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SANDOR'S INEQUALITY FOR PSEUDO INTEGRALS

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Abstract. In this paper, we show a Sandor type inequality for pseudo-integrals. Indeed, we prove a classic version of this inequality for pseudo-integrals. Some illustrative examples are given for the theorems. We continue by proving a strengthened version of Sandor's inequality for pseudo-integral.

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Key words. Sandor type inequality, fuzzy integral inequality, pseudo-integral, fuzzy measure.

1. INTRODUCTION

The theory of fuzzy measures and fuzzy integral (Sugeno integral) was introduced by Sugeno [22] in his Ph.D. thesis in 1974. The properties and applications of fuzzy integral have been studied by many authors. Ralescu and Adams studied in [21] several equivalent definitions of fuzzy integrals; Román Flores et al. started studying some fuzzy integral inequalities for monotone functions with applications for solving fuzzy integrals.

Since 2007, some authors have studied some other fuzzy integral inequalities (see [5, 7–10, 13]).

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([17], [18], [19]). Based on this structure, there were developed the concepts of $\oplus$ measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc.

Recently, Daraby et al. generalized Stolarsky, Hardy and Feng Qi type inequalities for pseudo-integrals ([4, 6, 11, 12]).

Sandor inequality in classical case has the following form.

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THEOREM 1.1 ([3]). *Let $f : [a, b] \to \mathbb{R}$ be a convex and non-negative function. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)\mathrm{d}x \leq \frac{1}{3}\left[f^2(a) + f(a)f(b) + f^2(b)\right], \tag{1}$$

holds.

Sandor's inequality is proved in some versions for Sugeno integrals, for more details of these versions, we refer reader to [3]. Also, Li et al. have proved Sandor type inequalities for Sugeno integral with respect to general (α, m, r)-convex functions in [14]. Moreover Yang et al. and Lu et al. have studied on Sandor's inequalities for fuzzy integrals respectively in [23] and [15].

In this paper, we express and prove the Sandor type inequality for pseudo integrals and illustrate it by some examples. Also, we prove a strengthened version of Sandor's inequality for pseudo-integral.

2. PRELIMINARY

Now, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details, we refer to [16, 24].

Let $[a, b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

DEFINITION 2.1. (Wang and Klir [24]). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

DEFINITION 2.2 (Wang and Klir [24]). The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty)$, i.e., pseudo-operations are given with:

$$x \oplus y = g^{-1}(g(x) + g(x)) \qquad \text{and} \qquad x \odot y = g^{-1}(g(x)g(x)). \tag{2}$$

Then, the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the $g-$integral

$$\int_{[c,d]}^{\oplus} f(x)\mathrm{d}x = g^{-1}\left(\int_c^d g(f(x))\mathrm{d}x\right). \tag{3}$$

More details on this structure as well as corresponding measures and integrals can be found in [16]. The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot \mathrm{d}m = \sup_{x\in\mathbb{R}} \left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additine. Then any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot \mathrm{d}m$ can be obtained as a limit of g-integrals.

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned} m(A) &= \text{ess sup}_{\mu}(x|x \in A) \\ &= \sup\left\{a|\mu(x|x \in A, x > a) > 0\right\}. \end{aligned}$$

Theorem 2.3 (Mesiar and Pap [16]). *Let m be a* sup-*measure on* $([0, \infty]$, $\mathbb{B}[0, \infty])$, *where* $\mathbb{B}([0, \infty])$ *is the Borel σ-algebra on* $[0, \infty]$,

$$m(A) = \text{ess sup}_{\mu}(\psi(x)|x \in A),$$

and $\psi : [0, \infty] \to [0, \infty]$ is a continuous function. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0, \infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power $\lambda, \lambda \in (0, \infty)$) such that $\lim\limits_{\lambda\to\infty} m_\lambda = m$.

Theorem 2.4 (Mesiar and Pap [16]). *Let $([0, \infty], \sup, \odot)$ be a semiring , when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0, \infty)$. Let m be the same as in Theorem 2.3, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$,*

$$\begin{aligned} \int^{\sup} f \odot \mathrm{d}m &= \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot \mathrm{d}m_\lambda \\ &= \lim_{\lambda\to\infty} (g^\lambda)^{-1}\left(\int g^\lambda(f(x))\mathrm{d}x\right). \end{aligned} \tag{4}$$

3. MAIN RESULT

In this section, we express and prove Sandor's inequality for pseudo-integrals.

3.1. Sandor type inequality for pseudo integrals.

THEOREM 3.1. *Let $f : [a,b] \to [c,d]$ be a continuous, convex and non-negative function and $g : [c,d] \to [0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^{2}(x)\mathrm{d}x\right) \leq \frac{1}{3} g\Big(\left[f_{\odot}^{2}(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^{2}(b)\right]\Big), \tag{5}$$

holds.

Proof. From the right side of inequality, we have

$$\begin{aligned}
&\frac{1}{3}\left[f_{\odot}^{2}(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^{2}(b)\right] \\
&\quad= \frac{1}{3}\left\{\left[g^{-1}(g(f)(a) \cdot g(f)(a)\right) \oplus g^{-1}\left(g(f)(a) \cdot g(f)(b)\right)\right] \\
&\quad\quad \oplus \left[g^{-1}\left(g(f)(b) \cdot g(f)(b)\right]\right\} \\
&\quad= \frac{1}{3}\Big\{g^{-1}\Big(g\left(g^{-1}\left(g(f)(a) \cdot g(f)(a)\right)\right) + g\left(g^{-1}\left(g(f)(a) \cdot g(f)(b)\right)\right)\Big) \\
&\quad\quad \oplus \left[g^{-1}\left(g(f)(b) \cdot g(f)(b)\right)\right]\Big\} \\
&\quad= \frac{1}{3}\Bigg\{g^{-1}\Big(g\Big[g^{-1}\Big(g\left(g^{-1}\left(g(f)(a) \cdot g(f)(a)\right)\right) \\
&\quad\quad + g\left(g^{-1}\left(g(f)(a) \cdot g(f)(b)\right)\right)\Big)\Big] + g\Big[g^{-1}\left(g(f)(b) \cdot g(f)(b)\right]\Big)\Big)\Bigg\} \\
&\quad= \frac{1}{3}\left\{g^{-1}\Big(g(f)^{2}(a) + g(f)(a) \cdot g(f)(b) + g(f)^{2}(b)\Big)\right\}.
\end{aligned}$$

From classical version of Sandor's inequality and Equality 3, we conclude that

$$\begin{aligned}
&\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b} g(f)^{2}(x)\mathrm{d}x\right]\right) \\
&\quad\quad \leq \frac{1}{3}\left(g^{-1}\left[g(f)^{2}(a) + g(f)(a) \cdot g(f)(b) + g(f)^{2}(b)\right]\right).
\end{aligned}$$

Continuting left side of the above mentioned inequality follows that:

$$\begin{aligned}
&\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b} g(f)(x) \cdot g(f)(x)\mathrm{d}x\right]\right) \\
&\quad= \frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b} gg^{-1}\left(g(f)(x) \cdot g(f)(x)\mathrm{d}x\right)\right]\right) \\
&\quad= \frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right)\int_{a}^{b} g\left(f_{\odot}^{2}(x)\right)\mathrm{d}x\right]\right)
\end{aligned}$$

$$= \frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right) gg^{-1}\int_a^b\int_a^b g\left(f_{\odot}^2(x)\right)\mathrm{d}x\right]\right)$$
$$= \frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)\right)\right]\right).$$

It follows that:

$$\frac{1}{3}\left(g^{-1}\left[\left(\frac{3}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x\right)\right]\right)$$
$$\leq \frac{1}{3}\left[f_{\odot}^2(a) \oplus f(a)\odot f(b) \oplus f_{\odot}^2(b)\right].$$

Then

$$g^{-1}\left[\left(\frac{3}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x\right)\right] \leq \left[f_{\odot}^2(a) \oplus f(a)\odot f(b) \oplus f_{\odot}^2(b)\right].$$

So,

$$\left(\frac{1}{b-a}\right) g\left(\int_{[a,b]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x\right) \leq \frac{1}{3} g\left(f_{\odot}^2(a) \oplus f(a)\odot f(b) \oplus f_{\odot}^2(b)\right).$$

Thereby, the proof is complete. □

When, we restrict our argument to semiring $([0,1],\oplus,\odot)$, we obtain the following theorem.

Corollary 3.2. *Let $f:[0,1]\to[c,d]$ be a continuous, convex and non-negative function and $g:[c,d]\to[0,\infty)$ be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g\left(\int_{[0,1]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x\right) \leq \frac{1}{3} g\left[f_{\odot}^2(0) \oplus f(0)\odot f(1) \oplus f_{\odot}^2(1)\right], \tag{6}$$

holds.

Example 3.3. Let f and g are defined from $[0,1]$ to $[0,1]$ by $f(x)=x^2$ and $g(x)=\sqrt{x}$. Then we have

$$\frac{1}{4} = \frac{1}{1-0}\int_{[0,1]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x \leq \frac{1}{3} g\left[f_{\odot}^2(0) \oplus f(0)\odot f(1) \oplus f_{\odot}^2(1)\right] = \frac{1}{3}.$$

Example 3.4. Let f and g are defined from $[0,1]$ to $[0,1]$ by $f(x)=x^2$, $g(x)=x^3$. Then we have

$$\frac{1}{13} = \frac{1}{1-0}\int_{[0,1]}^{\oplus} f_{\odot}^2(x)\mathrm{d}x \leq \frac{1}{3} g\left[f_{\odot}^2(0) \oplus f(0)\odot f(1) \oplus f_{\odot}^2(1)\right] = \frac{1}{3}.$$

The following example shows that convexity of f in Theorem 3.1 is necessary.

EXAMPLE 3.5. Suppose that $f(x) = \sqrt{x}$ and $g(x) = x^2$. Them simple calculaion show that

$$\frac{2}{3} = \frac{1}{1-0}\int_{[0,1]}^{\oplus} f_{\odot}^{2}(x)\mathrm{d}x \not\leq \frac{1}{3}g\left[f_{\odot}^{2}(0) \oplus f(0) \odot f(1) \oplus f_{\odot}^{2}(1)\right] = \frac{1}{3}.$$

We can not remove the assumption g is increasing in Theorem 3.1. The following example shows this fact.

EXAMPLE 3.6. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Then we have

$$\frac{5}{6} = \frac{1}{1-0}\int_{[0,1]}^{\oplus} f_{\odot}^{2}(x)\mathrm{d}x \not\leq \frac{1}{3}g\left[f_{\odot}^{2}(0) \oplus f(0) \odot f(1) \oplus f_{\odot}^{2}(1)\right] = \frac{1}{3}.$$

COROLLARY 3.7. *In the Theorem 3.1, if we suppose that $g(x) = x$, then the Inequality* (5) *follows that Inequality* (1).

Now, we generalize the Sandor type inequity by the semiring $([a, b], \sup, \odot)$, where $\lambda \in (0, \infty)$.

THEOREM 3.8. *Let $f : [a, b] \to [a, b]$ be a measurable comonotone function and $([a, b], \sup, \odot)$ be a simiring and m be the same as Theorems 2.3 and 2.4. If g is a continuous and increasing function, then the following inequality*

$$(7) \qquad \left(\frac{1}{b-a}\right) g\left(\int_{[a,b]}^{\sup} f_{\odot}^{2}(x)\mathrm{d}x\right) \leq \frac{1}{3}g\left[f_{\odot}^{2}(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^{2}(b)\right],$$

holds.

Proof. The proof is similar to the Theorem 3.1. □

Now, by an example we illustrate the validity of Theorem (3.8).

EXAMPLE 3.9. Let $g^{\lambda}(x) = e^{\lambda x}$. Then we have:

$$x \oplus y = \lim_{\lambda\to\infty} \ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x, y),$$
$$x \odot y = \lim_{\lambda\to\infty} \frac{1}{\lambda}\left(e^{\lambda x} \cdot e^{\lambda y}\right) = x + y.$$

Therefore (3.3) reduces on the following inequality:

$$\frac{1}{b-a}\sup\left(f^2(x) + \psi(x)\right) \leq \frac{1}{3}\left[f^2(a) \oplus f(a) \cdot f(b) \oplus f^2(b)\right].$$

where ψ is the same as in Theorem 2.3.

Note that the third important case $\oplus = max$ and $\odot = min$ has been studied in [3] and the pseudo-integral in a such a case yields the Sugeno integrals.

4. FURTHER DISCUSSION

In this section, we proved a strengthened version of Sandor's inequality for pseudo-integral.

THEOREM 4.1. *Let $f : [a, b] \to [c, d]$ be a measurable function. Let a generator $g : [c, d] \to [0, \infty)$ of the pseudo addition $\oplus$ and the pseudo multiplication $\odot$ be an increasing function. Let $\varphi : [c, d] \to [c, d]$ be a continuous and strictly increasing function such that commutes with $\odot$. Then the inequality*

$$\begin{aligned}(8)\quad &\varphi^{-1}\left(\frac{1}{b-a}g\left(\int_{[a,b]}^{\oplus}\varphi\left(f_{\odot}^{2}\right)(x)\mathrm{d}x\right)\right)\\ &\leq \varphi^{-1}\left(\frac{1}{3}g\left[(\varphi(f))_{\odot}^{2}(a)\oplus(\varphi(f))(a)\odot(\varphi(f))(b)\oplus(\varphi(f))_{\odot}^{2}(b)\right]\right).\end{aligned}$$

holds.

Proof. Since φ commutes with $\odot$, then we have

$$\begin{aligned}\varphi^{-1}\left(\frac{1}{b-a}\int_{[a,b]}^{\oplus}\varphi\left(f_{\odot}^{2}\right)(x)\mathrm{d}x\right) &= \varphi^{-1}\left(\frac{1}{b-a}\int_{[a,b]}^{\oplus}(\varphi(f\odot f))(x)\mathrm{d}x\right)\\ &= \varphi^{-1}\left(\frac{1}{b-a}\int_{[a,b]}^{\oplus}(\varphi(f)\odot\varphi(f))(x)\mathrm{d}x\right)\\ &= \varphi^{-1}\left(\frac{1}{b-a}\int_{[a,b]}^{\oplus}(\varphi(f))_{\odot}^{2}(x)\mathrm{d}x\right),\end{aligned}$$

and since f be a convex function and φ be a strictly increasing function, it follows that,

$$\begin{aligned}&\varphi^{-1}\left(\frac{1}{b-a}g\left(\int_{[a,b]}^{\oplus}\varphi\left(f_{\odot}^{2}\right)(x)\mathrm{d}x\right)\right)\\ &\quad\leq \varphi^{-1}\left(\frac{1}{3}g\left[(\varphi(f))_{\odot}^{2}(a)\oplus(\varphi(f))(a)\odot(\varphi(f))(b)\oplus(\varphi(f))_{\odot}^{2}(b)\right]\right).\end{aligned}$$

□

EXAMPLE 4.2. If we suppose that $\varphi(x) = x^s$ for any $s > 0$, then from Inequality (8) we have:

$$\begin{aligned}&\left(\frac{1}{b-a}\right)g\left(\int_{[a,b]}^{\oplus}\left(f_{\odot}^{2}\right)^{s}\mathrm{d}m\right)\\ &\quad\leq \frac{1}{3}g\left(\left(f_{\odot}^{2}\right)^{\frac{1}{s}}(a)\oplus f^{\frac{1}{s}}(a)\odot f^{\frac{1}{s}}(b)\oplus\left(f_{\odot}^{2}\right)^{\frac{1}{s}}(b)\right).\end{aligned}$$

Now, we generalize the above mentioned strengthened version by the semiring $([a, b], \sup, \odot)$, where $\lambda \in (0, \infty)$.

THEOREM 4.3. *If $f : [a,b] \to [c,d]$ be a measurable function and φ is a continuous and strictly function, then by using the Theorem 2.4, we have*

$$
\begin{aligned}
&\varphi^{-1}\left(\frac{1}{b-a} g\left(\int_{[a,b]}^{\sup} \varphi\left(f_{\odot}^{2}\right) \odot \mathrm{d}m\right)\right) \\
&\leq \varphi^{-1}\left(\frac{1}{3} g\left[(\varphi(f))_{\odot}^{2}(a) \oplus (\varphi(f))(a) \odot (\varphi(f))(b) \oplus (\varphi(f))_{\odot}^{2}(b)\right]\right).
\end{aligned} \tag{9}
$$

Proof. From the left side of (9), we have

$$
\begin{aligned}
&\varphi^{-1}\left(\frac{1}{b-a} \int_{[a,b]}^{\sup} \varphi\left(f_{\odot}^{2}\right) \odot \mathrm{d}m\right) \\
&\quad = \varphi^{-1}\left(\frac{1}{b-a} \lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1} \int_{a}^{b} g^{\lambda}\left(\varphi\left(f_{\odot}^{2}\right)\right)(x)\mathrm{d}x\right) \\
&\quad \leq \varphi^{-1}\left(\frac{1}{3} g\left[(\varphi(f))_{\odot}^{2}(a) \oplus (\varphi(f))(a) \odot (\varphi(f))(b) \oplus (\varphi(f))_{\odot}^{2}(b)\right]\right).
\end{aligned}
$$

□

5. CONCLUSION

In this paper, we have proved Sandor type inequality for pseudo integrals. More precisely: Let $f : [a,b] \to [c,d]$ be a continuous, convex and non-negative function and $g : [c,d] \to [0,\infty)$ be a continuous and increasing function. Then

$$
g\left(\frac{1}{b-a} g\left(\int_{[a,b]}^{\oplus} f_{\odot}^{2}(x)\mathrm{d}x\right)\right) \leq \frac{1}{3}\left(g\left[f_{\odot}^{2}(a) \oplus f(a) \odot f(b) \oplus f_{\odot}^{2}(b)\right]\right),
$$

holds. Also we have given some illustrate examples. Moreover, a strengthened version of Sandor's inequality for pseudo-integral is proved.

REFERENCES

[1] H. Agahi, R. Mesiar and Y. Ouyang, *Chebyshev type inequalities for pseudo-integrals*, Nonlinear Anal., **72** (2010), 2737–2743.

[2] P. S. Bullen, *A dictionary of inequalities*, CRC Press, 1998.

[3] J. Caballero and K. Sadaragani, *Sandor's inequality for Sugeno integrals*, Appl. Math. Comput., **218** (2011), 1617–1622.

[4] B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets and Systems, **194** (2012), 90–96.

[5] B. Daraby, *A review on some fuzzy integral inequalities*, Sahand Communications in Mathematical Analysis, **18** (2021), 153–185.

[6] B. Daraby, *Study on some integral inequalities for pseudo-integrals*, Sahand Communications in Mathematical Analysis, **19** (2022), 139–159.

[7] B. Daraby and L. Arabi, *Related Fritz Carlson type inequalities for Sugeno integrals*, Soft Computing, **17** (2013), 1745–1750.

[8] B. Daraby, H. G. Asll and I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Appl. Math. Comput., **305** (2017), 323–329.

[9] B. Daraby, H. Ghazanfary Asll and I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Appl. Math. Comput., **305** (2017), 323–329.

[10] B. Daraby, R. Mesiar, F. Rostampour and A. Rahimi, *Related Thunsdorff type and Frank-Pick type inequalities for Sugeno integral*, Appl. Math. Comput., **414** (2022), Article 126683, 1–6.

[11] B. Daraby, F. Rostampour and A. Rahimi, *Hardy's type inequality for pseudo integrals*, Acta Univ. Apulensis Math. Inform., **42** (2015), 53–65.

[12] B. Daraby, A. Shafiloo and A. Rahimi, *Generalizations of the Feng Qi type inequality for pseudo-integral*, Gazi University Journal of Science, **28** (2015), 695–702.

[13] A. Flores-Franulič and H. Román-Flores, *A Chebyshev type inequality for fuzzy integrals*, Appl. Math. Comput., **190** (2007), 1178–1184.

[14] D. Li, Y. Cheng, and X. Wang, *Sandor Type Inequalities for Sugeno Integral with respect to General* (α, m, r)*-Convex Functions*, J. Funct. Spaces, **2015** (2015), Article 460520, 1–13.

[15] W. Lu, X. Song and L. Huang , *Hermite and Sandaor type inequalities based on fuzzy integrals*, J. Shandong Univ. Nat. Sci., **51** (2016), 22–28.

[16] R. Mesiar and E. Pap, *Idempotent integral as limit of* g*-integrals*, Fuzzy Sets and Systems, **102** (1999), 385–392.

[17] E. Pap, *An integral generated by decomposable measure*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., **20** (1990), 135–144.

[18] E. Pap, *Pseudo-additive measures and their applications*, in *Handbook of Measure Theory*, Elsevier, Amsterdam, 2002, 1403–1465.

[19] E. Pap and N. Ralevič, *Pseudo-Laplace transform*, Nonlinear Analysis, **33** (1998), 553–560.

[20] E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Information Sci., **180** (2010), 543–548.

[21] D. Ralescu and G. Adams, *The fuzzy integral*, J. Math. Anal. Appl., **75** (1980), 562–570.

[22] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[23] X. Yang , X. Song and W. Lu, *Sandor's type inequality for fuzzy integral. Semiannual*, Journal of Mathematics of Nanjing University, **32** (2015), 144–156.

[24] Z. Wang , G.J. Klir , *Fuzzy Measure Theory*, Plenum Press, New York, (1992).

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