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A LIOUVILLE THEOREM FOR F-HARMONIC MAPS WITH MODERATE DIVERGENT ENERGY

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Abstract. In this paper, we obtain a Liouville theorem of F-harmonic maps between complete Riemannian manifolds with moderate divergent F-energy. We assume that F is a concave function and satisfies a differential inequality. We employ Ara's F-stress-energy tensor and the Hessian comparison theorem to prove the main result.

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Key words. Riemannian manifold, F-harmonic maps, Liouville theorem, stress-energy tensor.

1. INTRODUCTION

Harmonic maps [5, 11] are critical points of the energy functional defined on the space of smooth maps between Riemannian manifolds. In [1], the author introduced the theory of F-harmonic maps to unify various types of harmonic map, e.g. p-harmonic maps and exponentially harmonic maps. Liouville type properties of harmonic maps were studied by several authors.(cf. [3, 8, 9] and the reference therein). In this paper, we study Liouville type properties of F-harmonic maps with moderate divergent F-energy (see Definition 2.5). That is, we study conditions for which a F-harmonic map with moderate divergent F-energy u between two Riemannian manifolds (M, g) and (N, h) is a constant map. In [11, p. 46], the author obtained the following

THEOREM 1.1 ([11]). *Let M be a Cartan-Hadamard manifold of dimension m whose sectional varies in a small range, and u a harmonic map from M into any Riemannian manifold N with moderate divergent energy. If the dimension m of the domain manifold is greater than 2, then u has to be constant.*

Recall that a Cartan-Hadamard manifold is a complete simply-connected, non-positive sectional curvature Riemannian manifold. We want to know whether for F-harmonic map, we can have similar Liouville type properties. In [4], the authors performed deep analysis about the growth rate of F-energy and used asymptotic assumption of the map at infinity to obtain a series of

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Liouville theorems under various conditions e.g. pinching of radial curvature of the domain manifold. In [10], the author obtained some Liouville theorems under finiteness F-energy condition. To the best of our knowledge, Liouville theroem for F-harmonic maps with the moderate energy divergent condition (In the sense of Xin's works) have not appeared in the literature. A key ingredient in Xin's proof is the conservation law of u in terms of the the vanishing of the divergence of the stress energy tensor of harmonic map. Therefore, we need to find a conservation law for F-harmonic map with a suitably defined F-stress energy tensor. It is known that the Ara's stress-energy tensor (See (7)) for F-harmonic map is conserved [1] and we shall use this tensor in our study. Our main theorem is the following

THEOREM 1.2. *Let $(M^m, g), m > 2$ be a complete, simply connected with nonnegative sectional curvature Riemannian manifold which has a pole and (N^n, h) be any Riemannian manifold. Let $u : (M^m, g) \to (N^n, h)$ be F-harmonic map where $F : (0, \infty) \to [0, \infty)$ is a C^2 function such that $F(0) = 0$, $F' > 0$ on $(0, \infty)$ and F is a concave function satisfying the differential inequality ,*

$$tF'(t)/F(t) \leq C \tag{1}$$

where C is a positive constant. Assume that the sectional curvature of (M^m, g) varies in a small range, more precisely, we assume that the radial curvature k of the domain manifold M satisfies $-c_1^2 \leq k \leq -c_2^2 < 0$, where c_1, c_2 are positive constants, and with moderate divergent F-energy. Then the map u has to be constant.

To replace the condition finiteness of F-energy with the condition of moderate divergent F-energy, we assume that the function $F(t)$ is concave and satisfies a differential inequality. To prove the main result, we firstly obtain an identity about the divergence of the F-stress energy tensor and take integration to obtain Lemma 2.4; then we apply Hessian comparison theorem to get some key estimates. Throughout this paper, we will adopt Einstein summation convention: sum on repeated indices. This paper is organized as follows, in Section 2, we recall definition of F-harmonic maps, conservation law and stress energy tensor. In Section 3, we present the proof of the main results.

2. PRELIMINARIES

2.1. F-ENERGY AND F-HARMONIC MAPS

Let $F : \mathbb{R} \to \mathbb{R}$ be a C^∞ function. For a smooth map u between Riemannian manifolds (M, g) and (N, h), the F-energy $E_F(u)$ is defined by

$$E_F(u) = \int_M F\left(\frac{|\mathrm{d}u|^2}{2}\right) * 1. \tag{2}$$

While let us recall the energy of $E(u)$ is defined by

$$E(u) = \int_M \left(\frac{|\mathrm{d}u|^2}{2}\right) * 1 = \int_M \left(\frac{1}{2} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} g^{ij}\right) * 1. \tag{3}$$

Here, the volume element

$$*1 = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m. \tag{4}$$

A smooth map $u : (M, g) \to (N, h)$ is said to be F–harmonic if $u|_\Omega$ is a critical point of the F–energy $E_f(u)$ for every compact domain $\Omega \in M$.

2.2. CONSERVATION LAW FOR F-HARMONIC MAP

Let $u : (M, g) \to (N, h)$ be a smooth map. Let $\mathcal{E}$ and u^*h denote the energy density and the first fundamental form, respectively. Let $\mathcal{E}(u) := \frac{|\mathrm{d}u|^2}{2}, \mathcal{E}_F(u) := F(\frac{|\mathrm{d}u|^2}{2})$. For harmonic maps, the stress-energy(SE) tensor S_u is defined by

$$S_u = \mathcal{E}(u)g - u^*h, \tag{5}$$

which is a symmetric 2-tensor. Here, $\mathcal{E}(u)$ denotes the energy density function. A detailed treatment on stress-energy tensor of harmonic map can be found in [2, 5]. For harmonic maps between manifolds, the basic relation between the stress-energy tensor and harmonic maps is the following [5]

Proposition 2.1. $\mathrm{div} S_u = -\langle \tau_u, \mathrm{d}u \rangle$.

Definition 2.2. Let $u : (M^m, g) \to (N^n, h)$ be any smooth map. If u satisfies $\mathrm{div} S_u \equiv 0$, then we say that the map u satisfies the conservation law.

The Euler-Lagrange equation of F-harmonic map gives us,

$$\tau_u^F = F'(\mathcal{E}(u))\tau_u + u_* \left(\mathrm{grad}(F'(\mathcal{E}(u)))\right). \tag{6}$$

and we have the following F-stress-energy tensor [1],

$$S_u^F = F(\mathcal{E}(u))g - F'(\mathcal{E}(u))u_*h. \tag{7}$$

Then, it holds that

$$\mathrm{div} S_u^F = -\langle \tau_u^F, \mathrm{d}u \rangle. \tag{8}$$

Remark 2.3. We say that the F-stress energy tensor is conserved or satisfies a conservation law since it satisfies the equation(8).

Lemma 2.4. *Assume that $D \Subset M$ is a geodesic ball and its boundary ∂D is a geodesic sphere, and u is F-harmonic. Choose a local orthonormal frame field of M along ∂D such that $f_1, \ldots, f_{m-1} \in \Gamma(T\partial D)$, f_m is the normal vector of ∂D, $\mathbf{n} := f_m$, we have*

$$\int_{\partial D} \left(F(\mathcal{E}(u))\langle X, \mathbf{n}\rangle - F'(\mathcal{E}(u))\langle u_*X, u_*\mathbf{n}\rangle\right) * 1 = \int_D \langle S_u^F, \nabla X\rangle * 1, \tag{9}$$

where

$$\langle S_u^F, \nabla X\rangle = F(\mathcal{E}(u))\mathrm{div}X - F'(\mathcal{E}(u))\langle u_* e_i, u_* e_j\rangle_h \langle \nabla_{e_i} X, e_j\rangle_g. \tag{10}$$

Proof. Let $\{e_i\}$ be an orthonormal frame near a point $p \in M$ such that $\nabla_{e_i} e_j|_p = 0$. Let $X \in \Gamma(TM)$ be any vector field on M, we have

$$\begin{aligned} \mathrm{div}(F(\mathcal{E}(u)X) &= X(F(\mathcal{E}(u))) + F(\mathcal{E}(u))\mathrm{div}X \\ &= F'(\mathcal{E}(u))X(\mathcal{E}(u)) + F(\mathcal{E}(u))\mathrm{div}X \\ &= F'(\mathcal{E}(u))X(\mathcal{E}(u)) + \langle \nabla X, F(\mathcal{E}(u))g\rangle, \end{aligned} \tag{11}$$

where we use the notation

$$\nabla X(e_i, e_j) := \langle \nabla_{e_i} X, e_j\rangle. \tag{12}$$

In the following, we will use $\mathrm{d}u(e_i) = u_* e_i$ interchangeably for simplicity. We compute the first term in (11) as follows,

$$\begin{aligned} F'(\mathcal{E}(u))X(\mathcal{E}(u)) &= F'(\mathcal{E}(u))\frac{1}{2}\nabla_X \langle u_* e_i, u_* e_i\rangle \\ &= F'(\mathcal{E}(u))\langle \nabla_X u_* e_i, u_* e_i\rangle \\ &= F'(\mathcal{E}(u))\langle (\nabla_X \mathrm{d}u)e_i, u_* e_i\rangle \\ &= F'(\mathcal{E}(u))\langle (\nabla_{e_i} \mathrm{d}u)X, u_* e_i\rangle \\ &= \langle (\nabla_{e_i} \mathrm{d}u)X, F'(\mathcal{E}(u))u_* e_i\rangle \\ &= \langle \nabla_{e_i}(\mathrm{d}u(X)), F'(\mathcal{E}(u))u_* e_i\rangle - \langle \mathrm{d}u(\nabla_{e_i} X), F'(\mathcal{E}(u))u_* e_i\rangle \\ &= \nabla_{e_i}\langle \mathrm{d}u(X), F'(\mathcal{E}(u))u_* e_i\rangle - \langle \mathrm{d}u(X), \nabla_{e_i}(F'(\mathcal{E}(u))u_* e_i)\rangle \\ &\quad - \langle \mathrm{d}u(\nabla_{e_i} X), F'(\mathcal{E}(u))u_* e_i\rangle \\ &= \nabla_{e_i}\langle \mathrm{d}u(X), F'(\mathcal{E}(u))u_* e_i\rangle - \langle \mathrm{d}u(X), \nabla_{e_i}(F'(\mathcal{E}(u))u_* e_i)\rangle \\ &\quad - F'(\mathcal{E}(u))\langle \mathrm{d}u(\nabla_{e_i} X), u_* e_i\rangle \\ &= \nabla_{e_i}\langle \mathrm{d}u(X), F'(\mathcal{E}(u))u_* e_i\rangle - \langle \mathrm{d}u(X), \tau_u^F\rangle \\ &\quad - F'(\mathcal{E}(u))\langle \mathrm{d}u(\nabla_{e_i} X), u_* e_i\rangle \\ &= \mathrm{div}\left(F'(\mathcal{E}(u))\langle u_* X, u_* e_i\rangle e_i\right) - \langle u_* X, \tau_u^F\rangle \\ &\quad - F'(\mathcal{E}(u))\langle \nabla_{e_i} X, e_j\rangle \langle u_* e_i, u_* e_j\rangle \\ &= \mathrm{div}\left(F'(\mathcal{E}(u))\langle u_* X, u_* e_i\rangle e_i\right) - \langle u_* X, \tau_u^F\rangle \\ &\quad - F'(\mathcal{E}(u))\langle \nabla X, u^* h\rangle, \end{aligned}$$

substituting this into (11), we have

$$\begin{aligned} \mathrm{div}(F(\mathcal{E}(u)X) &= \mathrm{div}\left(F'(\mathcal{E}(u))\langle u_* X, u_* e_i\rangle e_i\right) \\ &\quad - \langle u_* X, \tau_u^F\rangle - F'(\mathcal{E}(u))\langle \nabla X, u^* h\rangle + \langle \nabla X, F(\mathcal{E}(u))g\rangle, \\ &= \mathrm{div}\left(F'(\mathcal{E}(u))\langle u_* X, u_* e_i\rangle e_i\right) \\ &\quad - \langle u_* X, \tau_u^F\rangle + \langle S_u^F, \nabla X\rangle. \end{aligned} \tag{13}$$

As $D \Subset M$ is a geodesic ball and its boundary is a geodesic sphere, and u is a F-harmonic map and $\{e_i\}$ be an orthonormal frame near a point $p \in \partial D$ such that $e_1, \dots, e_{m-1} \in \Gamma(T\partial D)$, e_m is the normal vector of ∂D, $\mathbf{n} := e_m$ and $\nabla_{e_i} e_j|_p = 0$, taking integration on D for (13), we obtain

$$\int_{\partial D} \big(F(\mathcal{E}(u))\langle X, \mathbf{n}\rangle - F'(\mathcal{E}(u))\langle u_* X, u_* \mathbf{n}\rangle)\big) * 1 = \int_D \langle S_u^F, \nabla X\rangle * 1.$$

The proof is completed. □

We modify the definition of the condition of moderate divergent energy in [11] for F-energy to obtain the following

Definition 2.5. If there is a positive function $\eta(r)$ and $R_0 > 0$ such that

$$\int_{R_0}^{\infty} \frac{\mathrm{d}r}{r\eta(r)} = \infty,$$

then we have

$$\lim_{R\to\infty} \int_{B_R(x_0)} \frac{F(\mathcal{E}(u))(x)}{\eta(r(x))} * 1 < \infty.$$

Then, we say that u is moderate F-energy divergent.

3. PROOF OF MAIN RESULTS

In this section, we will give the proof of Theorem 1.2.

Proof. Let $D = B_R(x_0)$ be a geodesic ball of radius R and centered at x_0 and its boundary is the geodesic sphere $\partial B_R(x_0)$. It is clear that the square of the distance function from x_0 in $B_R(x_0)$ is smooth. Let $\frac{\partial}{\partial r}$ be the unit radial vector field which is also the unit normal vector field $\mathbf{n}$ to $\partial B_R(x_0)$. Take $X = r\frac{\partial}{\partial r}$ in (9). L.H.S of (9) becomes

$$\begin{aligned}
&\int_{\partial B_R(x_0)} \big(F(\mathcal{E}(u))\langle X, \mathbf{n}\rangle - F'(\mathcal{E}(u))\langle u_* X, u_* \mathbf{n}\rangle)\big) * 1 \\
&= \int_{\partial B_R(x_0)} RF(\mathcal{E}(u)) * 1 - \int_{\partial B_R(x_0)} F'(\mathcal{E}(u)) R\langle u_* \frac{\partial}{\partial r}, u_* \frac{\partial}{\partial r}\rangle) * 1 \\
&\le R \int_{\partial B_R(x_0)} F(\mathcal{E}(u)) * 1.
\end{aligned}$$

We want to obtain a lower bound of the R.H.S. of (9), to this end, we compute the following first,

$$\nabla_{\frac{\partial}{\partial r}} X = \nabla_{\frac{\partial}{\partial r}} (r\frac{\partial}{\partial r}) = \frac{\partial}{\partial r}, \tag{14}$$

$$\begin{aligned}
\nabla_{e_s} X &= \nabla_{e_s}(r\frac{\partial}{\partial r}) = r\nabla_{e_s}\frac{\partial}{\partial r} \\
&= r\mathrm{Hess}(r)(e_s, e_t)e_t,
\end{aligned} \tag{15}$$

$$\mathrm{div} X = 1 + r\mathrm{Hess}(r)(e_s, e_s), \tag{16}$$

where $\{e_\alpha\} := \left\{e_s, \frac{\partial}{\partial r}\right\}$ is a local orthonormal frame near x_0 on $B_R(x_0)$. Using (15),(16), we have

$$
\begin{aligned}
&\langle u_* e_\alpha, u_* e_\beta\rangle\langle\nabla_{e_\alpha}X, e_\beta\rangle\\
&= \langle u_* e_s, u_* e_t\rangle\langle\nabla_{e_s}X, e_t\rangle + \langle u_*\frac{\partial}{\partial r}, u_* e_t\rangle\langle\nabla_{e_s}X, e_t\rangle\\
&+ \langle u_* e_s, u_*\frac{\partial}{\partial r}\rangle\langle\nabla_{e_s}X, \frac{\partial}{\partial r}\rangle + \langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle\langle\nabla_{\frac{\partial}{\partial r}}X, \frac{\partial}{\partial r}\rangle\\
&= \langle u_* e_s, u_* e_t\rangle\langle\nabla_{e_s}X, e_t\rangle + \langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle\\
&= r\mathrm{Hess}(r)(e_s, e_t)\langle u_* e_s, u_* e_t\rangle + \langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle.
\end{aligned}
\tag{17}
$$

Using (10),(17), we obtain

$$
\begin{aligned}
\langle S_u^F, \nabla X\rangle &= F(\mathcal{E}(u))\mathrm{div}X - F'(\mathcal{E}(u))\langle u_* e_\alpha, u_* e_\beta\rangle_h\langle\nabla_{e_\alpha}X, e_\beta\rangle_g\\
&= F(\mathcal{E}(u))(1 + r\mathrm{Hess}(r)(e_s, e_s))\\
&- F'(\mathcal{E}(u))\left(r\mathrm{Hess}(r)(e_s, e_t)\langle u_* e_s, u_* e_t\rangle + \langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle\right).
\end{aligned}
\tag{18}
$$

We consider the case that when the radial curvature k of the domain manifold M satisfies $-c_1^2 \leq k \leq -c_2^2 < 0,$ where c_1, c_2 are positive constants. Using the Hessian comparison theorem [7], we see that (18) becomes

$$
\begin{aligned}
&\langle S_u^F, \nabla X\rangle \geq F(\mathcal{E}(u))[1 + (m-1)(c_2 r)\coth(c_2 r)]\\
&- F'(\mathcal{E}(u))\left|u_*\frac{\partial}{\partial r}\right|^2 - F'(\mathcal{E}(u))(c_1 r)\coth(c_1 r)\langle u_* e_s, u_* e_s\rangle\\
&= F\left(\frac{1}{2}\langle u_* e_s, u_* e_s\rangle + \frac{1}{2}\langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle\right)[1 + (m-1)(c_2 r)\coth(c_2 r)]\\
&- F'(\mathcal{E}(u))\left|u_*\frac{\partial}{\partial r}\right|^2 - F'(\mathcal{E}(u))(c_1 r)\coth(c_1 r)\langle u_* e_s, u_* e_s\rangle\\
&\geq \left\{F\left(\frac{1}{2}\langle u_* e_s, u_* e_s\rangle\right) + F\left(\frac{1}{2}\langle u_*\frac{\partial}{\partial r}, u_*\frac{\partial}{\partial r}\rangle\right)\right\}\\
&[1 + (m-1)(c_2 r)\coth(c_2 r)]\\
&- F'(\mathcal{E}(u))\left|u_*\frac{\partial}{\partial r}\right|^2 - F'(\mathcal{E}(u))(c_1 r)\coth(c_1 r)\langle u_* e_s, u_* e_s\rangle\\
&\geq [F(\frac{1}{2}\langle u_* e_s, u_* e_s\rangle) + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)]\\
&[1 + (m-1)(c_2 r)\coth(c_2 r)]\\
&- F'(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)(\left|u_*\frac{\partial}{\partial r}\right|^2)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&- F'(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle)(c_1 r)\coth(c_1 r)(\langle u_*e_s, u_*e_s\rangle) \\
&\geq F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) + r\coth(c2r)[(m-1)c_2 F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) \\
&- \langle u_*e_s, u_*e_s\rangle c_1 F'(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle)] + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2) \\
&- F'(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)(\left|u_*\frac{\partial}{\partial r}\right|^2) \\
&\geq B_1[F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)]
\end{aligned}
$$

where B_1 is a positive constant. Using (9) and (19), we obtain

$$
\begin{aligned}
&R\int_{\partial B_R(x_0)} F(\mathcal{E}(u)) * 1 \\
&\geq \int_{B_R(x_0)} B_1[F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)] * 1.
\end{aligned} \tag{20}
$$

We will finish the proof by contradiction. If u is not a constant map then $F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)$ is not identically equal to zero. There exists $R_0 > 0$, such that whenever $R > R_0$, we have

$$
\int_{B_R(x_0)} [F(\frac{1}{2}\langle u_*e_s, u_*e_s\rangle) + F(\frac{1}{2}\left|u_*\frac{\partial}{\partial r}\right|^2)] * 1 \geq B_2, \tag{21}
$$

where B_2 is a positive constant. Combining (20) and (21), we have

$$
\int_{\partial B_R(x_0)} F(\mathcal{E}(u)) * 1 \geq \frac{B_1 B_2}{R}. \tag{22}
$$

As we assume that u is moderate F-energy divergent, we recall that if there is a positive function $\eta(r)$ and $R_0 > 0$ such that

$$
\int_{R_0}^{\infty} \frac{\mathrm{d}r}{r\eta(r)} = \infty, \tag{23}
$$

then we have

$$
\lim_{R\to\infty} \int_{B_R(x_0)} \frac{F(\mathcal{E}(u))(x)}{\eta(r(x))} * 1 < \infty. \tag{24}
$$

Now, by using (24), (22) implies that

$$
\begin{aligned}
\lim_{R\to\infty} \int_{B_R(x_0)} \frac{F(\mathcal{E}(u))(x)}{\eta(r(x))} * 1 &= \int_0^{\infty} \frac{\mathrm{d}r}{\eta(r)} \int_{B_R(x_0)} F(\mathcal{E}(u)) * 1 \\
&\geq B_1 B_2 \int_0^{\infty} \frac{\mathrm{d}r}{r\eta(r)} \\
&\geq B_1 B_2 \int_{R_0}^{\infty} \frac{\mathrm{d}r}{r\eta(r)} = \infty.
\end{aligned}
$$

This contradicts to the fact that u is moderate F-energy divergent. We conclude that u has to be a constant map. □

COROLLARY 3.1. *Assume that* $(M^m, g), m > 2$ *is a complete, simply connected with nonnegative sectional curvature Riemannian manifold which has a pole and* (N^n, h) *be any Riemannian manifold. Let* $u : (M^m, g) \to (N^n, h)$ *be* F*-harmonic map where* $F : (0, \infty) \to [0, \infty)$ *is a* C^2 *function such that* $F(0) = 0$*,* $F' < 0$ *Assume that the sectional curvature* k *of* (M^m, g) *satisfies:*$-c_1^2 \leq k \leq -c_2^2 < 0$*, where* c_1, c_2 *are positive constants., with moderate divergent energy. Then* u *has to be constant.*

Proof. The setup is similar to that of the previous proof. As before, using the Hessian comparison theorem, we see that (18) becomes

$$\begin{aligned}
&\langle S_u^F, \nabla X\rangle \geq F(\mathcal{E}(u))[1 + (m-1)(c_2 r)\coth(c_2 r)] \\
&- F'(\mathcal{E}(u)) \left| u_* \frac{\partial}{\partial r} \right|^2 - F'(\mathcal{E}(u))(c_1 r)\coth(c_1 r)\langle u_* e_s, u_* e_s\rangle \\
&\geq CF(\mathcal{E}(u)).
\end{aligned}$$

The remaining steps are similar to the proof of Theorem 1.2 and we omit them. □

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