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ALGEBRAIC IDENTITIES AND GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract. Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$. A map $\mathcal{F} : \mathcal{R} \longrightarrow \mathcal{R}$ is called a multiplicative generalized derivation associated with d (not necessarily additive) if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. In this paper, our intention is to study the commutativity of $\mathcal{R}$ using a multiplicative generalized derivation that satisfies some algebraic identities.

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1. INTRODUCTION AND SOME PRELIMINARIES

In the following, $\mathcal{R}$ always denotes an associative ring with the multiplicative center $Z(\mathcal{R})$, unless stated otherwise. As usual, the symbols $[s, t]$ denote the commutator $stts$, $x \circ y$ the anticommutator $st + ts$, $I_{\mathcal{R}}$ the mapping identity of $\mathcal{R}$, and $0_{\mathcal{R}}$ the mapping zero of $\mathcal{R}$. Recall that a ring R is prime if $x\mathcal{R}y = \{0\}$ implies $x = 0$ or $y = 0$, and $\mathcal{R}$ is semiprime if $x\mathcal{R}x = \{0\}$ implies $x = 0$. An additive mapping $\mathcal{H} : \mathcal{R} \longrightarrow \mathcal{R}$ is a left multiplier of $\mathcal{R}$ if $\mathcal{H}(xy) = \mathcal{H}(x)y$ for all $x, y \in \mathcal{R}$, and a mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ (not necessarily additive) is a multiplication derivation of a ring $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. Furthermore, a mapping $\mathcal{F}$ is said to be a multiplication-generalized derivation of $\mathcal{R}$ associated with d if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. Obviously, every generalized derivation is a multiplicative generalized derivation on $\mathcal{R}$, but the converse is not generally true (the multiplicative derivation d is a multiplicative generalized derivation associated with itself).

During the last decade there have been many results concerning the behavior of prime rings (commutativity, rang, dimension ...), especially the rings involved by additive and multiplicative maps, see for example ([7]). More precisely, the classical Posner's second theorem states that a prime ring must be commutative if it admits a non-zero derivation d satisfying $d(x)x - xd(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. In this context, in [4] Ashraf and Rehman proved that a nonzero ideal I of a prime ring $\mathcal{R}$ must be commutative if $\mathcal{R}$

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admits a nonzero derivation d satisfying $d(xy) \mp xy \in Z(\mathcal{R})$ for all $x, y \in I$. Moreover, Dhara and Rehman, in [5], generalized these identities to show that for a nonzero square closed Lie ideal I of a prime ring $\mathcal{R}$, if $\mathcal{R}$ admits nonzero generalized derivations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ satisfying $\mathcal{F}(x)\mathcal{G}(y) \mp \mathcal{H}(xy) \in Z(\mathcal{R})$ or $\mathcal{F}(x)\mathcal{F}(y) \mp \mathcal{H}(yx) \in Z(\mathcal{R})$ for all $x, y \in I$, then $I \subseteq Z(\mathcal{R})$.

In this paper we study the concept of multiplicative (generalized) derivation, also we study certain identities involving multiplicative (generalized) derivation in prime rings.

Finally, an example is given to show that the restrictions imposed on the hypothesis of our results are not superfluous.

Before presenting the main theorems, we give several well-known semiprime ring fundamental identities and results that will be used extensively in the following sections.

(i) $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in \mathcal{R}$.
(ii) $[xy, z] = [x, z]y + x[y, z]$ for all $x, y, z \in \mathcal{R}$.

LEMMA 1.1 ([6, Lemma 2]). Let $\mathcal{R}$ be a prime ring. Then for some $0 \neq a \in Z(\mathcal{R})$, if $ab \in Z(\mathcal{R})$, then $b \in Z(\mathcal{R})$. In particular, if $ab = 0$, then $b = 0$.

LEMMA 1.2. Let $\mathcal{R}$ be a ring and d be a multiplicative derivation of $\mathcal{R}$. Then $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$.

LEMMA 1.3 ([8, Theorem 2(ii)]). Let $\mathcal{R}$ be a prime ring and U be a nonzero ideal of $\mathcal{R}$. If there exists a derivation d of $\mathcal{R}$ such that $x[[d(x), x], x] = 0$ for all $x \in U$, then either $d = 0$ or $\mathcal{R}$ is commutative.

2. MAIN RESULTS

THEOREM 2.1. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a left multiplier $\mathcal{H}$ and multiplicative generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations $(f \neq 0)$ and g, respectively, if $\mathcal{G}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) + \mathcal{H}(yx) \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is an integral domain.

Proof. Assume that

$$\mathcal{G}(xy) + \mathcal{F}(x)\mathcal{F}(y) + \mathcal{H}(yx) \in Z(\mathcal{R}) \text{ for all } x, y \in U. \tag{1}$$

Replacing y by yz in (1), we get

$$(\mathcal{G}(xy) + \mathcal{F}(x)\mathcal{F}(y))z + xyg(z) + \mathcal{F}(x)yf(z) + \mathcal{H}(y)zx \in Z(\mathcal{R}) \tag{2}$$

for all $x, y, z \in U$. Which lead to

$$(\mathcal{G}(xy) + \mathcal{F}(x)\mathcal{F}(y) + \mathcal{H}(y)x)z + xyg(z) + \mathcal{F}(x)yf(z) + \mathcal{H}(y)[z, x] \in Z(\mathcal{R}). \tag{3}$$

This can be rewritten as

$$[xyg(z), z] + [\mathcal{F}(x)yf(z), z] + [\mathcal{H}(y)[z, x], z] = 0 \text{ for all } x, y, z \in U. \tag{4}$$

Replacing x by x^2 in (4), then for all $x, y, z \in U$, we obtain

$$[x^2yg(z), z] + [\mathcal{F}(x)xyf(z), z] + [xf(x)yf(z), z] + [\mathcal{H}(y)[z, x^2], z] = 0. \tag{5}$$

Replacing y by xy in (4), we arrive at

$$(6) \quad [x^2yg(z), z] + [\mathcal{F}(x)xyf(z), z] + [\mathcal{H}(x)y[z, x], z] = 0 \text{ for all } x, y, z \in U.$$

Calculate the deference between (5) and (6), then for all $x, y, z \in U$ we find

$$(7) \qquad [xf(x)yf(z), z] + [\mathcal{H}(y)[z, x^2], z] - [\mathcal{H}(x)y[z, x], z] = 0.$$

Substituting x in place of z in (7), we can easily arrive at

$$(8) \qquad [xf(x)yf(x), x] = 0 \text{ for all } x, y \in U.$$

Taking $yf(x)z$ in place of y in (8) and using it again, we find that

$$(9) \qquad xf(x)yf(x)[zf(x), x] = 0 \text{ for all } x, y, z \in U,$$

which implies that

$$xf(x)Uf(x)[zf(x), x] = \{0\} \text{ for all } x, y \in U.$$

The primeness of R gives the following expression $xf(x) = 0$ or $f(x)[zf(x), x] = 0$ for all $x, z \in U$, In any cases, the equation (8) lead to

$$(10) \qquad [xf(x), x]yf(x) = 0 \text{ for all } x, y \in U.$$

which implies that $[xf(x), x]\mathcal{R}xf(x) = \{0\}$ for all $x \in U$. By primness of $\mathcal{R}$, it yields for each $x \in U$ either $xf(x) = 0$ or $x[f(x), x] = 0$. In any cases, it follows that $[xf(x), x] = 0$ for all $x \in U$. By Lemma 1.3, we conclude that $\mathcal{R}$ is commutative by using the fact that $f \neq 0$. □

A generalized derivation is obviously a multiplicative generalized derivation as $\mp I_{\mathcal{R}}$, $0_{\mathcal{R}}$ are multipliers of $\mathcal{R}$. Additionally, we have equivalents for a $\mathcal{G}$ is a generalized derivation of $\mathcal{R}$ and a $\mathcal{G} \mp I_{\mathcal{R}}$ is a generalized derivation of $\mathcal{R}$. The next corollaries then follow as direct consequences of Theorem 2.1.

Corollary 2.2. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations $(f \neq 0)$ and g respectively.

(i) [1, Theorem 4] If $\mathcal{G}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \in Z(R)$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.

(ii) [10, Theorems 1 and 2] If $\mathcal{G}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \mp yx \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.

(iii) If $\mathcal{G}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \mp [x, y] \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.

(iv) If $\mathcal{G}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \mp x \circ y \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.

It's easy to see that $0_{\mathcal{R}}$ and $\mp I_{\mathcal{R}}$ are multiplicative generalized derivations of $\mathcal{R}$, so from Theorem 2.1, we can derive the following corollaries

Corollary 2.3. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation f. Then

(i) If $\mathcal{F}(x)\mathcal{F}(y) \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.
(ii) [3, Theorems 2.4 and 2.6] If $\mathcal{F}(x)\mathcal{F}(y) \mp yx \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.
(iii) [1, Theorem 4] If $\mathcal{F}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.
(iv) If $\mathcal{F}(xy) \mp \mathcal{F}(x)\mathcal{F}(y) \mp yx \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.
(v) [10, Theorems 1 and 2] If $\mathcal{F}(x)\mathcal{F}(y) \mp [x, y] \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.
(vi) [2, Corollary 2.20] If $\mathcal{F}(x)\mathcal{F}(y) \mp x \circ y \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is commutative.

THEOREM 2.4. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admit a left multiplier H and multiplicative generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations ($f \neq 0$) and g respectively. If $\mathcal{G}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) + \mathcal{H}(yx) \in Z(\mathcal{R})$ for all $x, y \in U$, then $\mathcal{R}$ is an integral domain.

Proof. Suppose that

$$\mathcal{G}(xy) + \mathcal{F}(y)\mathcal{F}(x) + \mathcal{H}(yx) \in Z(\mathcal{R}) \text{ for all } x, y \in U. \tag{11}$$

Replacing x by xz in (11), we get

$$\mathcal{G}(x)zy + xg(zy) + \mathcal{F}(y)\mathcal{F}(x)z + \mathcal{F}(y)xf(z) + \mathcal{H}(yx)z \in Z(\mathcal{R}). \tag{12}$$

Add and subtract $\mathcal{G}(xy)z$ to (12) lead to

$$\mathcal{G}(x)[z, y]+xg(zy)-xg(y)z+\mathcal{F}(y)xf(z)+(\mathcal{G}(xy)+\mathcal{F}(y)\mathcal{F}(x)+\mathcal{H}(yx))z. \tag{13}$$

Using our hypothesis, then for all $x, y, z \in U$, we have

$$[\mathcal{G}(x)[z, y], z] + [xg(zy) - xg(y)z, z] + [\mathcal{F}(y)xf(z), z] = 0. \tag{14}$$

Taking z^2 in place of y in (14), we get

$$[xz^2g(z), z] + [\mathcal{F}(z)zxf(z), z] + [zf(z)xf(z), z] = 0 \text{ for all } x, z \in U. \tag{15}$$

Replacing x by zx and y by z respectively in (14), we get

$$z[xzg(z), z] + [\mathcal{F}(z)zxf(z), z] = 0 \text{ for all } x, z \in U. \tag{16}$$

Deference between (15) and (16), we give

$$[[x, z]zg(z), z] + [zf(z)xf(z), z] = 0 \text{ for all } x, z \in U. \tag{17}$$

Substituting zx in place of x in (17), we arrive at

$$z[[x, z]zg(z), z] + [zf(z)zxf(z), z] = 0 \text{ for all } x, z \in U. \tag{18}$$

Left multiplying (17) by z and then subtracting from (18), we obtain

$$[z[f(z), z]xf(z), z] = 0 \text{ for all } x, z \in U. \tag{19}$$

Replacing again x by xz in (19), we find

$$[z[f(z), z]xzf(z), z] = 0 \text{ for all } x, z \in U. \tag{20}$$

Right multiplying (19) by z and then subtracting from (20), we obtain

$$[z[f(z),z]x[f(z),z],z] = 0 \text{ for all } x,z \in U \tag{21}$$

which implies that

$$z[f(z),z]x[f(z),z]z - z^2[f(z),z]x[f(z),z] = 0 \text{ for all } x,u,z \in U. \tag{22}$$

Substituting $xz[f(z),z]uz$ in place of x, where $u \in U$, we get

$$z[f(z),z]xz[f(z),z]uz[f(z),z]z - z^2[f(z),z]xz[f(z),z]uz[f(z),z] = 0. \tag{23}$$

Equations (22) and (23) give

$$z[f(z),z]xz^2[f(z),z]uz[f(z),z] - z[f(z),z]xz[f(z),z]zuz[f(z),z] = 0, \tag{24}$$

which implies that

$$[zf(z),z]x[z[f(z),z],z]u[zf(z),z] = 0 \text{ for all } x,u,z \in U. \tag{25}$$

Since $\mathcal{R}$ is prime ring, we get $[z[f(z),z],z] = 0$ for all $z \in U$, which gives $z[[f(z),z],z] = 0$ for all $z \in U$. Using Lemma 1.3 and the fact $f \neq 0$, we conclude that $\mathcal{R}$ is commutative. □

The following corollaries are immediate consequences of Theorem 2.4.

Corollary 2.5. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations ($f \neq 0$) and g respectively.

(i) If $\mathcal{G}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(ii) If [10, Theorems 1 and 2] If $\mathcal{G}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \mp yx \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative..
(iii) If $\mathcal{G}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \mp [x,y] \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(iv) If $\mathcal{G}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \mp x \circ y \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.

Corollary 2.6. Let $\mathcal{R}$ be a prime ring and U a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with nonzero derivation f.

(i) If $\mathcal{F}(y)\mathcal{F}(x) \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(ii) If $\mathcal{F}(y)\mathcal{F}(x) \mp yx \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(iii) [1, Theorem 5] If $\mathcal{F}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(iv) If $\mathcal{F}(xy) \mp \mathcal{F}(y)\mathcal{F}(x) \mp yx \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(v) [10, Theorem 1 and Theorem 2] If $\mathcal{F}(y)\mathcal{F}(x) \mp [x,y] \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.
(vi) $\mathcal{F}(y)\mathcal{F}(x) \mp x \circ y \in Z(\mathcal{R})$ for all $x,y \in U$, then $\mathcal{R}$ is commutative.

Corollary 2.7 ([9, Theorem 2.1]). Let $\mathcal{R}$ be a prime ring. Suppose that $\mathcal{R}$ admits a multiplicative generalized derivation F associated with a nonzero derivation f.

(i) If $\mathcal{F}$ acts as homomorphism of $\mathcal{R}$, then $\mathcal{R}$ is commutative.

(ii) If $\mathcal{F}$ acts as anti-homomorphism of $\mathcal{R}$, then $\mathcal{R}$ is commutative.

The following example proves that the primeness of $\mathcal{R}$ is essential in the theorem 2.1 and the theorem 2.4, and all their corollaries.

Example 2.8. Let $\mathbb{Z}$ be the set of integers, and define $\mathcal{R}$, and $\mathcal{F}:\mathcal{R}\to\mathcal{R}$ as follows:

$$\mathcal{R}=\left\{\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix} \mid x,y,z,0\in\mathbb{Z}\right\},U=\left\{\begin{pmatrix}0&0&y\\0&0&0\\0&0&0\end{pmatrix} \mid y,0\in\mathbb{Z}\right\}$$

$$\mathcal{G}\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&y\\0&0&0\\0&0&0\end{pmatrix} \text{ and } g\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&x&y^2\\0&0&z\\0&0&0\end{pmatrix}.$$

$$\mathcal{F}\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&y\\0&0&yz\\0&0&0\end{pmatrix} \text{ and } f\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-x&0\\0&0&z\\0&0&0\end{pmatrix}.$$

$$\mathcal{H}\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&x&y\\0&0&0\\0&0&0\end{pmatrix}.$$

We can verify that U is an ideal of $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{F}$ are multiplicative generalized derivations associated with derivation $f\neq 0$ and a map g respectively and $\mathcal{H}$ is a left multiplier.

(i) $\mathcal{G}(xy)\mp\mathcal{F}(x)\mathcal{F}(y)+\mathcal{H}(yx)\in Z(\mathcal{R})$ for all $x,y\in U$.

(ii) $\mathcal{G}(xy)\mp\mathcal{F}(y)\mathcal{F}(x)+\mathcal{H}(yx)\in Z(\mathcal{R})$ for all $x,y\in U$.

But $\mathcal{R}$ is not commutative.

Theorem 2.9. Let $\mathcal{R}$ be a prime ring and $\mathcal{F}$ be a multiplicative generalized derivation associated with derivation f such $f(Z(\mathcal{R}))\neq\{0\}$. If $a,b\notin Z(\mathcal{R})$, then there is no multiplicative generalized derivations $\mathcal{G}$, $\mathcal{H}$ associated respectively with derivations g,h satisfying $\mathcal{F}(x)-a\mathcal{G}(x)-\mathcal{H}(x)b\in Z(\mathcal{R})$ for all $x\in\mathcal{R}$.

Proof. Suppose there exist derivations $\mathcal{G}$ and $\mathcal{H}$ such that

$$\mathcal{F}(x)-a\mathcal{G}(x)(x)-\mathcal{H}(x)b\in Z(\mathcal{R}) \text{ for all } x\in\mathcal{R}. \tag{26}$$

Replacing x by xz in (26), where $z\in Z(\mathcal{R})$ and using it we get

$$xf(z)-axg(z)-xh(z)b\in Z(\mathcal{R}) \text{ for all } x\in\mathcal{R},z\in Z(\mathcal{R}). \tag{27}$$

Add and subtract $xag(z)$ in (27) we find that

$$x(f(z)-ag(z)-h(z)b)+[x,a]g(z)\in Z(\mathcal{R}) \text{ for all } x\in\mathcal{R},z\in Z(\mathcal{R}). \tag{28}$$

Replacing x by sx in (28) and using it, we arrive at

$$(29)\qquad s\big(x(f(z)-ag(z)-h(z)b)+[x,a]g(z)\big)+[s,a]xg(z)\in Z(\mathcal{R}).$$

From (27) and (29), we can easily find that

$$\big[[s,a]xg(z),s\big]=0 \text{ for all } x,s\in\mathcal{R}, z\in Z(\mathcal{R}).$$

When developing the last expression, one can find

$$(30)\qquad \big([[s,a],s]x+[s,a][x,s]\big)g(z)=0 \text{ for all } x,s\in\mathcal{R}, z\in Z(\mathcal{R}).$$

Which leads to

$$\big([[s,a],s]x+[s,a][x,s]\big)\mathcal{R}g(z)=\{0\} \text{ for all } s\in\mathcal{R}, z\in Z(\mathcal{R}).$$

Using the primness of $\mathcal{R}$, we find that

$$(31)\qquad [[s,a],s]x+[s,a][x,s]=0 \text{ or } g(z)=0 \text{ for all } x,s\in\mathcal{R}, z\in Z(\mathcal{R}).$$

Suppose the first case, and replacing x by xa and using it, we obtain

$$(32)\qquad [s,a]x[a,s]=0 \text{ for all } x,s\in\mathcal{R}.$$

Which implies that $[s,a]\mathcal{R}[s,a]=\{0\}$. By primness of $\mathcal{R}$, we get $[s,a]=0$ for all $x\in\mathcal{R}$, which forces that $a\in Z(\mathcal{R})$, contradiction.

Now assuming that $g(z)=0$, (27) leads us to

$$(33)\qquad x(f(z)-h(z)b)\in Z(\mathcal{R}) \text{ for all } x\in\mathcal{R}, z\in Z(\mathcal{R}).$$

By primness of $\mathcal{R}$, we find $f(z)=h(z)b$ for all $z\in Z(\mathcal{R})$ or $\mathcal{R}$ is commutative. The second conclusion gives a contradiction. Hence, we can easily arrive at

$$[b,x]h(z)=0 \text{ for all } x\in\mathcal{R}, z\in Z(\mathcal{R}).$$

Since $\mathcal{R}$ is prime and $b\notin Z(\mathcal{R})$, the last equation reduces to $h(z)=0$. Now replacing $h(z)=0$ in (33) we get $xf(z)\in Z(\mathcal{R})$ for all $x\in\mathcal{R}$, $z\in Z(\mathcal{R})$ by Lemma 1.1, we obtain $f(Z(\mathcal{R}))=\{0\}$, which gives a contradiction.

□

COROLLARY 2.10. Let $\mathcal{R}$ be a prime ring of $\mathcal{R}$. If $\mathcal{R}$ admits $(\mathcal{F},\mathcal{G},\mathcal{H})$ multiplicative generalized derivations associated respectively with multiplicative derivations (f,g,h), such that $\mathcal{F}(x)=a\mathcal{G}(x)-\mathcal{H}(x)b$ for all $x\in\mathcal{R}$. If $f(Z(\mathcal{R}))\neq\{0\}$, then $a\in Z(\mathcal{R})$ or $b\in Z(\mathcal{R})$.

Proof. Assume that $a\notin Z(\mathcal{R})$ and $b\notin Z(\mathcal{R})$ such that $\mathcal{F}(x)-a\mathcal{G}(x)-\mathcal{H}(x)b=0$ for all $x\in\mathcal{R}$. Since $f(Z(\mathcal{R}))\neq\{0\}$, using Theorem 2.9, we arrive at a contradiction. □

THEOREM 2.11. Let $\mathcal{R}$ be a 2-torsion free prime ring and multiplicative generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with multiplicative derivations f and g, respectively, such that $\mathcal{F}(x)x-x\mathcal{G}(x)\in Z(\mathcal{R})$ for all $x\in\mathcal{R}$, then $\mathcal{R}$ is an integral domain or $f(Z(\mathcal{R}))=g(Z(\mathcal{R}))$.

Proof. Assume that $\mathcal{F}(x)x - x\mathcal{G}(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. It follows that $[\mathcal{F}(x)x - x\mathcal{G}(x), y] = 0$ which leads to

$$[\mathcal{F}(x), y]x + \mathcal{F}(x)[x, y] - [x, y]\mathcal{G}(x) - x[\mathcal{G}(x), y] = 0. \tag{34}$$

Replacing x by xh in (34) with $h \in Z(\mathcal{R})$ and using it we find

$$[x, y]x(f(h) - g(h)) + x[x, y](f(h) - g(h)) = 0 \text{ for all } x, y \in \mathcal{R}, h \in Z(\mathcal{R}).$$

Which implies that $([x, y] \circ x)(f(h) - g(h)) = 0$ for all $x, y \in \mathcal{R}, h \in Z(\mathcal{R})$. Using the primness of $\mathcal{R}$, it is easy to see that $(f(h) - g(h)) = 0$ for all $h \in Z(\mathcal{R})$ or $[x, y] \circ x = 0$ for all $x, y \in \mathcal{R}$.
Now assume that

$$[x, y] \circ x = 0 \text{ for all } x, y \in \mathcal{R} \tag{35}$$

It is follow that

$$(xy - yx)x + x(xy - yx) = 0 \text{ for all } x, y \in \mathcal{R} \tag{36}$$

which implies that

$$x^2y - yx^2 = 0 \text{ for all } x, y \in \mathcal{R} \tag{37}$$

Equivalently, $x^2 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, which forces that the commutativity of $\mathcal{R}$. □

The following examples show that the condition primness of $\mathcal{R}$ in theorem 2.11 cannot be omitted.

EXAMPLE 2.12. Let the ring $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\mathcal{R}_1$ is a commutative ring and $\mathcal{R}_2$ is a noncommutative ring, as consequence $\mathcal{R}$ is not prime and not commutative, d any derivation of $\mathcal{R}_1$ and $g(x, y) = (d(x), 0)$, so we can prove that $\mathcal{G}$ defined in $\mathcal{R}$ by $\mathcal{G}(x, y) = (d(x) + x, y)$ is a multiplicative generalized derivation associated with a multiplication derivation g, $(\mathcal{G} = g + I_R)$ and we take $\mathcal{F} = I_{\mathcal{R}}$. Since $\mathcal{R}$ satisfy the identity $\mathcal{F}(X)X - X\mathcal{G}(X) \in Z(\mathcal{R})$ for all $X \in \mathcal{R}$ and $f(Z(\mathcal{R})) \neq g(Z(\mathcal{R}))$. But $\mathcal{R}$ is not an integral domain.

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