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ON THE REALIZABILITY OF THE SPECIAL LINEAR GROUP OVER THE MULTIVARIATE LAURENT POLYNOMIAL RING

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Abstract. We present an algorithm to determine the realization of matrices in SL_2 over the multivariate Laurant polynomial ring $R[x_1^{\pm}, ..., x_k^{\pm}]$. For this we have to generalize Park's algorithm. Thus the purpose is to express a matrix in $\mathrm{SL}_2(R[x_1^{\pm}, ..., x_k^{\pm}])$ as a product of elementary matrices. Furthermore, we extend this algorithm to some specific matrices in $\mathrm{SL}_3(R[x_1^{\pm}, ..., x_k^{\pm}])$. To illustrate our results, some examples are studied using special software which implements the proposed algorithms.

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Key words. Suslin's stability theorem, Laurent polynomial ring, monomial orders, Park's algorithm, S-pairs.

1. INTRODUCTION

Let R be a field, we consider the special linear group $\mathrm{SL}_n(R[x_1, .., x_k])$. It is proved that for $n \geq 3$ every matrix in $\mathrm{SL}_n(R[x_1, .., x_k])$ can be written as a product of elementary matrices [7], i.e., realizable; which led to the following theorem.

THEOREM 1.1 (Suslin's stability theorem [7]). *If R is a discrete field and $n \geq 3$, then every matrix in $\mathrm{SL}_n(R[x_1, .., x_k])$ is realizable, i.e.,*

$$\mathrm{SL}_n\left(R\left[x_1, \ldots, x_k\right]\right) = E_n\left(R\left[x_1, \ldots, x_k\right]\right).$$

But it is not the case when $k = n = 2$. A counterexample was established by P. M. Cohn in [3], i.e., the Cohn matrix,

$$C = \begin{pmatrix} 1 + xy & x^2 \\ -y^2 & 1 - xy \end{pmatrix}.$$

This matrix lies in $\mathrm{SL}_2(R[x, y])/E_2(R[x, y])$, thus it is nonrealizable.

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The question is, can these results remain valid for a wider class of the Laurent-polynomial ring $R[x_1^{\pm},..,x_k^{\pm}]$?

The answer is: for $n \geq 3$, Suslin proved that Theorem 1.1 can be extended to the Laurent polynomial ring so we state the following corollary.

COROLLARY 1.2. *If R is a field, then for $n \geq 3$ the group* $\mathrm{SL}_n(R[x_1^{\pm},..,x_k^{\pm}])$ *is generated by elementary matrices* [7].

Now for $n = 2$, the Cohn matrix is no longer a problem, it is proved to be realizable over the Laurent polynomial ring and an explicit factorization is given by Tolhuizen, Hollmann, Kalker in [8];

$$C = \begin{pmatrix} 1 & 0 \\ -\frac{y}{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{x}{y} & 1 \end{pmatrix}.$$

In the same article, they developed a realization algorithm and considered the following matrix to be nonrealizable over $\mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$,

$$A = \begin{pmatrix} 1 + (x+y)(x-y) & (x+y)^2 \\ (x-y)^2 & 1 + (x+y)(x-y) \end{pmatrix}.$$

In [5] it is proved that this conjecture is false by providing an explicit factorization of the matrix A into elementary matrices over $\mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$, by extending the realization algorithm of $\mathrm{SL}_2(R[x,y])$ proposed in the same article.

$$A = E_{12}(1)E_{21}\left(\frac{y}{2x}\right)E_{21}(-2)E_{21}\left(-x^2\right)E_{21}(2)E_{21}\left(-\frac{y}{2x}\right)E_{12}(-1).$$

Although this algorithm provides a factorization for the above matrix it cannot give a solution to every matrix in $\mathrm{SL}_2(R[x_1,..,x_k])$ so we do not have a complete solution to the problem.

The realization over the Laurent polynomial ring is an interesting issue, especially in signal processing, since many problems in this field can be expressed as Laurent polynomial matrices [4, 5, 8], so based on the previous research [1, 2, 5, 6, 9] we construct a new realization algorithm over $\mathrm{SL}_2(R[x_1^{\pm},..,x_k^{\pm}])$ using S-pairs with the appropriate monomial order.

Our work is divided into three parts. The first section is devoted to preliminaries, the second is about the realization of matrices over $\mathrm{SL}_2(R[x_1^{\pm},..,x_k^{\pm}])$; and then, in the third section, we extend our algorithm for matrices of the special form,

$$A = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ p & q & 1 \end{pmatrix} \in \mathrm{SL}_3(R[x_1^{\pm},..,x_k^{\pm}]).$$

2. PRELIMINARIES

Definition 2.1.

(1) $\mathrm{SL}_n(R)$ is the set of $n \times n$ matrices of determinant 1 whose entries are elements of R.

(2) An elementary matrix E_{ij} over R is defined as follows:

$$E_{ij}(a) = I_n + ae_{ij}, \text{ where } e_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{otherwise} \end{cases}$$

I_n is the identity matrix.

(3) $A \in \mathrm{SL}_n(R)$ is called realizable if it can be written as a product of elementary matrices.

Definition 2.2. A monomial ordering is a relation on $Z^n_{\geq 0}$ that verifies

(1) The relation $\geq$ is a total ordering;

(2) If $\alpha \geq \beta$, and $\gamma \in Z^n_{\geq 0}$ then $\alpha + \gamma \geq \beta + \gamma$;

(3) The relation $\geq$ is a well-ordering.

There are several term orderings. We list two of the most commonly used monomial orders. For $\alpha, \beta \in Z^n_{\geq 0}$:

Lexicographic ("dictionary"): Here $\alpha \geq_{\text{lex}} \beta$ if the left-most nonzero entry of $\alpha - \beta$ is positive. We write $x^\alpha \geq x^\beta$. The power of the first variable is used to determine the order.

Degree Lexicographic: Sort first by the total degree, then by the lexicographic degree. Here $\alpha \geq_{\text{dlex}} \beta$ if

$$|\alpha| := \sum_{k=1}^{n} \alpha_k \geq |\beta| := \sum_{k=1}^{n} \beta_k \text{ or } |\alpha| = |\beta| \text{ and } \alpha \geq_{\text{lex}} \beta.$$

Example 2.3. Consider the monomials $a = x^2y^2z^8$ and $b = x^3y^7z$. If the variables are ordered as $x > y > z$, then $a \geq_{\text{dlex}} b$ and $b \geq_{\text{lex}} a$.

Once a monomial order is given, we can talk about the leading monomial. It should be noted that if we change the monomial order, then we may have different leading terms for the same polynomial.

Definition 2.4. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $R[x]$ with $x := (x_1 ... x_n)$, $\alpha := (\alpha_1 ... \alpha_n) \in N^n$, $x^\alpha := (x_1^{\alpha_1} ... x_n^{\alpha_n})$ and let $\geq$ be a monomial order.

(1) The multidegree of the polynomial f is an element of N^n given by

$$\text{multideg}(f) = \max\{\alpha \in N^n : a_\alpha \neq 0\}.$$

(2) The leading coefficient of the polynomial f is an element of R given by

$$\text{LC}(f) = a_{\text{multideg}(f)}.$$

(3) The leading monomial of the polynomial f is a monomial of $R[x]$ given by

$$\mathrm{LM}(f) = x^{\mathrm{multideg}(f)}.$$

(4) The leading term of the polynomial f is a monomial of $R[x]$ given by;

$$\mathrm{LT}(f) = \mathrm{LC}(f)\mathrm{LM}(f).$$

Definition 2.5. For a fixed monomial order on $R[x]$, let $A \in \mathbb{M}_n(R[x])$. We define the matrix of its leading terms as $\mathrm{LT}(A) := (\mathrm{LT}(a_{ij}))$.

Example 2.6. Consider the polynomial $P = 2x^3y + x^2z + y^3x + z^4$. By fixing the lexicographic order and with the following order of variables we obtain:

$$\begin{aligned} x > y > z &\to \mathrm{LT}(P) = 2x^3y, \\ y > x > z &\to \mathrm{LT}(P) = y^3x, \\ z > x > y &\to \mathrm{LT}(P) = z^4. \end{aligned}$$

Notice that a particular order of the variables is assumed, by changing it, we obtain $n!$ nonequivalent lexicographic orderings. Since we are using the lexicographic order, we have to choose a proper variable order, in our case, the order is chosen as follows.

Proposition 2.7. *Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $R[x]$. For $i \neq j$, if $\sum_\alpha \deg(x_i^{\alpha_i}) \geq \sum_\alpha \deg(x_j^{\alpha_j})$ we choose the following order of variables $x_i > x_j$.*

By fixing the lexicographic order and applying Proposition 2.7 to Example 2.6 we obtain $x > z > y$ thus $\mathrm{LT}(P) = 2x^3y$.

Proposition 2.8. *Let $A \in \mathbb{M}_n(R[x])$, in this case, for $i \neq j$, if*

$$\sum_{k,l=1}^{n} \left(\sum_\alpha \deg(x_i^{\alpha_i}) \right)_{k,l} \geq \sum_{k,l=1}^{n} \left(\sum_\alpha \deg(x_j^{\alpha_j}) \right)_{k,l}$$

we choose the following order of variables $x_i > x_j$.

Example 2.9. Let

$$A = \begin{pmatrix} -\dfrac{y^2}{x} + \dfrac{1}{x} + \dfrac{y}{x^2} & 1 - \dfrac{y}{x} + \dfrac{1}{x^2} \\ xy - xy^3 & x + yx^2 - xy^2 \end{pmatrix}.$$

Based on Proposition 2.8 the order of variables is $y > x$ thus

$$\mathrm{LT}(A) = \begin{pmatrix} -\dfrac{y^2}{x} & -\dfrac{y}{x} \\ xy^3 & xy^2 \end{pmatrix}.$$

Definition 2.10 (S-pairs). Let $f, g \in R[x_1, x_2, ..., x_n]$, the S-pairs of this pair of polynomials is

$$S(f,g) = \frac{x^\gamma}{\mathrm{LT}(f)} f - \frac{x^\gamma}{\mathrm{LT}(g)} g$$

where $x^\gamma = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$ is the least common multiple of the leading monomials of the polynomials.

3. REALIZABILITY OVER $\mathrm{SL}_2(R[X_1^\pm, .., X_K^\pm])$

In this section we develop a new realization algorithm over the Laurent polynomial ring. The problem is that the Euclidean division algorithm is not valid for $R[x_1^\pm, ..., x_k^\pm]$, with $k \geq 1$. To solve this problem we use S-pairs with respect to the lexicographic order and Proposition 2.8.

Lemma 3.1. *Let*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x_1^\pm, .., x_k^\pm]).$$

If one of the entries of A is zero or invertible then A is realizable.

The explicit factorization of A in each case is given as follows:

(i) *If* $a_1 = 0$

$$A = E_{12}(-a_3^{-1})E_{21}(a_3)E_{12}(a_2 + sa_3^{-1}).$$

(ii) *If* $a_2 = 0$

$$A = E_{12}(-a_4^{-1})E_{21}(a_4)E_{12}(1 - a_1 - a_3^{-1}a_4)E_{21}(-1)E_{12}(1).$$

(iii) *If* $a_3 = 0$

$$A = E_{21}(-a_1^{-1})E_{12}(a_1)E_{21}(1 - a_4 - a_2a_1^{-1})E_{12}(-1)E_{21}(1)$$

(iv) *If* $a_4 = 0$

$$A = E_{21}(-a_2^{-1})E_{12}(a_2)E21(a_3 + a_1a_2^{-1}).$$

(v) *If* $a_1^{-1} \in \mathrm{SL}_2(R[x_1^\pm, .., x_k^\pm])$

$$A = E_{21}(a_1^{-1}(a_3 - 1))E_{12}(a_1 - 1)E_{21}(1)E_{12}(a_1^{-1}(1 - a_1 + a_2)).$$

(vi) *If* $a_2^{-1} \in \mathrm{SL}_2(R[x_1^\pm, .., x_k^\pm])$

$$A = E_{21}(a_2^{-1}(a_4 - 1))E_{12}(a_2)E_{21}(a_2^{-1}(a_1 - 1)). \tag{1}$$

(vii) *If* $a_3^{-1} \in \mathrm{SL}_2(R[x_1^\pm, .., x_k^\pm])$

$$A = E_{12}(a_3^{-1}(a_1 - 1))E_{21}(a_3)E_{12}(a_3^{-1}(a_4 - 1)). \tag{2}$$

(viii) *If* $a_4^{-1} \in \mathrm{SL}_2(R[x_1^\pm, .., x_k^\pm])$

$$A = E_{12}(a_4^{-1}(a_2 - 1))E_{21}(a_4 - 1)E_{12}(1)E_{21}(a_4^{-1}(1 - a_4 + a_3)).$$

EXAMPLE 3.2. The Cohn matrix is realizable in $\mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$: it has two explicit factorization that we can obtain by using the formula (1) or (2).
Let $C = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$. Then

$$C = E_{21}\left(-\frac{y}{x}\right) E_{12}\left(x^2\right) E_{12}\left(\frac{y}{x}\right)$$

or

$$C = E_{12}\left(-\frac{x}{y}\right) E_{21}\left(-y^2\right) E_{12}\left(\frac{x}{y}\right).$$

3.1. THE MAIN RESULT

For our main result, we start by recalling Park's necessary condition for realizability over $\mathrm{SL}_2(R[x_1, .., x_k])$.

THEOREM 3.3 (Park's theorem [5]). *Let*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x^{\pm}, .., x_k^{\pm}]),$$

for a fixed monomial order $' >'$. *If* A *is a nonconstant realizable matrix, then either* A *has a zero entry or one of the row vectors of* $\mathrm{LT}(A)$ *is a monomial multiple of the other row.*

COUNTEREXAMPLE 3.4. Let

$$A = \begin{pmatrix} \frac{1}{xy}+1 & xy \\ \frac{1}{x^4y^2}+\frac{2}{x^2y^2}+\frac{1}{x^3y}+\frac{1}{xy} & \frac{1}{x^2}+2 \end{pmatrix}.$$

By fixing the lexicographic order with $y > x$ we have

$$\mathrm{LT}(A) = \begin{pmatrix} 1 & xy \\ \frac{1}{xy} & 2 \end{pmatrix},$$

despite that neither of the two-row vectors $(1, xy)$ or $\left(\frac{1}{xy}, 2\right)$ is a monomial multiple of the other. We notice that the matrix A is realizable and by using formula (1) we obtain

$$A = E_{21}\left(\frac{1}{xy}\left(\frac{1}{x^2}+1\right)\right) E_{12}(xy) E_{21}\left(\frac{1}{x^2y^2}\right).$$

Therefore, Theorem 3.3 is not always valid for $\mathrm{SL}_2(R[x^{\pm}, .., x_k^{\pm}])$.

PROPOSITION 3.5. *Let* $A \in \mathrm{SL}_2(R[x_1^{\pm}, .., x_k^{\pm}])$, *for a fixed monomial order* $\det(\mathrm{LT}(A))$ *is not necessarily zero.*

Proof. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x_k^{\pm}, .., x_k^{\pm}])$$

be a non constant matrix. We have $\det(A) = a_1a_2 - a_3a_4 = 1$, thus $\mathrm{LT}(a_1a_2) = \mathrm{LT}(a_3a_4+1)$. We consider two cases.

Case 1. $\mathrm{LT}(a_3a_4+1) = \mathrm{LT}(a_3a_4)$
In this case we have $\mathrm{LT}(a_1a_2) = \mathrm{LT}(a_3a_4)$, thus $\det(\mathrm{LT}(A)) = 0$.

Case 2. $\mathrm{LT}(a_3a_4+1) \neq \mathrm{LT}(a_3a_4)$
In this case we have $\mathrm{LT}(a_1a_2) \neq \mathrm{LT}(a_3a_4)$, thus $\det(\mathrm{LT}(A)) \neq 0$. $\square$

THEOREM 3.6. *Let* $A \in \mathrm{SL}_2(R[x_1^{\pm}, .., x_k^{\pm}])$. *For a fixed monomial order* $>$, *the matrix* A *contains a monomial matrix* $T_0 \in \mathrm{SL}_2(R[x_1^{\pm}, .., x_k^{\pm}])$ *that verifies* $\det(T_0) = 0$.

Proof. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x_k^{\pm}, .., x_k^{\pm}])$,

Case 1. $\det(\mathrm{LT}(A)) = 0$
In this case $T_0 = \mathrm{LT}(A)$.

Case 2. $\det(\mathrm{LT}(A)) \neq 0$
In this case we suppose that $a_2 = \mathrm{LT}(a_2) + X$ with $X \in \mathrm{SL}_2(R[x_k^{\pm}, .., x_k^{\pm}])$, thus we obtain

$$A = \begin{pmatrix} a_1 & \mathrm{LT}(a_2) + X \\ a_3 & a_4 \end{pmatrix} \Rightarrow \det(A) = a_1a_4 - a_3\mathrm{LT}(a_2) - a_3X$$
$$\Rightarrow X = \frac{a_1a_4 - a_3\mathrm{LT}(a_2) - 1}{a_3}$$
$$\Rightarrow A = \begin{pmatrix} a_1 & \frac{a_1a_4 - 1}{a_3} \\ a_3 & a_4 \end{pmatrix}.$$

If $a_3^{-1} \in \mathrm{SL}_2(R[x_1^{\pm}, .., x_k^{\pm}])$ we can choose T_0 such that

$$T_0 = \begin{pmatrix} \mathrm{LT}(a_1) & \mathrm{LT}(\frac{a_1a_4}{a_3}) \\ \mathrm{LT}(a_3) & \mathrm{LT}(a_4) \end{pmatrix}.$$

It is clear that this matrix checks $\det(T_0) = 0$. If not, the choice of the elements of T_0 is based on the explicit form of each matrix so that the chosen monomials verify $\det(T_0) = 0$. $\square$

By Theorem 3.6 we introduce the following definitions.

DEFINITION 3.7. Let $A \in \mathrm{SL}_2(R[x_1^{\pm}, .., x_k^{\pm}])$, we introduce the following application

$$T: \ \mathrm{SL}_2(R[x_1^{\pm}, ..., x_k^{\pm}]) \to G$$
$$T(A) = \begin{cases} \mathrm{LT}(A), & \text{if } \det(\mathrm{LT}(A)) = 0 \\ T_0, & \text{if } \det(\mathrm{LT}(A)) \neq 0 \text{ and } \det(T_0) = 0, \end{cases}$$

where G is the set of $n \times n$ matrices of determinant 0 whose entries are monomials of $R[x_1^{\pm}, ..., x_k^{\pm}]$.

To generalize the notion of S-pairs for matrices in $\mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}])$ we combine Definition 2.10 and Proposition 2.8.

DEFINITION 3.8. The S_0-pairs of a matrix $A \in \mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}])$ are given as follows. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ and } T(A) = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$$

Then,

$$S_0\text{-pairs}(A) = \begin{cases} S_0(a_1, a_3) = \frac{x^\gamma}{t_1} a_1 - \frac{x^\gamma}{t_3} a_3 \\ S_0(a_2, a_4) = \frac{x^\gamma}{t_2} a_2 - \frac{x^\gamma}{t_4} a_4 \end{cases}$$

or

$$S_0\text{-pairs}(A) = \begin{cases} S_0(a_3, a_1) = \frac{x^\gamma}{t_3} a_3 - \frac{x^\gamma}{t_1} a_1 \\ S_0(a_4, a_2) = \frac{x^\gamma}{t_4} a_4 - \frac{x^\gamma}{t_2} a_2. \end{cases}$$

EXAMPLE 3.9. Let

$$A = \begin{pmatrix} 1 + \frac{1}{xy} - \frac{1}{x^2y^4} & \frac{1}{x^2} + \frac{1}{x^2y^2} - \frac{1}{x^3y^3} \\ -\frac{1}{y^2} + \frac{1}{x^2y^2} + \frac{1}{x^3y^3} - \frac{1}{x^4y^6} & 1 - \frac{1}{xy} + \frac{1}{x^4y^2} + \frac{1}{x^4y^4} - \frac{1}{x^5y^5} \end{pmatrix}$$

in $\mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$. By fixing the lexicographic order, in accordance with Proposition 3.5, we can see that

$$\mathrm{LT}(A) = \begin{pmatrix} 1 & \frac{1}{x^2} \\ -\frac{1}{y^2} & 1 \end{pmatrix},$$

We notice that

$$\det(\mathrm{LT}(A)) = \frac{1}{x^2y^2} \left(x^2y^2 + 1\right) \neq 0.$$

Thus

$$T(A) = T_0 = \begin{pmatrix} 1 & \frac{1}{x^2} \\ \frac{1}{x^2y^2} & \frac{1}{x^4y^2} \end{pmatrix}, \qquad \det(T_0) = 0,$$

We obtain

$$S_0\text{-pairs}(A) = \begin{cases} S_0(a_3, a_1) = -\frac{1}{y^2}. \\ S_0(a_4, a_2) = \frac{x^\gamma}{t_4} a_4 - \frac{x^\gamma}{t_2} a_2. \end{cases}$$

The following theorem is a generalization of Theorem 3.3. For notation purposes let

$$T(A) = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = \begin{pmatrix} T(a_1) & T(a_2) \\ T(a_3) & T(a_4) \end{pmatrix}.$$

THEOREM 3.10. *Let*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}]) \quad \textit{and} \quad T(A) = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix},$$

for a fixed monomial order $'>'$. *If* A *is a nonconstant realizable matrix, then either*

(i) *A has a zero entry;*
(ii) *one of the entry of A is invertible;*
(iii) $(t_1, t_2) = M\ (t_3, t_4)$, *with* $M = \frac{t_1}{t_3} = \frac{t_2}{t_4}$; *and* $M \in R[x_1^{\pm}, ..., x_k^{\pm}]$.

Proof. We suppose that A is realizable, so we can write it as follows

$$A = E_1 \cdots E_l$$

We consider two cases.

Case 1. If $l = 1$, we have either $A = E_{12}(a_2) = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$ or $A = E_{21}(a_3) = \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}$ in both cases, A has a zero entry.

Case 2. If $l > 1$, we have either $A = A'E_{12}(f)$ or $A = A'E_{21}(f)$ with $f \in R[x_1^{\pm}, ..., x_k^{\pm}]$ and $A' = E_1 \cdots E_{l-1}$.

We consider the case $A = A'E_{12}(f)$.

Using the S_0-pairs to reduce the degree of the elements of the matrix implies that:

$$\exists h \in R[x_1^{\pm}, ..., x_k^{\pm}] \setminus A' = \begin{pmatrix} a'_1 & a'_2 \\ a'_3 - ha'_1 & a'_4 - ha'_2 \end{pmatrix},$$

with

$$h = \frac{T(a'_3)}{T(a'_1)} = \frac{T(a'_4)}{T(a'_2)}.$$

Thus we have two cases to study.

Case 2.1. *A' is nonconstant.*

Since A' is realizable and based on the hypotheses we have

$\det(T(A')) = 0 \Rightarrow$

$$(3) \quad \begin{aligned} &(T(a'_1), T(a'_2)) = H\ \left(T(a'_3 - ha'_1), T(a'_4 - ha'_2)\right) \\ &H = \frac{T(a'_1)}{T(a'_3 - ha'_1)} = \frac{T(a'_2)}{T(a'_4 - ha'_2)} \in R[x_1^{\pm}, ..., x_k^{\pm}]. \end{aligned}$$

$$(4) \quad \begin{aligned} &A = A'E_{12}(f) \\ &A = \begin{pmatrix} a'_1 & a'_2 + fa'_1 \\ a'_3 - ha'_1 & a'_4 + fa'_3 - h(a'_2 + fa'_1) \end{pmatrix} \in \mathrm{SL}_2(R[x_1^{\pm}, ..., x_k^{\pm}]) \end{aligned}$$

From (3), (4) and the hypothesis we obtain, $\det(T(A)) = 0$

$$\begin{aligned} &\Rightarrow\ T(a'_1)T(a'_4 + fa'_3 - h(a'_2 + fa'_1)) = T(a'_3 - ha'_1)T(a'_2 + fa'_1) \\ &\Rightarrow\ H\ T(a'_3 - ha'_1)T(a'_4 + fa'_3 - h(a'_2 + fa'_1)) = T(a'_3 - ha'_1)T(a'_2 + fa'_1) \\ &\Rightarrow\ H\ T(a'_4 + fa'_3 - h(a'_2 + fa'_1)) = T(a'_2 + fa'_1). \end{aligned}$$

We have

$$(T(a_3), T(a_4)) = (T(a'_3 - ha'_1), T(a'_4 + fa'_3 - h(a'_2 + fa'_1))).$$

Hence

$$\begin{aligned}(T(a_1),T(a_2)) &= (T(a'_1),T(a'_2+fa'_1))\\ &= (H\ T(a'_3-ha'_1),T(a'_2+fa'_1))\\ &= (H\ T(a'_3-ha'_1),HT(a'_4+fa'_3-h(a'_2+fa'_1)))\\ &= H(\ T(a'_3-ha'_1),T(a'_4+fa'_3-h(a'_2+fa'_1)))\\ &= H(T(a_3),T(a_4))\\ &= \frac{T(a'_1)}{T(a'_3-ha'_1)}(T(a_3),T(a_4))\\ &= \frac{T(a_1)}{T(a_3)}(T(a_3),T(a_4))\\ &= M\ (T(a_3),T(a_4)).\end{aligned}$$

Since $\det(T(A))=0 \Rightarrow M=\dfrac{T(a_1)}{T(a_3)}=\dfrac{T(a_2)}{T(a_4)}$.

Case 2.2. *A' has a zero entry.*
Without loss of generality, suppose that $a'_2=0$. We also have two cases to study.

Case 2.2.1. $A=A'E_{12}(p)=\begin{pmatrix} a'_1 & pa'_1\\ a'_3-ha'_1 & a'_4+p\,(a'_3-ha'_1)\end{pmatrix}$.
We have $\det(A)=a'_1a'_4=1$.

$$\begin{aligned}&\det(T(A))=0\\ \Rightarrow\ & T(a'_1)T(a'_4+p(a'_3-ha'_1)=T(pa'_1)T(a'_3-ha'_1)\\ \Rightarrow\ & T(a'_4+p(a'_3-ha'_1)=T(p)T(a'_3-ha'_1).\end{aligned}$$

$$\begin{aligned}(T(a_3),T(a_4)) &= (T(a'_3-ha'_1),T(a'_4+p\left(a'_3-ha'_1\right)))\\ &= (T(a'_3-ha'_1),T(p)T(\left(a'_3-ha'_1\right)))\\ &= \frac{T(a'_3-ha'_1)}{T(a'_1)}(T(a'_1),T(pa'_1))\\ &= \frac{T(a_3)}{T(a_1)}(T(a_1),T(a_2))\end{aligned}$$

Therefore $(T(a_1),T(a_2))=M(T(a_3),T(a_4))$.

Case 2.2.2.

$$A=A'E_{21}(p)=\begin{pmatrix} a'_1 & 0\\ a'_3-ha'_1 & a'_4\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}=\begin{pmatrix} a'_1 & 0\\ a'_3-ha'_1+pa'_4 & a'_4\end{pmatrix},$$

where A has a zero entry.

□

3.2. REALIZATION ALGORITHM FOR $\mathrm{SL}_2(R[X_1^{\pm},...,X_K^{\pm}])$

We present the following algorithm for the factorization of a given matrix in $\mathrm{SL}_2(R[x_1^{\pm}...x_k^{\pm}])$ with respect to the fixed order and based on the S_0-pairs.

ALGORITHM 3.11. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ be a nonconstant matrix.
Define $A \in \mathrm{SL}_2(R[x_1^{\pm}...x_k^{\pm}])$, $i = 0$.

I. While $\det(A) = 1$ & $\det(T(A)) = 0$

$$\textbf{Step 1.} \begin{cases} \text{Calculate} \quad S_0(a_1, a_3) \ \& \ S_0(a_2, a_4). \\ \text{Do} \quad \begin{cases} a_1 = \dfrac{T(a_1)}{X^{\gamma}(a_1, a_3)} S_0(a_1, a_3) \\ a_2 = \dfrac{T(a_2)}{X^{\gamma}(a_2, a_4)} S_0(a_2, a_4) \\ \text{Update } i \\ \text{Update } A \\ E_i = E_{12}\left(\frac{T(a_1)}{T(a_3)}\right). \end{cases} \end{cases}$$

$$\textbf{Step 2.} \begin{cases} \text{Calculate} \quad S_0(a_3, a_1) \ \& \ S_0(a_4, a_2). \\ \text{Do} \quad \begin{cases} a_3 = \dfrac{T(a_3)}{X^{\gamma}(a_3, a_1)} S_0(a_3, a_1) \\ a_4 = \dfrac{T(a_4)}{X^{\gamma}(a_4, a_2)} S_0(a_4, a_2) \\ \text{Update } i \\ \text{Update } A \\ E_i = E_{21}\left(\frac{T(a_3)}{T(a_1)}\right). \end{cases} \end{cases}$$

$$\textbf{Step 3.} \begin{cases} \text{If one of the entries of } A \text{ is a monomial} \\ \qquad \text{use the formulas from Lemma 3.1.} \\ \text{If } A = I_{d_2},\ A \text{ is realizable the algorithm stops.} \end{cases}$$

II. If $\det(T(A)) \neq 0$ & $A \neq I_{d_2}$
Then one of the entries of A is a monomial therefore we use the formulas from Lemma 3.1.

REMARK 3.12. In this algorithm we can start with **Step 2** instead of **Step 1**, the choice is made based on the matrix.

The advantage of this algorithm is the fact that the choice of the matrix $T(A)$ depends on that of the previous step to ensure convergence.

The choice of the variables' order is determined at the beginning of the process and will not be changed. It is possible that we may find matrices with two initial leading terms, this case is treated in the Example 3.14.

Example 3.13. Let

$$A = \begin{pmatrix} 1+\frac{1}{xy}-\frac{1}{x^2y^4} & \frac{1}{x^2}+\frac{1}{x^2y^2}-\frac{1}{x^3y^3} \\ -\frac{1}{y^2}+\frac{1}{x^2y^2}+\frac{1}{x^3y^3}-\frac{1}{x^4y^6} & 1-\frac{1}{xy}+\frac{1}{x^4y^2}+\frac{1}{x^4y^4}-\frac{1}{x^5y^5} \end{pmatrix}$$

in $\mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$. We can see that for this matrix we have

$$\mathrm{LT}(A) = \begin{pmatrix} 1 & \frac{1}{x^2} \\ -\frac{1}{y^2} & 1 \end{pmatrix},$$

and

$$\det(\mathrm{LT}(A)) = \frac{1}{x^2y^2}\left(x^2y^2+1\right) \neq 0.$$

By applying the definition we obtain

$$T(A) = T_0 = \begin{pmatrix} 1 & \frac{1}{x^2} \\ \frac{1}{x^2y^2} & \frac{1}{x^4y^2} \end{pmatrix}, \qquad \det(T_0) = 0$$

and

$$A = \begin{pmatrix} \underline{1}+\frac{1}{xy}-\frac{1}{x^2y^4} & \underline{\frac{1}{x^2}}+\frac{1}{x^2y^2}-\frac{1}{x^3y^3} \\ -\frac{1}{y^2}+\underline{\frac{1}{x^2y^2}}+\frac{1}{x^3y^3}-\frac{1}{x^4y^6} & 1-\frac{1}{xy}+\underline{\frac{1}{x^4y^2}}+\frac{1}{x^4y^4}-\frac{1}{x^5y^5} \end{pmatrix}.$$

Therefore,

$$\begin{cases} S_{21} = -\frac{1}{y^2} \\ S_{22} = 1-\frac{1}{xy} \\ E1 = E_{21}(\frac{1}{x^2y^2}) \end{cases}$$

and

$$A1 = \begin{pmatrix} \frac{1}{xy}-\frac{1}{x^2y^4}+1 & \frac{1}{x^2y^2}-\frac{1}{x^3y^3}+\frac{1}{x^2} \\ -\frac{1}{y^2} & 1-\frac{1}{xy} \end{pmatrix}.$$

To obtain the explicit factorization, we use formula (2), thus

$$A = E_{21}\left(\frac{1}{x^2y^2}\right)E_{12}\left(-\frac{y}{x}+\frac{1}{x^2y^2}\right)E_{21}\left(-\frac{1}{y^2}\right)E_{12}\left(\frac{y}{x}\right).$$

Example 3.14. In this example for $y > x$ the matrix A has two leading terms. To solve the problem we proceed as follows:

$$A = \begin{pmatrix} -4y^4-8y^2-4 & x+2y^4+\frac{3}{2}+xy^2+3y^2 \\ 4xy^2-4y^2+4x-6 & -2xy^2-x^2+2y^2+2 \end{pmatrix}$$

$$\begin{aligned} A &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + (x+\frac{3}{2})y^0 \\ 4(x-1)y^2 + (4x-6)y^0 & -2y^2(x-1) + (-x^2+2)y^0 \end{pmatrix} \\ \mathrm{LT}(A) &= \begin{pmatrix} -4y^4 & 2y^4 \\ 4(x-1)y^2 & -2(x-1)y^2 \end{pmatrix} \end{aligned}$$

Now we apply the proposed algorithm on A

$$A = \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + \left(x+\frac{3}{2}\right)y^0 \\ 4(x-1)y^2 + (4x-6)y^0 & -2\left(x-1\right)y^2 + (-x^2+2)y^0 \end{pmatrix}$$

Hence,

$$A1 = \begin{pmatrix} \frac{1}{x-1}\left(2y^2 - 4xy^2 - 4x + 4\right) & \frac{1}{2(x-1)}\left(2x^2 + 4xy^2 + x - 2y^2 - 3\right) \\ 4x + 4xy^2 - 4y^2 - 6 & -x^2 - 2xy^2 + 2y^2 + 2 \end{pmatrix}$$

and

$$E_1 = E_{12}\left(\frac{y^2}{x-1}\right) \notin E_2(R[x^{\pm}, y^{\pm}]).$$

The problem is that we can't apply **Step 1** of our algorithm since $A_1 \notin \mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$. Thus we proceed differently by applying **Step 2**.

$$\begin{aligned} A &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + x + \frac{3}{2} \\ 4(x-1)y^2 + 4x - 6 & -2y^2\left(x-1\right) - x^2 + 2 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + x + \frac{3}{2} \\ 2 - 4x - 4\left(x-1\right)\frac{1}{y^2} & 2x - 1 + (\frac{1}{2}x + x^2 - \frac{3}{2})\frac{1}{y^2} \end{pmatrix}, \end{aligned}$$

with

$$E_1 = E_{21}\left(\left(x-1\right)\frac{1}{y^2}\right)$$

and

$$s\text{-pairs}\begin{cases} S_{11} = \frac{1}{2x-1}\left(2xy^2 + 2x - 1\right) \\ S_{12} = \frac{1}{8x-4}\left(4x^2 + 8xy^2 + 4x - 3\right) \end{cases}$$

Next,

$$A_2 = \begin{pmatrix} -\frac{4}{2x-1}\left(2xy^2 + 2x - 1\right) & \frac{1}{4x-2}\left(4x^2 + 8xy^2 + 4x - 3\right) \\ 2 - 4x - 4\left(x-1\right)\frac{1}{y^2} & 2x - 1 + (\frac{1}{2}x + x^2 - \frac{3}{2})\frac{1}{y^2} \end{pmatrix}$$

with

$$E_2 = E_{12}\left(\frac{-2}{2x-1}y^4\right).$$

Since $A_2 \notin \mathrm{SL}_2(R[x^{\pm}, y^{\pm}])$, we start by **Step 1** instead of **Step 2**.

$$\begin{aligned} A_1 &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + \left(x + \frac{3}{2}\right) y^0 \\ (2-4x)y^0 - 4\,(x-1)\,\frac{1}{y^2} & (2x-1)y^0 + (\frac{1}{2}x + x^2 - \frac{3}{2})\frac{1}{y^2} \end{pmatrix} \\ A_2 &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & 2y^4 + (x+3)y^2 + \left(x + \frac{3}{2}\right) y^0 \\ 4\frac{x}{y^2} + (4x-2)\frac{1}{y^4} & -2\frac{x}{y^2} + \left(\frac{3}{4} - x - x^2\right)\frac{1}{y^4} \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} E_2 &= E_{21}\left(-\frac{1}{4y^4}\,(4x-2)\right) \\ A_3 &= \begin{pmatrix} (-\frac{2}{x} - 4)y^2 - 4 & (\frac{3}{4x} + 2)y^2 + \left(x + \frac{3}{2}\right) y^0 \\ 4\frac{x}{y^2} + (4x-2)\frac{1}{y^4} & -2\frac{x}{y^2} + \left(\frac{3}{4} - x - x^2\right)\frac{1}{y^4} \end{pmatrix}, \end{aligned}$$

with

$$E_3 = E_{12}\left(\frac{1}{x}y^6\right).$$

In this step $T(A_3) \neq \mathrm{LT}(A_3)$ because

$$\mathrm{LT}(A_3) = \begin{pmatrix} (-\frac{2}{x} - 4)y^2 & (\frac{3}{4x} + 2)y^2 \\ 4\frac{x}{y^2} & -2\frac{x}{y^2} \end{pmatrix} \Rightarrow \det(\mathrm{LT}(A_3)) = 1 \neq 0.$$

Thus $T(A_3) = T_0$.

$$A_3 = \begin{pmatrix} (-\frac{2}{x} - 4)y^2 - 4 & (\frac{3}{4x} + 2)y^2 + \left(x + \frac{3}{2}\right) y^0 \\ 4\frac{x}{y^2} + (4x-2)\frac{1}{y^4} & -2\frac{x}{y^2} + \left(\frac{3}{4} - x - x^2\right)\frac{1}{y^4} \end{pmatrix}$$

and

$$s_0\text{-pairs} \begin{cases} S_{0,21} = \frac{1}{2x}\frac{y^2}{2x-1} \\ S_{0,22} = \frac{1}{2x}\frac{y^2}{2x-1} \end{cases}$$

with

$$\begin{aligned} T_0 &= \begin{pmatrix} -4 & x + \frac{3}{2} \\ 4\frac{x}{y^4} - \frac{2}{y^4} & \frac{3}{4y^4} - \frac{x}{y^4} - \frac{x^2}{y^4} \end{pmatrix} \\ A_4 &= \begin{pmatrix} -\frac{2}{x}y^2 - 4y^2 - 4 & x + \frac{3}{4x}y^2 + 2y^2 + \frac{3}{2} \\ \frac{1}{xy^2} & -\frac{3}{8xy^2} - \frac{1}{4y^2} \end{pmatrix} \end{aligned}$$

with

$$E_4 = E_{21}\left(\frac{1}{2y^4}\,(2x-1)\right)$$

Since the matrix A has an entry with only one term we can directly apply (2).

Therefore

$$\begin{aligned} A_4 &= \begin{pmatrix} -\frac{2}{x}y^2 - 4y^2 - 4 & x + \frac{3}{4x}y^2 + 2y^2 + \frac{3}{2} \\ \frac{1}{xy^2} & -\frac{3}{8xy^2} - \frac{1}{4y^2} \end{pmatrix} \\ &= E_{12}\left(-5xy^2 - 4xy^4 - 2y^4\right) E_{21}\left(\frac{1}{xy^2}\right) E_{12}\left(-xy^2 - \frac{1}{4}x - \frac{3}{8}\right). \end{aligned}$$

Thus

$$\begin{aligned} A &= \begin{pmatrix} -4y^4 - 8y^2 - 4 & x + xy^2 + 3y^2 + 2y^4 + \frac{3}{2} \\ 4x + 4xy^2 - 4y^2 - 6 & -x^2 - 2xy^2 + 2y^2 + 2 \end{pmatrix} \\ A &= E_{21}\left(-\frac{1}{y^2}(x-1)\right) E_{21}\left(\frac{1}{4y^4}(4x-2)\right) E_{12}\left(-\frac{1}{x}y^6\right) \\ &\quad E_{21}\left(-\frac{1}{2y^4}(2x-1)\right) E_{12}\left(-5xy^2 - 4xy^4 - 2y^4\right) E_{21}\left(\frac{1}{xy^2}\right) \\ &\quad E_{12}\left(-xy^2 - \frac{1}{4}x - \frac{3}{8}\right). \end{aligned}$$

EXAMPLE 3.15. The last example is when $A \in \mathrm{SL}_2(R[x^{\pm}, y^{\pm}, z^{\pm}, t^{\pm}])$

$$A = \begin{pmatrix} -t^2x^2 + yztx + zt + yx + 1 & -tx^2z + x^2 + yxz^2 + z^2 \\ xt^2y - t^2 - zty^2 - y^2 & -y^2z^2 + txyz - xy - tz + 1 \end{pmatrix}$$

$$A = E_{12}\left(-\frac{x}{y}\right) E_{21}\left(-y^2\right) E_{12}\left(\frac{x}{y}\right) E_{21}\left(-\frac{t}{z}\right) E_{12}\left(-z^2\right) E_{21}\left(\frac{t}{z}\right).$$

4. A REALIZATION ALGORITHM FOR $\mathrm{SL}_3(R[X_1^{\pm}, ..., X_K^{\pm}])$

In view of the results of the previous section we extend the realization algorithm for matrices of the special form

$$A = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ p & q & 1 \end{pmatrix} \in \mathrm{SL}_3(R[x_1^{\pm}, ..., x_k^{\pm}]).$$

The idea of choosing this matrix precisely comes from [6], and we reproduce their results in the Laurent polynomial ring.

$$A = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ p & q & 1 \end{pmatrix} \in \mathrm{SL}_3(R[x_1^{\pm}, ..., x_k^{\pm}]),$$

With $\tilde{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(R[x_1^{\pm}, ..., x_k^{\pm}])$.

By applying elementary operations on A we obtain

$$AE(-p)E(-q) = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(R[x_1^{\pm}, ..., x_k^{\pm}]).$$

The realization of $\begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(R[x_1^{\pm},...,x_k^{\pm}])$ is now reduced to the same problem but for $\tilde{A} \in \mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}])$. To express $\tilde{A}$ as a product of elementary matrices, we apply the algorithm obtained in the previous section.

$$A = \prod_{i=1}^{l} \begin{pmatrix} \tilde{E}_i & 0 \\ 0 & 1 \end{pmatrix} \times E_{31}(-p)E_{32}(-q) \in \mathrm{SL}_3(R[x_1^{\pm},...,x_k^{\pm}]),$$

with $\tilde{A} = \tilde{E}_1 \cdots \tilde{E}_l$, $\tilde{E} \in E(R[x^{\pm}, y^{\pm}])$ and $E \in E(R[x^{\pm}, y^{\pm}])$.

Example 4.1. Consider $A \in \mathrm{SL}_3(R[x^{\pm}, y^{\pm}])$,

$$A \quad = \begin{pmatrix} xy+1 & x^2y^2+xy+x & 0 \\ y & xy^2+1 & 0 \\ 2x^3 & 2x^4y & 1 \end{pmatrix}.$$

We have

$$\tilde{A} \quad = \begin{pmatrix} xy+1 & x^2y^2+xy+x \\ y & xy^2+1 \end{pmatrix} \in \mathrm{SL}_2(R[x^{\pm}, y^{\pm}]),$$

By applying the realization algorithm on $\tilde{A}$ we obtain

$$A = \tilde{E}_{12}(x)\tilde{E}_{21}(y)\tilde{E}_{12}(xy),$$

Therefore

$$A = E_{12}(x)E_{21}(y)E_{12}(xy)E_{31}(-2x^3)E_{32}(-2x^4y).$$

5. CONCLUSIONS

The realization algorithm in this paper is based on the one established by Park in [5]. As we have seen, the new algorithm can be applied on both $\mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}])$ and $\mathrm{SL}_3(R[x_1^{\pm},...,x_k^{\pm}])$. The key ingredient of this method is the use of the generalized notion of S-pairs for matrices in $\mathrm{SL}_2(R[x_1^{\pm},...,x_k^{\pm}])$ since it cancels the leading term of the elements of the matrix. The use of the proposed monomial order with the proper ordering is very crucial for the convergence of the algorithm. This algorithm can be a very useful tool in Signal Processing and it surely gives satisfying results.

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