{ and } S \cap G = \emptyset\}$. Clearly, $F \in \Omega$, and so $\Omega \neq \emptyset$. Moreover, $(\Omega, \subseteq)$ is a partial order. It is easy to see that $\Omega$ is closed under taking unions of chains and so $\Omega$ has at least one maximal element by Zorn's Lemma, say $\mathbf{p}$. Since $S \cap \mathbf{p} = \emptyset$ and $0 \in S$, we see that $0 \notin \mathbf{p}$ and $\mathbf{p} \subsetneqq \pounds$. It remains to show that $\mathbf{p}$ is a prime filter by Example 2.2 (b).

Now let $a, b \notin \mathbf{p}$; we must show that $a \vee b \notin \mathbf{p}$ for some elements $a, b \in \pounds$. Since $a \notin \mathbf{p}$, we have $F \subseteq \mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{a\})$. By the maximality of $\mathbf{p}$ in $\Omega$, we must have $S \cap (\mathbf{p} \wedge T(\{a\})) \neq \emptyset$, and so there exist $s \in S$, $t \in \pounds$ and $e \in \mathbf{p}$ such that $s = e \wedge (a \vee t)$. Similarly, there exist $s' \in S$, $t' \in \pounds$ and $e' \in \mathbf{p}$ such that $s' = e' \wedge (b \vee t')$. Put $u = a \vee t$ and $v = b \vee t'$. Then $s \vee s' = ((e \wedge u) \vee e') \wedge ((e \wedge u) \vee v) = ((e \wedge u) \vee e') \wedge (e \vee v) \wedge (a \vee b \vee t \vee t')$. Since $s \vee s' \in S$ and $((e \wedge u) \vee e') \wedge (e \vee v) \in \mathbf{p}$, we must have $a \vee b \notin \mathbf{p}$ since $S \cap \mathbf{p} = \emptyset$. Thus $\mathbf{p}$ is a prime filter of $\pounds$. □

Compare the next theorem with Proposition 5 in [13].

Theorem 2.13. *If $S$ is a strongly join closed subset of $\pounds$, then each filter of $\pounds$ disjoint with $S$ is contained in a minimal $S$-prime filter of $\pounds$.*

*Proof.* Let $F$ be a filter of $\pounds$ with $F \cap S = \emptyset$ and let $\Omega$ be the set of $S$-prime filters containing $F$. By Lemma 2.12, $\Omega \neq \emptyset$. Moreover, $(\Omega, \supseteq)$ is a partial order and $\Omega$ is inductive. Indeed, if $\{\mathbf{p_i}\}_{i \in \Lambda}$ is a chain of elements of $\Omega$, then by Theorem 2.11, $\mathbf{p} = \cap_{i \in \Lambda} \mathbf{p_i}$ is an $S$-prime filter of $\pounds$ which contains $F$; hence $\mathbf{p} \in \Omega$ is an upper bound for the chain. Then by Zorn's Lemma, $\Omega$ has a maximal element for "$\supseteq$" and so $F$ is contained in a minimal $S$-prime filter of $\pounds$. □

Definition 2.14. Assume that $S$ is a join closed subset of $\pounds$ and let $\mathbf{p}$ be a filter of $\pounds$ disjoint with $S$. Then $\mathbf{p}$ is said to be an $S$-maximal filter if there exists a fixed $s \in S$ and whenever $\mathbf{p} \subseteq \mathbf{q}$ for some filter $\mathbf{q}$ of $\pounds$, then either $s \vee \mathbf{q} \subseteq \mathbf{p}$ or $\mathbf{q} \cap S \neq \emptyset$.

Classically, in a lattice $\pounds$ every maximal filter is a prime filter, but its $S$-version has the following property. Compare it with Proposition 10 in [15].

Proposition 2.15. *Let $S$ be a join closed subset of $\pounds$. Then every $S$-maximal filter of $\pounds$ is an $S$-prime filter.*

*Proof.* Suppose that $\mathbf{p}$ is an $S$-maximal filter. Then there exists a fixed $s \in S$, and $\mathbf{p} \subseteq \mathbf{q}$ for some filter $\mathbf{q}$ of $\pounds$, which implies that $s \vee \mathbf{q} \subseteq \mathbf{p}$ or $\mathbf{q} \cap S \neq \emptyset$. Now, we will show that $\mathbf{p}$ is an $S$-prime filter. Let $x \vee y \in \mathbf{p}$ for some $x, y \in \pounds$. It suffices to show that $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$.

On the contrary, assume that $s \vee x \notin \mathbf{p}$ and $s \vee y \notin \mathbf{p}$. This shows that $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{x\})$ and $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{y\})$. Since $\mathbf{p}$ is $S$-maximal, we conclude that

$s \vee (\mathbf{p} \wedge T(\{x\})) \subseteq \mathbf{p}$ or $(\mathbf{p} \wedge T(\{x\})) \cap S \neq \emptyset$. If $s \vee (\mathbf{p} \wedge T(\{x\})) \subseteq \mathbf{p}$, then $s \vee x = s \vee (1 \wedge (0 \vee x)) \in \mathbf{p}$ which is impossible. So $(\mathbf{p} \wedge T(\{x\})) \cap S \neq \emptyset$. Likewise, $(\mathbf{p} \wedge T(\{y\})) \cap S \neq \emptyset$. Then there exist $s_1, s_2 \in S$ such that $s_1 = p_1 \wedge (a \vee x)$ and $s_2 = p_2 \wedge (b \vee y)$ for some $p_1, p_2 \in \mathbf{p}$ and $a, b \in \pounds$. Then we have $s_1 \vee s_2 = ((p_1 \wedge (a \vee x)) \vee p_2) \wedge (p_1 \vee b \vee y) \wedge (x \vee y \vee a \vee b) \in S \cap \mathbf{p}$, as $\mathbf{p}$ is a filter, which is a contradiction. Thus $\mathbf{p}$ is an $S$-prime filter of $\pounds$. □

We continue this section with the investigation of the stability of $S$-prime filters in various lattice-theoretic constructions.

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If $F$ is a filter of a lattice $(\pounds, \leq)$, we define a relation on $\pounds$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\pounds$, and we denote the equivalence class of $a$ by $a \wedge F$ and the collection of all equivalence classes by $\frac{\pounds}{F}$. We set up a partial order $\leq_Q$ on $\frac{\pounds}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{\pounds}{F}$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. The following notation below will be kept in this paper: It is straightforward to check that $(\frac{\pounds}{F}, \leq_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \frac{\pounds}{F}$. Note that $f \wedge F = F$ if and only if $f \in F$. Let $y \in \pounds$. We denote by $\bar{y}$ the equivalence class of $y$ in $\frac{\pounds}{F}$.

An element $x$ of $\pounds$ is called *the identity join of a lattice* $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\pounds$ is denoted by $\mathrm{Id}(\pounds)$.

Proposition 2.16. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *The following hold:*

(i) $S_Q = \{\bar{s} : s \in S\}$ *is a join closed subset of* $\frac{\pounds}{\mathbf{p}}$;

(ii) *If* $\mathrm{Id}(\frac{\pounds}{\mathbf{p}}) \cap S_Q = \emptyset$, *then prime and* $S$*-prime filters coincide.*

*Proof.* (i) Clearly, $\bar{0} \in S_Q$. If $\bar{s_1}, \bar{s_2} \in S_Q$ for some $s_1, s_2 \in S$ (so $s_1 \vee s_2 \in S$), then $\bar{s_1} \vee_Q \bar{s_2} = (s_1 \vee s_2) \wedge \mathbf{p} \in S_Q$, i.e. (i) holds.

(ii) It suffices to show that $P = (P :_\pounds s)$ for all $s \in S$ by Proposition 2.3. Since the inclusion $P \subseteq (P :_\pounds s)$ is clear, we will prove the reverse inclusion. Let $s \in S$ and $x \in (P :_\pounds s)$. Then $s \vee x \in \mathbf{p}$ gives $(s \wedge \mathbf{p}) \vee_Q (x \wedge \mathbf{p}) = (s \vee x) \wedge \mathbf{p} = 1 \wedge \mathbf{p}$. Since $\mathrm{Id}(\frac{\pounds}{\mathbf{p}}) \cap S_Q = \emptyset$, we conclude that $x \wedge \mathbf{p} = 1 \wedge \mathbf{p}$; so $x \wedge p_1 = 1 \wedge p_2 = p_2$ for some $p_1, p_2 \in \mathbf{p}$. This implies that $x \in \mathbf{p}$, by Lemma 1.1. □

Proposition 2.17. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$. *The following hold:*

(i) *Let* $\mathbf{q}$ *be a filter of* $\pounds$ *such that* $\mathbf{q} \cap S \neq \emptyset$. *If* $\mathbf{p}$ *is an* $S$*-prime filter of* $\pounds$, *then* $\mathbf{p} \vee \mathbf{q}$ *is an* $S$*-prime filter of* $\pounds$;

(ii) *Let* $\pounds \subseteq \pounds'$ *be an extension of lattices. If* $\mathbf{q}$ *is an* $S$*-prime filter of* $\pounds'$, *then* $\mathbf{q} \vee \pounds$ *is an* $S$*-prime filter of* $\pounds$;

(iii) *Let $f : \pounds \to \pounds'$ be a lattice homomorphism such that $f(0) = 0$ and $f(s) \neq 1$ for all $1 \neq s \in \pounds$. Then $f(S)$ is a join closed subset of $\pounds'$ and if $\mathbf{q}$ is an $f(S)$-prime filter of $\pounds'$, then $\mathbf{p} = f^{-1}(\mathbf{q})$ is an $S$-prime filter of $\pounds$.*

*Proof.* (i) Suppose that $q \in S \cap \mathbf{q}$ and let $x, y \in \pounds$ such that $x \vee y \in \mathbf{p} \vee \mathbf{q} \subseteq \mathbf{p}$. Then there is an element $s \in S$ such that $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$ which gives $s \vee q \vee x \in \mathbf{p}$ or $s \vee q \vee y \in \mathbf{p}$, where $s \vee q \in S$, i.e. (i) holds.

(ii) Let $x, y \in \pounds$ such that $x \vee y \in \mathbf{q} \vee \pounds \subseteq \mathbf{q}$. Then there is an element $s \in S$ such that $s \vee x \in \mathbf{q}$ or $s \vee y \in \mathbf{q}$ which implies that $s \vee x \in \mathbf{q} \vee \pounds$ or $s \vee y \in \mathbf{q} \vee \pounds$. This completes the proof.

(iii) Clearly, $f(S)$ is a join closed subset of $\pounds'$. By assumption, there exists $s \in S$ such that for all $x, y \in \pounds'$ if $x \vee y \in \mathbf{q}$, then $f(s) \vee x \in \mathbf{q}$ or $f(s) \vee y \in \mathbf{q}$. It is clear that $\mathbf{p} \cap S = \emptyset$. Let $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$; so $f(a \vee b) = f(a) \vee f(b) \in \mathbf{q}$ which gives $f(s) \vee f(a) = f(s \vee a) \in \mathbf{q}$ or $f(s) \vee f(b) = f(s \vee b) \in \mathbf{q}$. This implies that $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$, as required. □

Proposition 2.18. *Assume that $\mathbf{q}$ is a filter of $\pounds$ and let $S$ be a join closed subset of $\pounds$ disjoint with $\mathbf{q}$. Let $\mathbf{p}$ be a proper filter of $\pounds$ containing $\mathbf{q}$ such that $(\frac{\mathbf{p}}{\mathbf{q}}) \cap S_Q = \emptyset$. Then $\mathbf{p}$ is an $S$-prime filter of $\pounds$ if and only if $\frac{\mathbf{p}}{\mathbf{q}}$ is an $S_Q$-prime filter of $\frac{\pounds}{\mathbf{q}}$.*

*Proof.* Let $\mathbf{p}$ be an $S$-prime filter of $\pounds$. Then there is an element $s \in S$ such that for all $x, y \in \pounds$, if $x \vee y \in \mathbf{p}$, then $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. Let $a \wedge \mathbf{q}, b \wedge \mathbf{q} \in \frac{\pounds}{\mathbf{q}}$ such that $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) = (a \vee b) \wedge \mathbf{q} \in \frac{\mathbf{p}}{\mathbf{q}}$ which gives $a \vee b \in \mathbf{p}$ by [4, Lemma 4.3]; hence $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$. Therefore $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ or $(s \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$. Thus $\frac{\mathbf{p}}{\mathbf{q}}$ is an $S_Q$-prime filter of $\frac{\pounds}{\mathbf{q}}$. Conversely, if $\mathbf{p} \cap S \neq \emptyset$, then $(\frac{\mathbf{p}}{\mathbf{q}}) \cap S_Q \neq \emptyset$, which is impossible. Hence $S \cap \mathbf{p} = \emptyset$. Since $\frac{\mathbf{p}}{\mathbf{q}}$ is an $S_Q$-prime filter of $\frac{\pounds}{\mathbf{q}}$, we conclude that there exists $s \in S$ such that for all $x \wedge \mathbf{q}, y \wedge \mathbf{q} \in \frac{\pounds}{\mathbf{q}}$ with $(x \wedge \mathbf{q}) \vee_Q (y \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$, we get $(s \wedge \mathbf{q}) \vee_Q (x \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ or $(s \wedge \mathbf{q}) \vee_Q (y \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$.

Now, let $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$. Then $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ gives $(s \wedge \mathbf{q}) \vee_Q (a \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ or $(s \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$; hence $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$, i.e. the result holds. □

Definition 2.19. Let $S$ be a join closed subset of $\pounds$. We say that a filter $F$ of $\pounds$ is $S$-finite if $s \vee F \subseteq G \subseteq F$ for some finitely generated filter $G$ of $\pounds$ and some $s \in S$. We say that $\pounds$ is $S$-Noetherian if each filter of $\pounds$ is $S$-finite.

Lemma 2.20. *Assume that $S$ is a join closed subset of $\pounds$ and let $\mathbf{p}$ be a filter of $\pounds$ which is maximal among all non-$S$-finite filters of $\pounds$. Then $\mathbf{p}$ is a prime filter of $\pounds$.*

*Proof.* Notice that $\pounds$ is $S$-finite since $s \vee \pounds \subseteq T(\{s\}) \subseteq \pounds$ for every $s \in S \cap \pounds$. If $\mathbf{p}$ is not prime, let $a, b \notin \mathbf{p}$ with $a \vee b \in \mathbf{p}$. Since $\mathbf{p} \subsetneq \mathbf{p} \wedge T(\{a\})$, we conclude that $\mathbf{p} \wedge T(\{a\})$ is $S$-finite by maximality of $\mathbf{p}$; hence

$$s \vee (\mathbf{p} \wedge T(\{a\}) \subseteq T(\{p_1 \wedge (a \vee s_1), \cdots, p_n \wedge (a \vee s_n)\})$$

for some $s \in S$, $p_1, \cdots, p_n \in \mathbf{p}$ and $s_1, \cdots, s_n \in \pounds$. Also, $(\mathbf{p} :_{\pounds} a)$ is $S$-finite, so $t \vee (\mathbf{p} :_{\pounds} a) \subseteq T(\{q_1, \cdots, q_k\})$ for some $t \in S$ and $q_1, \cdots, q_n \in (\mathbf{p} :_{\pounds} a)$. Now let $x \in \mathbf{p}$ (so $x \vee s \in s \vee (\mathbf{p} \wedge T(\{a\}))$. Then

$$\begin{aligned} s \vee x &= (s \vee x) \vee \wedge_{i=1}^{n} (p_i \wedge (a \vee s_i)) \\ &= \wedge_{i=1}^{n} (s \vee x \vee p_i) \wedge (\wedge_{i=1}^{n} (s \vee x \vee a \vee s_i)), \end{aligned}$$

so $y = \wedge_{i=1}^{n} (s \vee x \vee s_i) \in (\mathbf{p} :_{\pounds} a)$ (so $t \vee y \in (\mathbf{p} :_{\pounds} a)$) which gives

$$t \vee y = (t \vee y) \vee (\wedge_{i=1}^{k} q_i) = \wedge_{i=1}^{k} (t \vee y \vee q_i).$$

Therefore

$$\begin{aligned} s \vee x \vee t &= \wedge_{i=1}^{n} (s \vee x \vee p_i \vee t) \wedge (\wedge_{i=1}^{n} (s \vee x \vee t \vee s_i \vee a)) \\ &= \wedge_{i=1}^{n} (s \vee x \vee p_i \vee t) \wedge (a \vee t \vee y) \\ &= \wedge_{i=1}^{n} (s \vee x \vee p_i \vee t) \wedge (\wedge_{i=1}^{k} (a \vee q_i \vee t \vee y)). \end{aligned}$$

So $(s \vee t) \vee \mathbf{p} \subseteq T(A) \subseteq \mathbf{p}$, where $A = \{p_1 \vee t, \cdots, p_n \vee t, a \vee q_1, \cdots, a \vee q_k\} \subseteq \mathbf{p}$; hence $\mathbf{p}$ is $S$-finite, which is a contradiction. Thus $\mathbf{p}$ is a prime filter of $\pounds$. □

In the following theorem we give an $S$-version of Cohen's Theorem ([12, Theorem 2]). Compare it with Proposition 4 in [2]).

Theorem 2.21. *Let $S$ be a join closed subset of $\pounds$. Then $\pounds$ is $S$-Noetherian if and only if every prime filter of $\pounds$ (disjoint from $S$) is $S$-finite.*

*Proof.* One side is clear. To see the other side, assume that $\mathbf{p}$ is $S$-finite for each prime filter $\mathbf{p}$ of $\pounds$ disjoint from $S$. Suppose That $\pounds$ is not $S$-Noetherian and we look for a contradiction.

The set $\Omega$ of all non-$S$-finite filters of $\pounds$ is inductively ordered under inclusion. By Zorn's Lemma, choose $\mathbf{q}$ maximal in $\Omega$. Then Lemma 2.20 shows that $\mathbf{q}$ is a prime filter. If $\mathbf{q} \cap S \neq \emptyset$, then $s \vee \mathbf{q} \subseteq T(\{s\}) \subseteq \mathbf{q}$ for every $s \in \mathbf{q} \cap S$ gives that $\mathbf{q}$ is $S$-finite, which is a contradiction. Thus $\mathbf{q} \cap S = \emptyset$.

Now, by the hypothesis, $\mathbf{q}$ is $S$-finite which is impossible since $\mathbf{q} \in \Omega$. Thus $\pounds$ is $S$-Noetherian. □

Theorem 2.22. *Let $S$ be a join closed subset of $\pounds$. The following assertions are equivalent:*

(i) *$\pounds$ is $S$-Noetherian;*

(ii) *Every $S$-prime filter of $\pounds$ is $S$-finite;*

(iii) *Every prime filter of $\pounds$ is $S$-finite.*

*Proof.* (i)⇒(ii) This is clear.

(ii)⇒(iii) Let $\mathbf{p}$ be a prime filter of $\pounds$. If $\mathbf{p} \cap S \neq \emptyset$, then $s \vee \mathbf{p} \subseteq T(\{s\}) \subseteq \mathbf{P}$ for every $s \in \mathbf{p} \cap S$ gives $\mathbf{p}$ is $S$-finite. If $\mathbf{p} \cap S = \emptyset$, then $\mathbf{p}$ is an $S$-prime filter of $\pounds$; so by the hypothesis, $\mathbf{p}$ is $S$-finite.

(iii)⇒(i) Follows from Theorem 2.21. □

Lemma 2.23. (i) *If* $A = \{a_1, \cdots, a_n\} \subseteq \pounds$, *then* $T(A) = \wedge_{i=1}^{n} T(\{a_i\})$.

(ii) *Assume that* $F$ *is a filter of* $\pounds$ *and let* $G$ *be a finitely generated filter. Then* $\frac{G \wedge F}{F}$ *is a finitely generated filter of* $\frac{\pounds}{F}$.

*Proof.* (i) Since for each $i \in \{1, \cdots, n\}$, $\{a_i\} \subseteq A \subseteq T(A)$, we conclude that $T(\{a_i\}) \subseteq T(A)$; so $\wedge_{i=1}^{n} T(\{a_i\}) \subseteq T(A)$. If $x \in T(A)$, then

$$x = (x \vee a_1) \wedge \cdots \wedge (x \vee a_n) \in \wedge_{i=1}^{n} T(\{a_i\}),$$

and so we have equality.

(ii) By assumption, there exists a finite set $A = \{a_1, \cdots, a_n\} \subseteq G$ such that $G = T(A)$. Let $\bar{y} = y \wedge F \in \frac{G \wedge F}{F}$. Then there exist

$$x = (x \vee a_1) \wedge \cdots \wedge (x \vee a_n) \in G$$

and $f \in F$ such that

$$\bar{y} = \bar{x} = (\bar{x} \vee_Q \bar{a_1}) \wedge_Q \cdots \wedge_Q (\bar{x} \vee_Q \bar{a_n}) \in \wedge_{i=1}^{n} T(\{\bar{a_i}\}) = T(\{\bar{a_1}, \cdots, \bar{a_n}\}),$$

by (i). This shows that $\frac{G \wedge F}{F}$ is a finitely generated filter of $\frac{\pounds}{F}$. □

Proposition 2.24. *Suppose that* $F$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *such that* $S \cap F = \emptyset$. *If* $\pounds$ *is* $S$*-Noetherian, then* $\frac{\pounds}{F}$ *is* $Q_S$*-Noetherian.*

*Proof.* Let $H$ be a filter of $\frac{\pounds}{F}$. Then by the hypothesis and [4, Lemma 4.2], there exists an $S$-finite filter $G$ of $\pounds$ such that $H = \frac{G}{F}$; so $s \vee G \subseteq K \subseteq G$ for some finitely generated filter $K = T(A)$ of $\pounds$ and some $s \in S$, where $A = \{a_1, \cdots, a_k\} \subseteq G$. Then by Lemma 2.23, $\frac{K \wedge F}{F} = T(\{\bar{a_1}, \cdots, \bar{a_n}\})$ is finitely generated. An inspection will show that $\bar{s} \vee_Q \frac{G}{F} \subseteq \frac{K \wedge F}{F} \subseteq \frac{G}{F}$ which implies that $\frac{G}{F}$ is a $Q_S$-finite filter of $\frac{\pounds}{F}$, i.e. $\frac{\pounds}{F}$ is $Q_S$-Noetherian. □

Compare the next theorem with Theorem 5 in [13].

Theorem 2.25. *Let* $S$ *be a join closed subset of* $\pounds$ *and* $F$ *a filter of* $\pounds$ *disjoint with* $S$. *If* $\pounds$ *is* $S$*-Noetherian, then there exist an* $s \in S$ *and* $S$*-prime filters* $\mathbf{p_1}, \cdots, \mathbf{p_n}$ *of* $\pounds$ *containing* $F$ *such that* $s \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \subseteq F$.

*Proof.* At first, we show that the set $\Omega = \{F \in \mathbb{F}(\pounds) :$ for all $s \in S$, for all $S$-prime filters $\mathbf{p_1}, \cdots, \mathbf{p_n}$ of $\pounds$, we have $s \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \nsubseteq F\}$ is empty.

On the contrary, assume that $\Omega \neq \emptyset$. Clearly, $(\Omega, \subseteq)$ is a partial order. Let $\{F_i\}_{i \in \Lambda}$ be a chain of elements of $\Omega$ and set $G = \cup_{i \in \Lambda} F_i$. It is clear that $G$ is a filter of $\pounds$.

We claim that $G \in \Omega$. Assume on the contrary, that $G \notin \Omega$. Then there exist $s \in S$ and $S$-prime filters $\mathbf{p_1}, \cdots, \mathbf{p_n}$ of $\pounds$ such that $s \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \subseteq G$. Since $\pounds$ is $S$-Noetherian, for all $i \in \{1, \cdots, n\}$, there exist $p_{1,i}, \cdots, p_{m_i,i} \in \pounds$ and $s_i \in S$ such that $s_i \vee \mathbf{p_i} \subseteq T(\{p_{1,i}, \cdots, p_{m_i,i}\}) \subseteq \mathbf{p_i}$.

Therefore,

$$\begin{aligned}&s \vee s_1 \vee \cdots \vee s_n \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n})\\ &\subseteq s \vee (T(\{p_{1,1}, \cdots, p_{m_1,1}\}) \vee \cdots \vee T(\{p_{1,n}, \cdots, p_{m_n,n}\})\\ &\subseteq s \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n})\\ &\subseteq G.\end{aligned}$$

Set $T(\{p_{1,1}, \cdots, p_{m_1,1}\}) \vee \cdots \vee T(\{p_{1,n}, \cdots, p_{m_n,n}\}) = T(\{c_1, \cdots, c_t\})$ for some $c_1, \cdots, c_t \in \pounds$. Since $T(\{c_1, \cdots, c_t\}) \subseteq G$, there is an element $j \in \Lambda$ such that $T(\{c_1, \cdots, c_t\}) \subseteq F_j$; hence

$$s \vee s_1 \vee \cdots \vee s_n \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \subseteq T(\{c_1, \cdots, c_t\}) \subseteq F_j \in \Omega,$$

which is impossible.

Thus $G$ is an upper bound of the chain $\{F_i\}_{i\in\Lambda}$; hence by Zorn's Lemma, $\Omega$ has a maximal element $\mathbf{p}$. If $s \in S \cap \mathbf{p}$, then $\mathbf{p}$ is a filter, which implies that for all $S$-prime filters $\mathbf{p}'$ of $\pounds$, $s \vee \mathbf{p}' \subseteq \mathbf{p} \in \Omega$, which is a contradiction.

Thus $\mathbf{p} \cap S = \emptyset$. Since $\mathbf{p} \in \Omega$, we conclude that $\mathbf{p}$ is not $S$-prime. So for all $s \in S$, there are elements $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$, but $s \vee a \notin \mathbf{p}$ and $s \vee b \notin \mathbf{p}$.

Since $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{s \vee a\})$ and $\mathbf{p} \subsetneqq \mathbf{p} \wedge T(\{s \vee b\})$, then by the maximality of $\mathbf{p}$ there are $u, v \in S$ and $\mathbf{p_1}, \cdots, \mathbf{p_n}, \mathbf{q_1}, \cdots, \mathbf{q_m}$ $S$-prime filters of $\pounds$ such that $u \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \subseteq \mathbf{p} \wedge T(\{s \vee a\})$ and $v \vee (\mathbf{q_1} \vee \cdots \vee \mathbf{q_m}) \subseteq \mathbf{p} \wedge T(\{s \vee b\})$.

Put $A = T(\{s \vee a\})$ and $B = T(\{s \vee b\})$. Therefore

$$\begin{aligned}&(u \vee v) \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n} \vee \mathbf{q_1} \vee \cdots \vee \mathbf{q_m})\\ &\subseteq (\mathbf{p} \wedge A) \vee (\mathbf{p} \wedge B) = \mathbf{p} \wedge (A \vee B)\\ &\subseteq \mathbf{p} \wedge T(\{a \vee b\})\\ &\subseteq \mathbf{p} \in \Omega,\end{aligned}$$

which is a contradiction. Thus $\Omega = \emptyset$.

Let $F$ be a filter of $\pounds$ disjoint with $S$. By Proposition 2.24, $\frac{\pounds}{F}$ is $Q_S$-Noetherian; so there exist $s \in S$ and $S_Q$-prime filters $\frac{\pounds}{F}$ which are of the form $\frac{\mathbf{q_i}}{F}$ by [4, Lemma 4.2], where $\mathbf{q_i}$ are $S$-prime filters of $\pounds$ containing $F$, by Proposition 2.18, such that

$$(s \wedge F) \vee_Q (\frac{\mathbf{q_1}}{F} \vee_Q \cdots \vee_Q \frac{\mathbf{q_n}}{F}) = \frac{s \vee (\mathbf{q_1} \vee \cdots \vee \mathbf{q_n})}{F} \subseteq \{\bar{1}\} = \frac{F}{F};$$

thus $s \vee (\mathbf{q_1} \vee \cdots \vee \mathbf{q_n}) \subseteq F$. This completes the proof. □

Corollary 2.26. *Let $F$ be a proper filter of a Noetherian lattice $\pounds$. Then there exist prime filters $\mathbf{p_1}, \cdots, \mathbf{p_n}$ of $\pounds$ containing $F$ such that*

$$\mathbf{p_1} \vee \cdots \vee \mathbf{p_n} \subseteq F.$$

*Proof.* Take $S = \{0\}$ in Theorem 2.25. □

Corollary 2.27. *Assume that $S$ is a finite join closed subset of $\pounds$ and let $F$ be a filter of $\pounds$ disjoint with $S$. Suppose that $\pounds$ is $S$-Noetherian. The following hold:*

(i) *There exist a minimal $S$-prime filter $\mathbf{q}$ over $F$ and an $S$-prime filter $\mathbf{p}$ of $\pounds$ containing $F$ such that $\mathbf{q} = (\mathbf{p} \vee T(\{s\}) \wedge F$;*

(ii) *Each filter of $\pounds$ disjoint with $S$ has a finite number of minimal $S$-prime filters. In particular, the set of minimal $S$-prime filters of $\pounds$ is finite.*

*Proof.* (i) By Example 2.10 and Theorem 2.13, there exists a minimal $S$-prime filter $\mathbf{q}$ over $F$. Also, by Theorem 2.25, there exists a $t \in S$ and $S$-prime filters $\mathbf{p_1}, \cdots, \mathbf{p_n}$ of $\pounds$ such that $t \vee (\mathbf{p_1} \vee \cdots \vee \mathbf{p_n}) \subseteq F \subseteq \mathbf{q}$. By Corollary 2.5, $s \vee \mathbf{p_i} \subseteq \mathbf{q}$, for some $s \in S$ and some $i \in \{1, \cdots, n\}$. Since $s = s \wedge 1 \in T(\{s\}) \wedge F$, we conclude that $(\mathbf{p} \vee T(\{s\}) \wedge F = (F \wedge T(\{s\}) \vee \mathbf{p_i}$ is an $S$-prime filter of $\pounds$ containing $F$, by Proposition 2.17. But $\mathbf{q}$ is minimal in the set of $S$-prime filters containing $F$, this shows that $\mathbf{q} = (\mathbf{p} \vee T(\{s\}) \wedge F$.

(ii) This follows from (i) and its proof. □

Corollary 2.28. *If $\pounds$ is Noetherian, then the set of minimal prime filters of $\pounds$ is finite.*

*Proof.* Take $S = \{0\}$ in Corollary 2.27. □

Assume that $(\pounds_1, \leq_1), (\pounds_2, \leq_2), \cdots, (\pounds_n, \leq_n)$ are lattices $(n \geq 2)$ and let $\pounds = \pounds_1 \times \pounds_2 \times \cdots \times \pounds_n$. We set up a partial order $\leq_c$ on $\pounds$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \pounds$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$.

The following notation below will be kept in this paper: It is straightforward to check that $(\pounds, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\pounds$ is a decomposable lattice.

Proposition 2.29. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where $S_i$ is a join closed subset of $\pounds_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \mathbf{p_2}$ is a filter of $\pounds$. The following statements are equivalent:*

(i) *$\mathbf{p}$ is an $S$-prime filter of $\pounds$;*

(ii) *$\mathbf{p_1}$ is an $S_1$-prime filter of $\pounds_1$ and $\mathbf{p_2} \cap S_2 \neq \emptyset$ or $\mathbf{p_2}$ is an $S_2$-prime filter of $\pounds_2$ and $\mathbf{p_1} \cap S_1 \neq \emptyset$.*

*Proof.* (i)⇒(ii) Let $\mathbf{p}$ be an $S$-prime filter of $\pounds$. Since $(0,1) \vee_c (1,0) = (1,1) \in \mathbf{p}$, there exists $s = (s_1, s_2) \in S$ such that $s \vee_c (0,1) = (s_1, 1) \in \mathbf{p}$ or $s \vee_c (1,0) = (1, s_2) \in \mathbf{p}$ and hence $\mathbf{p_1} \cap S_1 \neq \emptyset$ or $\mathbf{p_2} \cap S_2 \neq \emptyset$.

Without any loss of generality, we can assume that $\mathbf{p_1} \cap S_1 \neq \emptyset$. Since $\mathbf{p} \cap S = \emptyset$, we conclude that $\mathbf{p_2} \cap S_2 = \emptyset$. Let $x \vee y \in \mathbf{p_2}$ for some $x, y \in \pounds_2$. As $(1,x) \vee_c (1,y) = (1, x \vee y) \in \mathbf{p}$ and $\mathbf{p}$ is an $S$-prime filter, we obtain either $t \vee_c (1,x) = (1, t_2 \vee x) \in \mathbf{p}$ or $t \vee_c (1,y) = (1, t_2 \vee y) \in \mathbf{p}$ for some $t = (t_1, t_2) \in S$

and this yields $t_2 \vee x \in \mathbf{p_2}$ or $t_2 \vee y \in \mathbf{p_2}$. Hence $\mathbf{p_2}$ is a $S_2$-prime filter of $\pounds_2$. In the other case, one can similarly show that $\mathbf{p_1}$ is an $S_1$-prime filter of $\pounds_1$.

(ii)$\Rightarrow$(i) Suppose that $\mathbf{p_1} \cap S_1 \neq \emptyset$ and $\mathbf{p_2}$ is an $S_2$-prime filter of $\pounds_2$. Consider $s_1 \in \mathbf{p_1} \cap S_1$. Let $(a, b) \vee_c (c, d) \in \mathbf{p}$ for some $a, c \in \pounds_1$ and $b, d \in \pounds_2$. This shows that $b \vee d \in \mathbf{p_2}$ and hence there exists $s_2 \in S_2$ such that $s_2 \vee b \in \mathbf{p_2}$ or $s_2 \vee d \in \mathbf{p_2}$. Set $s = (s_1, s_2) \in S$. Then we have $s \vee_c (a, b) = (s_1 \vee a, s_2 \vee b) \in \mathbf{p}$ or $s \vee_c (c, d) = (s_1 \vee c, s_2 \vee d) \in \mathbf{p}$. Thus $\mathbf{p}$ is an $S$-prime filter of $\pounds$. In the other case, one can similarly prove that $\mathbf{p}$ is an $S$-prime filter of $\pounds$. □

THEOREM 2.30. *Let* $\pounds = \pounds_1 \times \cdots \times \pounds_n$ *be a decomposable lattice and let* $S = S_1 \times \cdots \times S_n$*, where* $S_i$ *is a join closed subset of* $\pounds_i$*. Suppose that* $\mathbf{p} = \mathbf{p_1} \times \cdots \times \mathbf{p_n}$ *is a filter of* $\pounds$*. The following statements are equivalent:*

(i) $\mathbf{p}$ *is an* $S$*-prime filter of* $\pounds$*;*

(ii) $\mathbf{p_i}$ *is an* $S_i$*-prime filter of* $\pounds_i$ *for some* $i \in \{1, \cdots, n\}$ *and* $\mathbf{p_j} \cap S_j \neq \emptyset$ *for all* $j \in \{1, \cdots, n\} \setminus \{i\}$*.*

*Proof.* We use induction on $n$. For $n = 1$, the result is true. If $n = 2$, then (i) and (ii) are equivalent by Proposition 2.29.

Assume that (i) and (ii) are equivalent when $k < n$. Set

$$\mathbf{p}' = \mathbf{p_1} \times \cdots \times \mathbf{p_{n-1}}, \quad S' = S_1 \times \cdots \times S_{n-1} \quad \text{and} \quad \pounds' = \pounds_1 \times \cdots \times \pounds_{n-1}.$$

Then by Proposition 2.29, $\mathbf{p} = \mathbf{p}' \times \mathbf{p_n}$ is an $S$-prime filter of $\pounds$ if and only if $\mathbf{p}'$ is an $S'$-prime filter of $\pounds'$ and $\mathbf{p_n} \cap S_n \neq \emptyset$ or $\mathbf{p}' \cap S' \neq \emptyset$ and $\mathbf{p_n}$ is a $S_n$-prime filter of $\pounds_n$. Now the assertion follows from the induction hypothesis. □

### 3. CHARACTERIZATION OF WEAKLY $S$-PRIME FILTERS

In this section, the concept of weakly $S$-prime filter is introduced and investigated. We remind the reader the following definition.

DEFINITION 3.1. (a) A proper filter $\mathbf{p}$ of a lattice $\pounds$ is called *weakly prime* if whenever $x, y \in \pounds$ and $1 \neq x \vee y \in \mathbf{p}$, then either $x \in \mathbf{p}$ or $y \in \mathbf{p}$.

(b) Let $S$ be a join closed subset of $\pounds$. A filter $\mathbf{p}$ of $\pounds$ satisfying $S \cap \mathbf{p} = \emptyset$ is said to be weakly $S$-prime if there exists an element $s \in S$ such that, whenever $x, y \in \pounds$, $1 \neq x \vee y \in \mathbf{p}$ implies $x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$.

EXAMPLE 3.2. (a) A weakly prime filter $\mathbf{p}$ of $\pounds$ is weakly $S$-prime for each join closed subset $S$ of $\pounds$ such that $S \cap \mathbf{p} = \emptyset$.

(b) Assume that $\pounds$ is the lattice as in Example 2.2 (c) and let $S = \{0, a\}$. Then $\mathbf{p} = \{1, c\}$ is a weakly $S$-prime filter of $\pounds$. Also, $\mathbf{p}$ is not a weakly prime filter of $\pounds$ because $1 \neq a \vee b = c \in \mathbf{p}$ and $a, b \notin \mathbf{p}$. Thus a weakly $S$-prime filter need not be a weakly prime filter.

(c) If $S = \{0\}$, then the weakly prime and the weakly $S$-prime filters of $\pounds$ are the same.

(d) It is easy to see that every $S$-prime filter is a weakly $S$-prime filter.

(e) Let $D = \{a, b, c\}$. Then $\pounds = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element $D$ and least element $\emptyset$ (note that if $x, y \in \pounds$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $\mathbf{p} = \{1\}$ and $S = \{\{a\}, \emptyset\}$. Then $S$ is a join closed subset of $\pounds$ and $\mathbf{p}$ is clearly a weakly $S$-prime filter of $\pounds$. Since $\{a, b\} \vee \{c\} \in \mathbf{p}$, $\{a\} \vee \{a, b\} \notin \mathbf{p}$ and $\{a\} \vee \{c\} \notin \mathbf{p}$, it follows that $\mathbf{p}$ is not a $S$-prime filter of $\pounds$. Thus a weakly $S$-prime filter need not be an $S$-prime filter.

Proposition 3.3. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$ *and* $\mathrm{Id}(\pounds) \cap S = \emptyset$. *The following assertions are equivalent:*

(i) $\mathbf{p}$ *is a weakly* $S$*-prime filter of* $\pounds$*;*

(ii) $(P :_{\pounds} s)$ *is a weakly prime filter of* $\pounds$ *for some* $s \in S$.

*Proof.* (i)⇒(ii) Since $\mathbf{p}$ is a weakly $S$-prime, then there is an element $s \in S$ such that for all $x, y \in \pounds$ with $1 \neq x \vee y \in \mathbf{p}$, we have $x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$.

Now, we show that $(\mathbf{p} :_{\pounds} s)$ is a weakly prime filter of $\pounds$. Let $x, y \in \pounds$ such that $1 \neq x \vee y \in (P :_{\pounds} s)$. Then $x \vee y \vee s \in \mathbf{p}$ (so $x \vee y \vee s \neq 1$, as $\mathrm{Id}(\pounds) \cap S = \emptyset$), which gives $x \vee s \vee s = x \vee s \in \mathbf{p}$ or $y \vee s \in \mathbf{p}$, which means that $x \in (P :_{\pounds} s)$ or $y \in (P :_{\pounds} s)$. Thus $(P :_{\pounds} s)$ is a weakly prime filter of $\pounds$.

(ii)⇒(i) Clear. □

Theorem 3.4. *Let* $S$ *be a join closed subset of* $\pounds$*, and* $\mathbf{p}$ *be a weakly* $S$*-prime filter of* $\pounds$*. If* $\mathbf{p}$ *is not* $S$*-prime, then* $\mathbf{p} = \{1\}$.

*Proof.* By assumption, there exists $s \in S$ such that, whenever $x, y \in \pounds$, $1 \neq x \vee y \in \mathbf{p}$ implies $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. On the contrary, assume that $\mathbf{p} \neq \{1\}$. We show that $\mathbf{p}$ is $S$-prime. Let $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$. If $1 \neq a \vee b \in \mathbf{p}$, then $\mathbf{p}$ is weakly $S$-prime gives $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$.

Now, suppose that $a \vee b = 1$. Since $\mathbf{p} \neq \{1\}$, there exists $p' \in \mathbf{p}$ such that $p' \neq 1$. Then $1 \neq (a \wedge p') \vee (b \wedge p') = p' \in \mathbf{p}$ gives $s \vee (a \wedge p') = (s \vee a) \wedge (s \vee p') \in \mathbf{p}$ or $(s \vee b) \wedge (s \vee p') \in \mathbf{p}$. Therefore, $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$ by Lemma 1.1 (i). This shows that $\mathbf{p}$ is an $S$-prime filter, as required. □

Corollary 3.5. *If* $\mathbf{p}$ *is a weakly prime filter that is not prime, then*

$$\mathbf{p} = \{1\}.$$

*Proof.* Take $S = \{0\}$ in Theorem 3.4. □

Compare the next theorem with Theorem 7 in [1].

Theorem 3.6. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$*. The following assertions are equivalent:*

(i) $\mathbf{p}$ *is a weakly* $S$*-prime filter of* $\pounds$*;*

(ii) *There exists* $s \in S$ *such that for each* $x \notin (\mathbf{p} :_{\pounds} s)$ *we have either* $(\mathbf{p} :_{\pounds} x) \subseteq (\mathbf{p} :_{\pounds} s)$ *or* $(\mathbf{p} :_{\pounds} x) = (1 :_{\pounds} x)$*;*

(iii) *There exists* $s \in S$ *such that for all* $F$ *and* $G$*, two filters of* $\pounds$*, if* $\{1\} \neq F \vee G \subseteq \mathbf{p}$*, then* $s \vee F \subseteq \mathbf{p}$ *or* $s \vee G \subseteq \mathbf{p}$.

*Proof.* (i)⇒(ii) By assumption, there is an element $s \in S$ such that, whenever $x, y \in \pounds$, $x \vee y \in \mathbf{p}$ implies $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. Let $x \in \pounds \setminus (\mathbf{p} :_{\pounds} s)$ (so $x \vee s \notin \mathbf{p}$) and suppose that $(\mathbf{p} :_{\pounds} x) \neq (1 :_{\pounds} x)$. Since $(1 :_{\pounds} x) \subsetneqq (\mathbf{p} :_{\pounds} x)$, we conclude that there exists $a \in (\mathbf{p} :_{\pounds} x)$ such that $a \vee x \neq 1$. So $1 \neq a \vee x \in \mathbf{p}$ gives $a \vee s \in \mathbf{p}$, as $\mathbf{p}$ is weakly $S$-prime. Let $z \in (\mathbf{p} :_{\pounds} x)$. If $x \vee z \neq 1$, then $s \vee z \in \mathbf{p}$, and so $z \in (\mathbf{p} :_{\pounds} s)$.

Now, suppose that $x \vee z = 1$. Then $1 \neq a \vee x = (a \vee x) \wedge (x \vee z) = x \vee (a \wedge z) \in \mathbf{p}$ implies that $s \vee (a \wedge z) = (s \vee a) \wedge (s \vee z) \in \mathbf{p}$; hence $s \vee z \in \mathbf{p}$ by Lemma 1.1 (i). Thus $z \in (\mathbf{p} :_{\pounds} s)$, i.e. $(\mathbf{p} :_{\pounds} x) \subseteq (\mathbf{p} :_{\pounds} s)$.

(ii)⇒(i) Let $x, y \in \pounds$ such that $1 \neq x \vee y \in \mathbf{p}$ with $s \vee x \notin \mathbf{p}$; so $x \notin (\mathbf{p} :_{\pounds} s)$. Since $y \in (\mathbf{p} :_{\pounds} x)$ and $x \vee y \neq 1$, we conclude that $(\mathbf{p} :_{\pounds} x) \subseteq (\mathbf{p} :_{\pounds} s)$ by (ii); hence $y \vee s \in \mathbf{p}$. This shows that $\mathbf{p}$ is weakly $S$-prime.

(ii)⇒(iii) Assume that $F$ and $G$ are filters of $\pounds$ such that $F \vee G \subseteq \mathbf{p}$ and, for the element $s \in S$ of (ii), we have $s \vee F \nsubseteq \mathbf{p}$ and $s \vee G \nsubseteq \mathbf{p}$. We claim that $F \vee G = \{1\}$. Let $x \in F \setminus (\mathbf{p} :_{\pounds} s)$. Then $x \vee G \subseteq \mathbf{p}$ gives $G \subseteq (\mathbf{p} :_{\pounds} x)$. Since $G \nsubseteq (\mathbf{p} :_{\pounds} s)$, we conclude that $G \subseteq (\mathbf{p} :_{\pounds} x) = (1 :_{\pounds} x)$; so $x \vee G = \{1\}$.

Now, assume that $x \in F \cap (\mathbf{p} :_{\pounds} s)$. Let $z \in G$. If $z \notin (\mathbf{p} :_{\pounds} s)$ then, as previously, $z \vee F = \{1\}$, and so $z \vee x = 1$. So we may assume that $z \in (\mathbf{p} :_{\pounds} s)$. Consider $g \in G$ such that $s \vee g \notin \mathbf{p}$. Then $s \vee (z \wedge g) = (s \vee g) \wedge (s \vee z) \notin \mathbf{p}$, by Lemma 1.1 (i), which implies that

$$g \in (\mathbf{p} :_{\pounds} x) \setminus (\mathbf{p} :_{\pounds} s) \text{ and } z \wedge g \in (\mathbf{p} :_{\pounds} x) \setminus (\mathbf{p} :_{\pounds} s).$$

Hence, $x \vee g = 1$ and $1 = x \vee (z \wedge g) = (x \vee g) \wedge (x \vee z) = x \vee z$. Thus, $x \vee G = \{1\}$. This shows that $F \vee G = \{1\}$.

(iii)⇒(i) Let $x, y \in \pounds$ such that $1 \neq x \vee y \in \mathbf{p}$. Then

$$\{1\} \neq T(\{x\}) \vee T(\{y\}) \subseteq \mathbf{p}$$

gives $s \vee x \in s \vee T(\{x\}) \subseteq \mathbf{p}$ or $s \vee y \in s \vee T(\{y\}) \subseteq \mathbf{p}$ by (iii), and so we have $\mathbf{p}$ is a weakly $S$-prime filter of $\pounds$. □

Corollary 3.7. *For proper filter* $\mathbf{p}$ *of* $\pounds$*, The following assertions are equivalent:*

(i) $\mathbf{p}$ *is a weakly prime filter of* $\pounds$*;*

(ii) *For each* $x \notin \mathbf{p}$ *we have either* $(\mathbf{p} :_{\pounds} x) = \mathbf{p}$ *or* $(\mathbf{p} :_{\pounds} x) = (1 :_{\pounds} x)$*;*

(iii) *For filters* $F$ *and* $G$ *of* $\pounds$ *with* $\{1\} \neq F \vee G \subseteq \mathbf{p}$*, either* $F \subseteq \mathbf{p}$ *or* $G \subseteq \mathbf{p}$.

*Proof.* Take $S = \{0\}$ in Theorem 3.6. □

Proposition 3.8. *Assume that* $\mathbf{p}$ *is a filter of* $\pounds$ *and let* $S$ *be a join closed subset of* $\pounds$ *disjoint with* $\mathbf{p}$*. If* $\mathbf{q}$ *is a filter of* $\pounds$ *such that* $\mathbf{q} \cap S \neq \emptyset$ *and* $\mathbf{p}$ *is a weakly* $S$*-prime, then* $\mathbf{p} \vee \mathbf{q}$ *is a weakly* $S$*-prime filter of* $\pounds$.

*Proof.* Since $(\mathbf{p} \vee \mathbf{q}) \cap S \subseteq \mathbf{p} \cap S = \emptyset$, we have $(\mathbf{p} \vee \mathbf{q}) \cap S = \emptyset$. Consider $t \in \mathbf{q} \cap S$. Let $a, b \in \pounds$ such that $1 \neq a \vee b \in \mathbf{p} \vee \mathbf{q} \subseteq \mathbf{p}$. Then there exists $s \in S$ such that $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$ which gives $s \vee t \vee a \in \mathbf{p} \vee \mathbf{q}$ or $s \vee t \vee b \in \mathbf{p} \vee \mathbf{q}$, where $s \vee t \in S$, as desired. □

PROPOSITION 3.9. *Suppose that $S$ is a join closed subset of $\pounds$. The following assertions are equivalent:*

(i) *Every weakly $S$-prime filter of $\pounds$ is prime;*

(ii) *$\pounds$ is a $\pounds$-domain and every $S$-prime filter of $\pounds$ is prime.*

*Proof.* (i)⇒(ii) Since $\{1\}$ is a weakly $S$-prime filter, we conclude that it is a prime filter by (i) which implies that $\pounds$ is a $\pounds$-domain. Finally, since every $S$-prime filter $\mathbf{p}$ of $\pounds$ is weakly $S$-prime, we get $\mathbf{p}$ is prime by (i).

(ii)⇒(i) Suppose that $\mathbf{p}$ is a weakly $S$-prime filter; we show that $\mathbf{p}$ is $S$-prime. Let $a, b \in \pounds$ such that $a \vee b \in \mathbf{p}$. If $a \vee b \neq 1$, then there exists $s \in S$ such that $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$. If $a \vee b = 1$, then $a = 1$ or $b = 1$; so $s \vee a = 1 \in \mathbf{p}$ or $s \vee b = 1 \in \mathbf{p}$, for every $s \in S$. Consequently, every weakly $S$-prime filter of $\pounds$ is prime by (ii). □

We close this section with the investigation of the stability of weakly $S$-prime filters in various lattice-theoretic constructions.

THEOREM 3.10. *Suppose that $S$ is a join closed subset of $\pounds$. The following assertions are equivalent:*

(i) *$\pounds$ is $S$-Noetherian;*

(ii) *Every weakly $S$-prime filter of $\pounds$ is $S$-finite;*

(iii) *Every $S$-prime filter of $\pounds$ is $S$-finite;*

(iv) *Every prime filter of $\pounds$ is $S$-finite.*

*Proof.* (i)⇒(ii)⇒(iii) Obvious.

(iii)⇒(iv) follows from Theorem 2.22.

(iv)⇒(i) follows from Theorem 2.21. □

EXAMPLE 3.11. Let $S' \subseteq S$ be join closed subsets of $\pounds$ and $\mathbf{p}$ a filter of $\pounds$ disjoint with $S$. It s clear that if $\mathbf{p}$ is a weakly $S'$-prime filter of $\pounds$, then $\mathbf{p}$ is a weakly $S$-prime filter. However, the converse is not true in general. Indeed, assume that $\pounds$ is the lattice as in Example 2.2 (c) and let $S' = \{0\} \subseteq S = \{0, a\}$. Then $\mathbf{p} = \{1, c\}$ is a weakly $S$-prime filter of $\pounds$ but not a weakly $S'$-prime filter of $\pounds$.

PROPOSITION 3.12. *Let $S' \subseteq S$ be join closed subsets of $\pounds$ such that for any $s \in S$, there exists $t \in S$ satisfying $s \vee t \in S'$. If $\mathbf{p}$ is a weakly $S$-prime filter of $\pounds$, then $\mathbf{p}$ is a weakly $S'$-prime filter of $\pounds$.*

*Proof.* Let $x, y \in \pounds$ such that $1 \neq x \vee y \in \mathbf{p}$. Then there exists $s \in S$ such that $s \vee x \in \mathbf{p}$ or $s \vee y \in \mathbf{p}$. By the hypothesis, there is $t \in S$ such that $s \vee t \in S'$ and then $s \vee t \vee \vee x \in \mathbf{p}$ or $s \vee t \vee y \in \mathbf{p}$, as $\mathbf{p}$ is a filter. This shows that $\mathbf{p}$ is a weakly $S'$-prime filter. □

Let $\pounds$ be a lattice. If $x \in \pounds$, then a complement of $x$ in $\pounds$ is an element $y \in \pounds$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\pounds$ is complemented if every element of $\pounds$ has a complement in $\pounds$.

LEMMA 3.13. *Let $\pounds_1$ and $\pounds_2$ be lattices and $f : \pounds_1 \to \pounds_2$ be a lattice homomorphism such that $f(1) = 1$. The following hold:*

(i) $\mathrm{Ker}(f) = \{x \in \pounds_1 : f(x) = 1\}$ *is a filter of $\pounds_1$;*

(ii) *If $f$ is injective, then* $\mathrm{Ker}(f) = \{1\}$*;*

(iii) *If $\pounds_1$ is a complemented lattice, then $f$ is injective if and only if* $\mathrm{Ker}(f) = \{1\}$.

*Proof.* (i) Straightforward.

(ii) If $x \in \mathrm{Ker}(f)$, then $f(x) = 1 = f(1)$; so $x = 1$, as required.

(iii) One side is clear. To see the other side, let $a, b \in \pounds_1$ such that $f(a) = f(b)$. There exist $a', b' \in \pounds_1$ such that $a \vee a' = 1 = b \vee b'$ and $a \wedge a' = 0 = b \wedge b'$. Then $f(a \vee b') = f(b) \vee f(b') = 1$ gives $a \vee b' \in \mathrm{Ker}(f) = \{1\}$; so $a \vee b' = 1$. Similarly, $b \vee a' = 1$. This shows that $a = a \wedge (a' \vee b) = a \wedge b$ which implies that $a \leq b$. Similarly, $b \leq a$, as needed. □

Compare the next theorem with Proposition 22 in [1].

THEOREM 3.14. *Let $f : \pounds \to \pounds'$ be a lattice homomorphism such that $f(1) = 1$ and $S$ a join closed subset of $\pounds$. The following hold:*

(i) *Let $\pounds$ be a complemented lattice. If $f$ is a epimorphism and $\mathbf{p}$ is a weakly $S$-prime filter with* $\mathrm{Ker}(f) \subseteq \mathbf{p}$*, then $f(\mathbf{p})$ is a weakly $f(S)$-prime filter of $\pounds'$;*

(ii) *If $f$ is a monomorphism and $\mathbf{p}'$ is a weakly $f(S)$-prime filter of $\pounds'$, then $\mathbf{p} = f^{-1}(\mathbf{p}')$ is a weakly $S$-prime filter of $\pounds$.*

*Proof.* (i) Let $u \in f(S) \cap f(\mathbf{p})$. Then $u = f(p) = f(s)$ for some $p \in \mathbf{p}$ and $s \in S$.

By assumption, there exists $p' \in \pounds$ such that $p \vee p' = 1$ and $p \wedge p' = 0$ which gives $f(s \vee p') = f(p) \vee f(p') = 1$; hence $s \vee p' \in \mathrm{Ker}(f) \subseteq \mathbf{p}$. Since $s = s \vee (p \wedge p') = (s \vee p') \wedge (s \vee p) \in \mathbf{p}$, we conclude that $s \in S \cap \mathbf{p}$ is a contradiction.

Thus $f(S) \cap f(\mathbf{p}) = \emptyset$. Let $x, y \in \pounds'$ such that $1 \neq x \vee y \in f(\mathbf{p})$. Then there exist $a, b \in \pounds$ such that $x = f(a)$, $y = f(b)$ and $1 \neq f(a \vee b) = x \vee y \in f(\mathbf{p})$ (so $a \vee b \neq 1$) which implies that $f(a \vee b) = f(q)$ for some $q \in \mathbf{p}$.

By the hypothesis, $q \vee q' = 1$ and $q \wedge q' = 0$ for some $q' \in £$. Since $f(a \vee b \vee q') = 1$, we conclude that $a \vee b \vee q' \in \mathrm{Ker}(f) \subseteq \mathbf{p}$; hence

$$1 \neq a \vee b = (a \vee b) \vee (q \wedge q') = (a \vee b \vee q) \wedge (a \vee b \vee q') \in \mathbf{p},$$

as $\mathbf{p}$ is a filter. This implies that $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$ for some $s \in S$. It means that $f(s) \vee x \in f(\mathbf{p})$ or $f(s) \vee y \in f(\mathbf{p})$. Therefore, $f(\mathbf{p})$ is a weakly $f(S)$-prime filter of $£'$.

(ii) By assumption, there exists $s \in S$ such that for all $x, y \in £'$, $x \vee y \in \mathbf{p'}$ implies $f(s) \vee x \in \mathbf{p'}$ or $f(s) \vee y \in \mathbf{p'}$. Clearly, $\mathbf{p} \cap S = \emptyset$. Let $a, b \in £$ such that $1 \neq a \vee b \in \mathbf{p}$. Since $\mathrm{Ker}(f) = \{1\}$ by Lemma 3.13 (ii), we conclude that $1 \neq f(a \vee b) = f(a) \vee f(b) \in \mathbf{p'}$; so $f(s) \vee f(a) = f(s \vee a) \in \mathbf{p'}$ or $f(s) \vee f(b) = f(s \vee b) \in \mathbf{p'}$. Hence, $s \vee a \in \mathbf{p}$ or $s \vee b \in \mathbf{p}$, and so $\mathbf{p} = f^{-1}(\mathbf{p'})$ is a weakly $S$-prime filter of $£$. □

Corollary 3.15. *Let $S$ be a join closed subset of $£$. The following hold:*

(i) *If $F \subseteq H$ are two filters of $£$ and $H$ is a weakly $S$-prime filter of $£$, then $\frac{H}{F}$ is a weakly $Q_S$-prime of $\frac{£}{F}$;*

(ii) *If $£$ is a sublattice of $£'$ and $H'$ is a weakly $S$-prime filter of $£'$, then $H' \cap £$ is a weakly $S$-prime filter of $£$.*

*Proof.* (i) Let $f : £ \to \frac{£}{F}$ such that $f(a) = a \wedge F$. Then $f$ is a lattice homomorphism from $£$ onto $\frac{£}{F}$ and $f(1) = 1$. Suppose that $H$ is a weakly $S$-prime filter of $£$. Since $\mathrm{Ker}(f) = F \subseteq H$ and $f$ is onto, we conclude that $f(H) = \frac{H}{F}$ is a $Q_S$-prime filter of $\frac{£}{F}$ by Theorem 3.14 (i).

(ii) It suffices to apply Theorem 3.14 (ii) to the natural injection $\iota : £ \to £'$ since $\iota^{-1}(H') = H' \cap £'$. □

Theorem 3.16. *Let $£ = £_1 \times £_2$ be a decomposable lattice and $S = S_1 \times S_2$, where $S_i$ is a join closed subset of $£_i$. Suppose that $\mathbf{p} = \mathbf{p_1} \times \mathbf{p_2}$ is a filter of $£$, where $\mathbf{p_1} \neq \{1\}$ and $\mathbf{p_2} \neq \{1\}$. The following statements are equivalent:*

(i) *$\mathbf{p}$ is a weakly $S$-prime filter of $£$;*

(ii) *$\mathbf{p_1}$ is an $S_1$-prime filter of $£_1$ and $\mathbf{p_2} \cap S_2 \neq \emptyset$ or $\mathbf{p_2}$ is an $S_2$-prime filter of $£_2$ and $\mathbf{p_1} \cap S_1 \neq \emptyset$,*

(iii) *$\mathbf{p}$ is an $S$-prime filter of $£$.*

*Proof.* (i)⇒(ii) Let $(1, 1) \neq (a, b) \in \mathbf{p}$. Then $(1, 1) \neq (a, 0) \vee_c (0, b) \in \mathbf{p}$. So there exists $s = (s_1, s_2) \in S$ such that $s \vee_c (a, 0) = (s_1 \vee a, s_2) \in \mathbf{p}$ or $s \vee_c (0, b) = (s_1, s_2 \vee b) \in \mathbf{p}$ which implies that $S_1 \cap \mathbf{p_1} \neq \emptyset$ or $S_2 \cap \mathbf{p_2} \neq \emptyset$. Suppose that $S_2 \cap \mathbf{p_2} \neq \emptyset$. Since $S \cap \mathbf{p} = \emptyset$, we have $S_1 \cap \mathbf{p_1} = \emptyset$.

Now, we claim that $\mathbf{p_1}$ is an $S_1$-prime filter of $£_1$. Let $x, y \in £_1$ such that $x \vee y \in \mathbf{p_1}$. Consider $1 \neq t \in S_2 \cap \mathbf{p_2}$. Then $(1, 1) \neq (x, t) \vee_c (y, 0) = (x \vee y, t) \in \mathbf{p}$ gives $s \vee_c (x, t) = (s_1 \vee x, s_2 \vee t) \in \mathbf{p}$ or $s \vee_c (y, 0) = (s_1 \vee y, s_2) \in \mathbf{p}$ which implies that $s_1 \vee x \in \mathbf{p_1}$ or $s_1 \vee y \in \mathbf{p_1}$, as needed.

(ii)⇒(iii) Follows directly from Proposition 2.29.

(iii)⇒(i) Obvious. □

Corollary 3.17. *Let £ be a decomposable lattice and $S$ a join closed subset of £. A proper filter* $\mathbf{p}$ *of £ disjoint with $S$ is weakly $S$-prime if and only if* $\mathbf{p} = \{1\}$ *or* $\mathbf{p}$ *is $S$-prime.*

*Proof.* This follows from Theorem 3.16. □

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Received August 18, 2023
Accepted February 24, 2024

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