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EXPLICIT CRITERIA FOR THE OSCILLATION OF CAPUTO TYPE FRACTIONAL-ORDER DELAY DIFFERENTIAL EQUATIONS

RAJASEKAR DEEPALAKSHMI, SIVAGANDHI SARAVANAN, ETHIRAJU THANDAPANI, and ERCAN TUNÇ

Abstract. New explicit criteria for the oscillation of all solutions of first and second-order delay differential equations including the Caputo fractional derivative are presented. Examples illustrating the importance and novelty of the main results are included.

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1. INTRODUCTION

This paper is concerned with the oscillatory behavior of all solutions of the fractional-order delay differential equations of the form

$$x'(t) + p\ {}^{C}D_0^{\alpha}x(t) + qx(t-\tau) = 0, \tag{1}$$

and

$$x''(t) + p\ {}^{C}D_0^{\alpha}x(t) + qx(t-\tau) = 0, \tag{2}$$

where $t > 0$, $p, q, \tau \in \mathbb{R}^+$, $\alpha \in (0,1)$, and ${}^{C}D_0^{\alpha}x$ denotes the Caputo fractional derivative of x of order α, that is,

$${}^{C}D_0^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)\mathrm{d}s,\ t \geq 0;$$

for the definition of the Caputo derivative of order $\alpha \in (n-1, n)$, $n \geq 1$, see [3, 6, 13, 15] for details.

As usual, a nontrivial solution of a differential equation is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory.* If all the solutions of an equation are oscillatory, then this equation itself is called oscillatory.

In recent years, fractional-order differential equations have gained considerably more attention among the researchers due to these type of equations find

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Corresponding author: Ercan Tunç.

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applications in various fields of the natural sciences and engineering. Readers interested in the applications of equations of this kind can refer to [11, 12, 15].

Caputo fractional derivative is widely used in the control theory and to describe economic processes because of its good explanation about the memory of characteristics of economic variables [12, 17]. It also is useful to note that the Laplace transform of Caputo's derivative has a similar transformation as that of integer-order derivatives. Next, we recall some facts about the Laplace transform, which play a significant role in the proofs of the main results. Let $x : [0, \infty) \to \mathbb{R}$ be a real-valued function. The Laplace transform of $x(t)$ is denoted by $\mathcal{L}[x(t)]$ or $X(s)$ and is given by the improper integral

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty \mathrm{e}^{-st} x(t) \mathrm{d}t, \tag{3}$$

and the abscissa of convergence of the Laplace transform $X(s)$ of $x(t)$ is defined as

$$\delta_0 = \inf\{\delta \in \mathbb{R} : X(\delta) \text{ exists}\}.$$

Therefore, $X(s)$ exists for $Re(s) > \delta_0$. The Laplace transform of the Caputo fractional derivative of $x(t)$ of order $\alpha \in (0, 1)$ is given by (see [4, 15])

$$\mathcal{L}[{}^C D_0^\alpha x(t)] = s^\alpha X(s) - s^{\alpha-1} x(0),$$

and the integer order derivative are

$$\mathcal{L}[x'(t)] = sX(s) - x(0),$$

and

$$\mathcal{L}[x''(t)] = s^2 X(s) - sx(0) - x'(0).$$

The oscillation theory of fractional-order differential equations received less attention compared to integer-order differential equations; we can refer the reader to [1, 2, 5, 7, 8, 14, 16, 18] and the references therein for some typical results, where the authors provide collection of oscillation results obtained for various types of fractional-order differential equations.

In [18], the authors considered the equation of the form

$${}^C D_t^\alpha x(t) + qx(t - \tau) = 0, \quad t > 0, \tag{4}$$

and obtained explicit criteria for the oscillation of all solutions of (4). In [5], the author studied equations (1) and (2) and provided oscillation criteria using characteristic equations of (1) and (2). It is known fact that finding the roots of such equations are extremely difficult since those equations involves transcendental function.

Therefore in this paper, we obtain easily verifiable conditions to get the oscillation of all solutions of the studied equations. Examples are provided to show the novelty of our results.

2. OSCILLATION RESULTS

THEOREM 2.1. *Let $p, q, \tau \in \mathbb{R}^+$ and let $\alpha \in (0,1)$ be the ratio of two odd integers. If the equation*

$$\lambda + p\lambda^\alpha + q\mathrm{e}^{-\lambda\tau} = 0 \tag{5}$$

has no real roots, then every solution of (1) *oscillates.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1), that is, there exists a positive constant T such that $x(t) > 0$ for all $t \geq T$. As equation (1) is autonomous, we may assume $x(t) > 0$ for $t \geq -\tau$. Taking Laplace transform on both sides of (1) yields, for $Re(s) > \delta_0$,

$$sX(s) + ps^\alpha X(s) + q\mathrm{e}^{-s\tau}X(s) - x(0) - ps^{\alpha-1}x(0) + q\mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st}x(t)\mathrm{d}t = 0,$$

that is,

$$(s + ps^\alpha + q\mathrm{e}^{-s\tau})X(s) = x(0)(1 + ps^{\alpha-1}) - q\mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st}x(t)\mathrm{d}t. \tag{6}$$

Let

$$F(s) = s + ps^\alpha + q\mathrm{e}^{-s\tau}, \tag{7}$$

and

$$\Omega(s) = (1 + ps^{\alpha-1})x(0) - q\mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st}x(t)\mathrm{d}t. \tag{8}$$

It follows from (6)–(8) that

$$F(s)X(s) = \Omega(s), \quad \mathrm{Re}\,(s) > \delta_0. \tag{9}$$

Clearly, the functions $F(s)$ and $\Omega(s)$ are entire functions. In view of (5), we have $F(s) \neq 0$ for all $s \in \mathbb{R}$, and so, it follows from (9) that

$$X(s) = \frac{\Omega(s)}{F(s)}, \quad \mathrm{Re}\,(s) > \delta_0. \tag{10}$$

Since $\delta_0 = -\infty$ (see [9, Theorem 2.1.1]), equality (10) becomes

$$X(s) = \frac{\Omega(s)}{F(s)} \quad \text{for all} \quad s \in \mathbb{R}. \tag{11}$$

By taking $s \to -\infty$, we see that (11) leads to a contradiction since $X(s)$ and $F(s)$ are always positive whereas $\Omega(s)$ becomes eventually negative by (8). The positivity of $F(s)$ follows from (7), $F(0) = q > 0$ and equation (5) has no real roots. The positivity $X(s)$ follows from (3) and $x(t) > 0$ for $t \geq 0$. By the positivity of $x(t)$ on $[-\tau, 0]$ and $\lim_{s\to-\infty} s^{\alpha-1} \to 0$, we have $\lim_{s\to-\infty} \Omega(s) = -\infty$. The proof of the theorem is complete. □

THEOREM 2.2. *Let $p, q, \tau \in \mathbb{R}^+$ and let $\alpha \in (0,1)$ be the ratio of two odd integers. If the equation*

$$\lambda^2 + p\lambda^\alpha + q\mathrm{e}^{-\lambda\tau} = 0 \tag{12}$$

has no real roots, then every solution of (2) oscillates.

Proof. The proof is similar that of Theorem 2.1 and hence it is omitted. □

LEMMA 2.3 ([10]). *Let $a \in \mathbb{R}^+$ and $0 < \alpha \leq 1$. Then*

$$a^\alpha \leq \alpha a + (1-\alpha), \tag{13}$$

and equality holds if $\alpha = 1$.

Having the above results, we will derive explicit sufficient conditions for the oscillation of equations (1) and (2).

THEOREM 2.4. *Let $p, q, \tau \in \mathbb{R}^+$ and let $\alpha \in (0,1)$ be the quotient of odd integers. If*

$$(qe\tau - 1)(q + p\alpha - p)^{1-\alpha} - p(1+p\alpha)^{1-\alpha} > 0, \tag{14}$$

then (1) oscillates.

Proof. In view of Theorem 2.1, our goal is to prove that equation (5) has no real roots. For the sake of contradiction that equation (5) has a negative real root λ. In fact, if $\lambda \geq 0$, then $\lambda + p\lambda^\alpha + q\mathrm{e}^{-\lambda\tau} > 0$. Let $\lambda_1 = -\lambda > 0$. Since α is the ratio of odd integers, it follows from (5) that

$$\lambda_1 + p\lambda_1^\alpha = q\mathrm{e}^{\lambda_1\tau} \geq q. \tag{15}$$

Using (13) in (15), we obtain

$$\lambda_1 + p\alpha\lambda_1 + p(1-\alpha) \geq \lambda_1 + p\lambda_1^\alpha \geq q$$

or

$$\lambda_1(1+p\alpha) \geq q - p(1-\alpha). \tag{16}$$

From (15) and the inequality $\mathrm{e}^x \geq ex$ for $x \geq 0$, we observe that $\lambda_1 + p\lambda_1^\alpha \geq \lambda_1 qe\tau$, or $p\lambda_1^\alpha \geq \lambda_1(qe\tau - 1)$, or $p \geq \lambda_1^{1-\alpha}(qe\tau - 1)$, or

$$p \geq \left(\frac{q + p\alpha - p}{1+p\alpha}\right)^{1-\alpha}(qe\tau - 1),$$

where we have used (16). This contradicts with (14). The proof of the theorem is complete. □

COROLLARY 2.5. *Let $p, q, \tau \in \mathbb{R}^+$ and let $\alpha \in (0,1)$ be the quotient of odd integers. If (14) holds and*

$$\liminf_{t\to\infty} q(t) = q > 0, \tag{17}$$

then every solution of the equation

$$x'(t) + p\,{}^C D_0^\alpha x(t) + q(t)x(t-\tau) = 0, \quad t > 0, \tag{18}$$

where $q \in C\left((0,\infty), \mathbb{R}^+\right)$, oscillates.

Proof. Let $x(t)$ be an eventually positive solution of (18). Then there exists $T > 0$ sufficiently large such that $x(t) > 0$ and $x(t-\tau) > 0$ for all $t \geq T$. It follows from (17) and (18) that, for $t \geq T$,

$$\begin{aligned} 0 &= x'(t) + p\, {}^C D_0^\alpha x(t) + q(t)x(t-\tau) \\ &\geq x'(t) + p\, {}^C D_0^\alpha x(t) + \liminf_{t\to\infty} q(t)x(t-\tau) \\ &= x'(t) + p\, {}^C D_0^\alpha x(t) + qx(t-\tau). \end{aligned}$$

Now, we see that the eventually positive solution $x(t)$ satisfies the inequality

$$x'(t) + p\, {}^C D_0^\alpha x(t) + qx(t-\tau) \leq 0, \quad t \geq T. \tag{19}$$

In view of (14) and Theorem 2.4, equation (5) has no real roots. Therefore, similarly to the proof of Theorem 2.1, inequality (19) has no eventually positive solution, which implies that every solution of (18) oscillates. The proof of the corollary is complete. □

THEOREM 2.6. *Let $p, q, \tau \in \mathbb{R}^+$ and let $\alpha \in (0,1)$ be the quotient of odd integers. If*

$$q^{\frac{1}{\alpha}} \mathrm{e}\tau - p^{\frac{1}{\alpha}} > 0, \tag{20}$$

then every solution of (2) *oscillates.*

Proof. In view of Theorem 2.2, our goal is to prove that equation (12) has no real roots. For the sake of contradiction that equation (12) has a negative real root λ. In fact, if $\lambda \geq 0$, then $\lambda^2 + p\lambda^\alpha + q\mathrm{e}^{-\lambda\tau} > 0$. Let $\lambda_1 = -\lambda > 0$. Since α is the ratio of odd integers, it follows from (12) that

$$p\lambda_1^\alpha = \lambda_1^2 + q\mathrm{e}^{\lambda_1\tau} \geq q. \tag{21}$$

Using the fact that $\mathrm{e}^x \geq ex$ for $x \geq 0$ in (21), we obtain

$$p\lambda_1^\alpha = \lambda_1^2 + q\mathrm{e}^{\lambda_1\tau} \geq \lambda_1 q\mathrm{e}\tau. \tag{22}$$

Combining (21) and (22) leads to $p^{\frac{1}{\alpha}} \geq q^{\frac{1}{\alpha}}\mathrm{e}\tau$, which contradicts to (20). This completes the proof of the theorem. □

3. NUMERICAL EXAMPLES

We present two examples to show the importance and novelty of our main results.

EXAMPLE 3.1. Consider the first-order delay differential equation with fractional-order derivative of the form

$$x'(t) + {}^C D_0^{1/3} x(t) + \sqrt{3}x\left(t - \frac{2\pi}{3}\right) = 0, \tag{23}$$

where $\alpha = 1/3$, $p = 1$, $q = \sqrt{3}$ and $\tau = 2\pi/3$. By doing a simple calculation, we see that condition (14) gives $8.0313 > 0$. That is, condition (14) is satisfied.

Therefore, by Theorem 2.4, equation (23) is oscillatory. In fact, one such oscillatory solution of (23) is $x(t) = \sin t$.

EXAMPLE 3.2. Consider the second-order delay differential equation with fractional-order derivative of the form

$$x''(t) + p\,{}^{C}D_0^{3/5}x(t) + qx\left(t - \frac{17\pi}{10}\right) = 0, \tag{24}$$

where $\alpha = 3/5$, $p = \sqrt{2}\sqrt{5-\sqrt{5}}/4$, $q = (\sqrt{5}+1)/4$ and $\tau = 17\pi/10$. By doing a simple calculation, we see that condition (20) yields $9.7846 > 0$. That is, condition (20) is satisfied. Therefore, by Theorem 2.6, equation (24) is oscillatory. In fact, one such oscillatory solution of (24) is $x(t) = \cos\ t$.

REMARK 3.3. Notice that when $p = 0$, we see that equations (1) and (2) reduce to equations without fractional derivative. That is, condition (14) reduces to $q\tau > \frac{1}{e}$ and condition (20) reduce to $q > 0$, which are well-known conditions for the oscillation of the equations $x'(t) + qx(t-\tau) = 0$, and $x'' + qx(t-\tau) = 0$, respectively.

4. CONCLUSION

In this paper, based on parameters and fractional exponent, we have provided explicit sufficient conditions for the oscillation of all solutions of the studied equations (1) and (2). The obtained results extended and generalized that of in [18]. Further our results are advantageous over that of in [5], since our conditions are explicit and easy to verify. Thus, the results obtained here are further contribution to the oscillation theory of fractional differential equations.

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Tamil Nadu Dr. Ambedkar Law University
Department of Interdisciplinary Studies
Chennai-600 028, India
E-mail: profdeepalakshmi@gmail.com
https://orcid.org/0000-0002-8310-2859

Madras School of Economics
Chennai-600 025, India
E-mail: profsaran11@gmail.com
https://orcid.org/0000-0003-4562-2500

University of Madras
Ramanujan Institute for Advanced Study in Mathematics
Chennai–600 005, India
E-mail: ethandapani@yahoo.co.in
https://orcid.org/0000-0001-6801-4191

Tokat Gaziosmanpasa University
Faculty of Arts and Sciences
Department of Mathematics
Tokat–60240, Turkey
E-mail: ercantunc72@yahoo.com
https://orcid.org/0000-0001-8860-608X