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ON THE SOLUTIONS OF A STURM-LIOUVILLE TYPE SYSTEM OF DIFFERENTIAL INCLUSIONS WITH NONLOCAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. We consider a Sturm-Liouville type system of differential inclusions with nonlocal integral boundary conditions and we obtain an existence result for this problem in the case when the set-valued maps have nonconvex values.

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Key words. Differential inclusion, boundary value problem, measurable selection.

1. INTRODUCTION

In the elementary theory of ordinary differential equations it is well known that any linear second-order differential equation can be rewritten in the self-adjoint form $(p(t)x')' = r(t)x$. Together with boundary conditions of the form $a_1x(0) = a_2x'(0)$, $b_1x(T) = b_2x'(T)$ this problem is called the Sturm-Liouville problem. At the same time, a differential inclusion of the form $(p(t)x')' \in F(t,x)$ with any boundary conditions is usually called a Sturm-Liouville type differential inclusion.

In this note we are concerned with the following system

$$\begin{cases} (p(t)u'(t))' \in F(t,u(t),v(t)), & \text{a.e. } t \in [a,b], \\ (q(t)v'(t))' \in G(t,u(t),v(t)), & \text{a.e. } t \in [a,b], \end{cases} \tag{1}$$

with nonlocal integral boundary conditions of the form

$$\begin{cases} a_1u(a) + a_2u(b) = \alpha_1 \int_a^\eta v(s)\mathrm{d}s, & a_3u'(a) + a_4u'(b) = \alpha_2 \int_a^\eta v'(s)\mathrm{d}s, \\ b_1v(a) + b_2v(b) = \alpha_3 \int_\xi^b u(s)\mathrm{d}s, & b_3v'(a) + b_4v'(b) = \alpha_4 \int_\xi^b u'(s)\mathrm{d}s, \end{cases} \tag{2}$$

where $a < \eta < \xi < b$, $p(.) : [a,b] \to (0,\infty)$, $q(.) : [a,b] \to (0,\infty)$ are continuous functions, $a_i, b_i, \alpha_i \in \mathbf{R}_+$, $i = 1,2,3,4$ and $F(.,.,.) : [a,b] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$, $G(.,.,.) : [a,b] \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ are given set-valued maps.

In a recent paper [1], problem (1)–(2) is studied and two existence results for this problem are obtained by using fixed point techniques. Our aim is to improve a result in [1]; namely, we treat the situation when the values of

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the set-valued maps are not convex, but these set-valued maps are assumed to be Lipschitz in the second and third variable. In this case we establish an existence result for problem (1)–(2). Our result use Filippov's technique ([12]); more exactly, the existence of solutions is obtained by starting from a pair of given "quasi" solutions. In addition, the result provides an estimate between the "quasi" solutions and the solutions obtained.

Such kind of results for "simple" Sturm-Liouville differential inclusions may be found in the literature [3–7]. A similar result for a Sturm-Liouville type system of differential inclusions is obtained in [11], but the problem in [11] involves nonlocal multi-point boundary conditions which can't be covered by boundary conditions as in (2). Also, we point out that the approach presented here may be found at coupled systems of fractional differential inclusions [8–10]. Finally, we underline that even if the method we use here is known in the theory of differential inclusions it is largely ignored by the authors that are dealing with such problems in favor of fixed point approaches.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. PRELIMINARIES

We set by I the interval $[a,b]$. We denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_a^b |x(t)|\mathrm{d}t$.

The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset \mathbf{R}$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, where $d^*(A,B) = \sup\{d(a,B); a \in A\}$ and $d(x,B) = \inf_{y\in B} d(x,y)$.

Next the following notations will be used.

$$M = (a_1 + a_2)(b_1 + b_2) - \alpha_1\alpha_3(\eta - a)(b - \xi), \quad K = K_1K_4 - K_2K_3,$$

where

$$K_1 = \frac{a_3}{p(a)} + \frac{a_4}{p(b)}, \quad K_2 = \int_a^\eta \frac{\alpha_2}{q(s)}\mathrm{d}s, \quad K_3 = \int_\xi^b \frac{\alpha_4}{p(s)}\mathrm{d}s, \quad K_4 = \frac{b_3}{q(a)} + \frac{b_4}{q(b)}.$$

The next technical result is proved in [1].

Lemma 2.1. *Let $f(.) : [a,b] \to \mathbf{R}$, $g(.) : [a,b] \to \mathbf{R}$ be continuous mappings and assume that $M \neq 0$, $K \neq 0$. Then the solution of the linear system*

$$\begin{cases} (p(t)u'(t))' = f(t) & t \in [a,b], \\ (q(t)v'(t))' = g(t) & t \in [a,b] \end{cases}$$

with boundary conditions (2) *is given by*

$$(3)\quad \begin{cases} u(t)=\int_a^t(\frac{1}{p(u)}\int_a^u f(\tau)\mathrm{d}\tau)\mathrm{d}u+\frac{1}{M}[-a_2(b_1+b_2)\int_a^b(\frac{1}{p(u)}\int_a^u f(\tau)\mathrm{d}\tau)\mathrm{d}u \\ +\alpha_1(b_1+b_2)\int_a^\eta(\int_a^s\frac{1}{q(u)}(\int_a^u g(\tau)\mathrm{d}\tau)\mathrm{d}u)\mathrm{d}s-\alpha_1b_2(\eta-a)\int_a^b\frac{1}{q(u)} \\ (\int_a^u g(\tau)\mathrm{d}\tau)\mathrm{d}u+\alpha_1\alpha_3(\eta-a)\int_\xi^b(\int_a^s\frac{1}{p(u)}(\int_a^u f(\tau)\mathrm{d}\tau)\mathrm{d}u)\mathrm{d}s] \\ +\frac{1}{KM}[(K_1a_2(b_1+b_2)\int_a^b\frac{1}{p(u)}\mathrm{d}u-K_3\alpha_1(b_1+b_2)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s \\ +K_3\alpha_1b_2(\eta-a)\int_a^b\frac{1}{q(u)}\mathrm{d}u-K_4\alpha_1\alpha_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s \\ -MK_4\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\frac{a_4}{p(b)}\int_a^b f(u)\mathrm{d}u)+(-K_4a_2(b_1+b_2)\int_a^b\frac{1}{p(u)}\mathrm{d}u \\ +K_3\alpha_1(b_1+b_2)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s-K_3\alpha_1b_2(\eta-a)\int_a^b\frac{1}{q(u)}\mathrm{d}u \\ +K_4\alpha_1\alpha_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s+MK_4\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\int_a^\eta\frac{\alpha_2}{q(s)} \\ \int_a^s g(u)\mathrm{d}u)\mathrm{d}s+(K_2a_2(b_1+b_2)\int_a^b\frac{1}{p(u)}\mathrm{d}u-K_1\alpha_1(b_1+b_2) \\ \int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s+K_1\alpha_1b_2(\eta-a)\int_a^b\frac{1}{q(u)}\mathrm{d}u-K_2\alpha_1\alpha_3(\eta-a) \\ \int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s-MK_2\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\frac{b_4}{q(b)}\int_a^b g(u)\mathrm{d}u)+(-K_2a_2(b_1+b_2) \\ \int_a^b\frac{1}{p(u)}\mathrm{d}u+K_1\alpha_1(b_1+b_2)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s-K_1\alpha_1b_2(\eta-a) \\ \int_a^b\frac{1}{q(u)}\mathrm{d}uK_2\alpha_1\alpha_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s \\ +MK_2\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\int_\xi^b\frac{\alpha_4}{p(s)}\int_a^s f(u)\mathrm{d}u\mathrm{d}s)] \end{cases}$$

$$(4)\quad \begin{cases} v(t)=\int_a^t(\frac{1}{q(u)}\int_a^u g(\tau)\mathrm{d}\tau)\mathrm{d}u+\frac{1}{M}[-a_2\alpha_3(b-\xi)\int_a^b(\frac{1}{p(u)}\int_a^u f(\tau)\mathrm{d}\tau)\mathrm{d}u \\ +\alpha_1\alpha_3(b-\xi)\int_a^\eta(\int_a^s\frac{1}{q(u)}(\int_a^u g(\tau)\mathrm{d}\tau)\mathrm{d}u)\mathrm{d}s-b_2(a_1+a_2)\int_a^b\frac{1}{q(u)} \\ (\int_a^u g(\tau)\mathrm{d}\tau)\mathrm{d}u+\alpha_3(a_1+a_2)\int_\xi^b(\int_a^s\frac{1}{p(u)}(\int_a^u f(\tau)\mathrm{d}\tau)\mathrm{d}u)\mathrm{d}s] \\ +\frac{1}{KM}[(K_4a_2\alpha_3(b-\xi)\int_a^b\frac{1}{p(u)}\mathrm{d}u-K_3\alpha_1\alpha_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s \\ +K_3b_2(a_1+a_2)\int_a^b\frac{1}{q(u)}\mathrm{d}u-K_4\alpha_3(a_1+a_2)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s \\ -MK_3\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\frac{a_4}{p(b)}\int_a^b f(u)\mathrm{d}u)+(-K_4a_2\alpha_3(b-\xi)\int_a^b\frac{1}{p(u)}\mathrm{d}u \\ +K_3\alpha_1\alpha_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s-K_3b_2(a_1+a_2)\int_a^b\frac{1}{q(u)}\mathrm{d}u \\ +K_4\alpha_3(a_1+a_2)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s+MK_3\int_a^t\frac{1}{p(u)}\mathrm{d}u)\cdot(\int_a^\eta\frac{\alpha_2}{q(s)}\int_a^s g(u) \\ \mathrm{d}u\mathrm{d}s)+(K_2a_2\alpha_3(b-\xi)\int_a^b\frac{1}{p(u)}\mathrm{d}u-K_1\alpha_1\alpha_3(b-\xi)\int_a^\eta \\ \int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s+K_1b_2(a_1+a_2)\int_a^b\frac{1}{q(u)}\mathrm{d}u-K_2\alpha_3(a_1+a_2)\int_\xi^b\int_a^s\frac{1}{p(u)} \\ \mathrm{d}u\mathrm{d}s-MK_1\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\frac{b_4}{q(b)}\int_a^b g(u)\mathrm{d}u)+(-K_2a_2\alpha_3(b-\xi)\int_a^b\frac{1}{p(u)} \\ \mathrm{d}u+K_1\alpha_1\alpha_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(u)}\mathrm{d}u\mathrm{d}s-K_1b_2(a_1+a_2) \\ \int_a^b\frac{1}{q(u)}\mathrm{d}u+K_2\alpha_3(a_1+a_2)\int_\xi^b\int_a^s\frac{1}{p(u)}\mathrm{d}u\mathrm{d}s \\ +MK_1\int_a^t\frac{1}{p(u)}\mathrm{d}u)(\int_\xi^b\frac{\alpha_4}{p(s)}\int_a^s f(u)\mathrm{d}u\mathrm{d}s)]. \end{cases}$$

Definition 2.2. $(u(.),v(.))\in C(I,\mathbf{R})^2$ is said to be a solution of problem (1)–(2) if there exists $f(.),g(.)\in L^1(I,\mathbf{R})$ such that $f(t)\in F(t,u(t),v(t))$ a.e. (I), $g(t)\in G(t,u(t),v(t))$ a.e. (I) and $u(.)$ and $v(.)$ are given by (3)–(4).

In what follows $\chi_A(\cdot)$ denotes the characteristic function of the set $A\subset\mathbf{R}$.

REMARK 2.3. Let us introduce the following notations

$c_1(t) = \frac{1}{KM}[K_4 a_2(b_1+b_2)\int_a^b \frac{1}{p(u)}\mathrm{d}u - K_3\alpha_1(b_1+b_2)\int_a^\eta\int_a^s \frac{1}{q(u)}\mathrm{d}u\mathrm{d}s +$
$K_3\alpha_1 b_2(\eta-a)\int_a^b \frac{1}{q(u)}\mathrm{d}u - K_4\alpha_1\alpha_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(u)}\mathrm{d}u\mathrm{d}s - MK_4\int_a^t \frac{1}{p(u)}\mathrm{d}u],$

$c_2(t) = \frac{1}{KM}[K_2 a_2(b_1+b_2)\int_a^b \frac{1}{p(u)}\mathrm{d}u - K_1\alpha_1(b_1+b_2)\int_a^\eta\int_a^s \frac{1}{q(u)}\mathrm{d}u\mathrm{d}s +$
$K_1\alpha_1 b_2(\eta-a)\int_a^b \frac{1}{q(u)}\mathrm{d}u - K_2\alpha_1\alpha_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(u)}\mathrm{d}u\mathrm{d}s - MK_2\int_a^t \frac{1}{p(u)}\mathrm{d}u],$

$c_3(t) = \frac{1}{KM}[(K_4 a_2\alpha_3(b-\xi)\int_a^b \frac{1}{p(u)}\mathrm{d}u - K_3\alpha_1\alpha_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(u)}\mathrm{d}u\mathrm{d}s +$
$K_3 b_2(a_1+a_2)\int_a^b \frac{1}{q(u)}\mathrm{d}u - K_4\alpha_3(a_1+a_2)\int_\xi^b\int_a^s \frac{1}{p(u)}\mathrm{d}u\mathrm{d}s - MK_3\int_a^t \frac{1}{p(u)}\mathrm{d}u],$

$c_4(t) = \frac{1}{KM}[(K_2 a_2\alpha_3(b-\xi)\int_a^b \frac{1}{p(u)}\mathrm{d}u - K_1\alpha_1\alpha_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(u)}\mathrm{d}u\mathrm{d}s +$
$K_1 b_2(a_1+a_2)\int_a^b \frac{1}{q(u)}\mathrm{d}u - K_2\alpha_3(a_1+a_2)\int_\xi^b\int_a^s \frac{1}{p(u)}\mathrm{d}u\mathrm{d}s - MK_1\int_a^t \frac{1}{p(u)}\mathrm{d}u],$

$$\mathcal{S}_1(t,z) = \frac{a_4}{p(b)}c_1(t) - c_2(t)\int_\xi^b \frac{\alpha_4}{p(u)}\mathrm{d}u + (\int_z^t \frac{1}{p(u)}\mathrm{d}u)\chi_{[a,t]}(z) - \frac{a_2(b_1+b_2)}{M}\int_z^b \frac{1}{p(u)}\mathrm{d}u + \frac{\alpha_1\alpha_3(\eta-a)}{M}\int_\xi^b(\int_z^s \frac{1}{p(u)}\mathrm{d}u)\chi_{[a,s]}(z)\mathrm{d}s,$$

$$\mathcal{S}_2(t,z) = \frac{b_4}{q(b)}c_2(t) - c_1(t)(\int_z^\eta \frac{\alpha_2}{q(u)}\mathrm{d}u)\chi_{[a,\eta]}(z) + \frac{\alpha_1(b_1+b_2)}{M}\int_a^\eta(\int_z^s \frac{1}{q(u)}\mathrm{d}u)\chi_{[a,s]}(z)\mathrm{d}s - \frac{\alpha_1 b_2(\eta-a)}{M}\int_\xi^b \frac{1}{q(u)}\mathrm{d}u,$$

$$\mathcal{S}_3(t,z) = \frac{a_4}{p(b)}c_3(t) - c_4(t)\int_\xi^b \frac{\alpha_4}{p(u)}\mathrm{d}u - \frac{\alpha_3 a_2(b-\xi)}{M}\int_z^b \frac{1}{p(u)}\mathrm{d}u + \frac{\alpha_3(a_1+a_2)}{M}\int_\xi^b(\int_z^s \frac{1}{p(u)}\mathrm{d}u)\chi_{[a,s]}(z)\mathrm{d}s,$$

$$\mathcal{S}_4(t,z) = \frac{b_4}{q(b)}c_4(t) - c_3(t)(\int_z^\eta \frac{\alpha_2}{q(u)}\mathrm{d}u)\chi_{[a,\eta]}(z) + (\int_z^t \frac{1}{q(u)}\mathrm{d}u)\chi_{[a,t]}(z) + \frac{\alpha_1\alpha_3(b-\xi)}{M}\int_a^\eta(\int_z^s \frac{1}{q(u)}\mathrm{d}u)\chi_{[a,s]}(z)\mathrm{d}s - \frac{b_2(a_1+a_2)}{M}\int_z^b \frac{1}{q(u)}\mathrm{d}u.$$

Then the solutions $(u(.), v(.))$ in Lemma 1 may be put as

$$u(t) = \int_a^b \mathcal{S}_1(t,\tau)f(\tau)\mathrm{d}\tau + \int_a^b \mathcal{S}_2(t,\tau)g(\tau)\mathrm{d}\tau, \quad t \in I$$
$$v(t) = \int_a^b \mathcal{S}_3(t,\tau)f(\tau)\mathrm{d}\tau + \int_a^b \mathcal{S}_4(t,\tau)g(\tau)\mathrm{d}\tau, \quad t \in I.$$

Moreover, if we define $C_i := \max_{t\in I}|c_i(t)|$, $i = 1,2,3,4$, $M_1 := \max_{t\in I}\frac{1}{p(t)}$, $M_2 := \max_{t\in I}\frac{1}{q(t)}$, for any $t, z \in I$ we have the following estimates

$$|\mathcal{S}_1(t,z)| \le \frac{a_4 C_1}{p(b)} + M_1 C_2\alpha_4(b-\xi) + M_1(b-a) + \frac{a_2(b_1+b_2)}{|M|}M_1(b-a) + \frac{\alpha_1\alpha_3(\eta-a)}{|M|}[\frac{(b-a)^2}{2} - \frac{(\xi-a)^2}{2}] =: s_1,$$

$$|\mathcal{S}_2(t,z)| \le \frac{b_4 C_2}{q(b)} + M_2 C_1\alpha_2(\eta-a) + \frac{\alpha_1(b_1+b_2)M_2}{|M|}\frac{(\eta-a)^2}{2} + \frac{\alpha_1 b_2 M_2(\eta-a)(b-\xi)}{|M|} =: s_2,$$

$$|\mathcal{S}_3(t,z)| \le \frac{a_4 C_3}{p(b)} + M_1 C_4\alpha_4(b-\xi) + \frac{\alpha_3 a_2 M_1(b-a)(b-\xi)}{|M|} + \frac{\alpha_3(a_1+a_2)}{|M|}M_1[\frac{(b-a)^2}{2} - \frac{(\xi-a)^2}{2}] =: s_3,$$

$$|\mathcal{S}_4(t,z)| \le \frac{b_4 C_4}{q(b)} + M_2 C_3\alpha_2(\eta-a) + M_2(b-a) + \frac{\alpha_1\alpha_3(b-\xi)M_2}{|M|}\frac{(\eta-a)^2}{2} + \frac{b_2(a_1+a_2)M_2}{|M|} =: s_4.$$

Finally, in the proof of our main result we need the following classical selection result for set-valued maps (e.g., [1]).

Lemma 2.4. *Let Z be a separable Banach space, B its closed unit ball, $A : I \to \mathcal{P}(Z)$ is a set-valued map whose values are nonempty closed and $b : I \to Z, c : I \to \mathbf{R}_+$ are two measurable functions. If*

$$A(t) \cap (b(t) + c(t)B) \neq \emptyset \quad a.e.\ (I),$$

then the set-valued map $t \to A(t) \cap (b(t)+c(t)B)$ admits a measurable selection.

3. THE MAIN RESULT

We are working under the following assumptions.

Hypothesis. i) $F : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $G : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ have nonempty closed values and the set-valued maps $F(., u, v)$, $G(., u, v)$ are measurable for any $u, v \in \mathbf{R}$.

ii) There exist $L_1(.), L_2(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t, ., .)$ is $L_1(t)$-Lipschitz and $G(t, ., .)$ is $L_2(t)$-Lipschitz; i.e.,

$$d_H(F(t, u_1, v_1), F(t, u_2, v_2)) \leq L_1(t)(|u_1 - u_2| + |v_1 - v_2|) \quad \forall\, u_1, u_2, v_1, v_2 \in \mathbf{R},$$

$$d_H(G(t, u_1, v_1), G(t, u_2, v_2)) \leq L_2(t)(|u_1 - u_2| + |v_1 - v_2|) \quad \forall\, u_1, u_2, v_1, v_2 \in \mathbf{R}.$$

We use the notation $L(t) = s_1L_1(t) + s_2L_2(t) + s_3L_1(t) + s_4L_2(t)$, $t \in I$.

Theorem 3.1. *Assume that $M, K \neq 0$, Hypothesis is satisfied and $|L(.)|_1 < 1$. For the mappings $(x(.), y(.)) \in C(I, \mathbf{R})^2$ there exist $r_1(.), r_2(.) \in L^1(I, \mathbf{R})$ with $d((p(t)x'(t))', F(t, x(t), y(t))) \leq r_1(t)$ a.e. $t \in I$, $d((q(t)y'(t))', G(t, x(t), y(t))) \leq r_2(t)$ a.e. $t \in I$, $a_1x(a) + a_2x(b) = \alpha_1 \int_a^\eta y(s)\mathrm{d}s$, $a_3x'(a) + a_4y'(b) = \alpha_2 \int_a^\eta y'(s)\mathrm{d}s$, $b_1y(a) + b_2y(b) = \alpha_3 \int_\xi^b x(s)\mathrm{d}s$, $b_3x'(a) + b_4x'(b) = \alpha_4 \int_\xi^b y'(s)\mathrm{d}s$.*

Then there exists $(u(.), v(.)) \in C(I, \mathbf{R})^2$ a solution of problem (1)–(2) satisfying for all $t \in I$

$$(5) \qquad |u(t) - x(t)| + |v(t) - y(t)| \leq \frac{(s_1 + s_3)|r_1(.)|_1 + (s_2 + s_4)|r_2(.)|_1}{1 - |L(.)|_1}.$$

Proof. By the assumption of theorem we have

$$F(t, x(t), y(t)) \cap \{(p(t)x'(t))' + r_1(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. } (I),$$
$$G(t, x(t), y(t)) \cap \{(q(t)y'(t))' + r_2(t)[-1, 1]\} \neq \emptyset \quad \text{a.e. } (I).$$

From Lemma 2 there exist measurable selections $f_1(t) \in F(t, x(t), y(t))$, $g_1(t) \in G(t, x(t), y(t))$ a.e. (I) such that

$$|f_1(t) - (p(t)x'(t))'| \leq r_1(t), \quad |g_1(t) - (q(t)y'(t))'| \leq r_2(t) \quad \text{a.e. } (I).$$

Define

$$u_1(t) = \int_a^b \mathcal{S}_1(t, \tau)f_1(\tau)\mathrm{d}\tau + \int_a^b \mathcal{S}_2(t, \tau)g_1(\tau)\mathrm{d}\tau, \quad t \in I$$
$$v_1(t) = \int_a^b \mathcal{S}_3(t, \tau)f_1(\tau)\mathrm{d}\tau + \int_a^b \mathcal{S}_4(t, \tau)g_1(\tau)\mathrm{d}\tau, \quad t \in I.$$

We have the estimates

$$\begin{array}{ll}|u_1(t)-x(t)|\leq s_1|r_1(.)|_1+s_2|r_2(.)|_1 & \forall\, t\in I,\\ |v_1(t)-y(t)|\leq s_3|r_1(.)|_1+s_4|r_2(.)|_1 & \forall\, t\in I,\end{array}$$

and so,

$$|u_1(t)-x(t)|+|v_1(t)-y(t)|\leq (s_1+s_3)|r_1(.)|_1+(s_2+s_4)|r_2(.)|_1=:s.$$

Next we construct the sequences $u_n(.), v_n(.)\in C(I,\mathbf{R})$ and $f_n(.)$, $g_n(.)\in L^1(I,\mathbf{R})$, $n\geq 1$ with the following properties

$$\begin{array}{ll}u_n(t)=\int_a^b \mathcal{S}_1(t,\tau)f_n(\tau)\mathrm{d}\tau+\int_a^b \mathcal{S}_2(t,\tau)g_n(\tau)\mathrm{d}\tau, & t\in I\\ v_n(t)=\int_a^b \mathcal{S}_3(t,\tau)f_n(\tau)\mathrm{d}\tau+\int_a^b \mathcal{S}_4(t,\tau)g_n(\tau)\mathrm{d}\tau, & t\in I.\end{array}\tag{6}$$

$$f_n(t)\in F(t,u_{n-1}(t),v_{n-1}(t)),\quad g_n(t)\in G(t,u_{n-1}(t),v_{n-1}(t))\quad \text{a.e.}\,(I),\tag{7}$$

$$\begin{array}{l}|f_{n+1}(t)-f_n(t)|\leq L_1(t)(|u_n(t)-u_{n-1}(t)|+|v_n(t)-v_{n-1}(t)|)\text{a.e. }(I),\\ |g_{n+1}(t)-g_n(t)|\leq L_2(t)(|u_n(t)-u_{n-1}(t)|+|v_n(t)-v_{n-1}(t)|)\text{a.e. }(I).\end{array}\tag{8}$$

In what follows we prove that from (6)–(8) it follows

$$|u_{n+1}(t)-u_n(t)|+|v_{n+1}(t)-v_n(t)|\leq s(|L(.)|_1)^n\quad \text{a.e. }(I)\quad \forall n\in\mathbf{N}.\tag{9}$$

The situation when $n=0$ is already shown. Now, we assume that (9) is true for $n-1$. For almost all $t\in I$,

$$|u_{n+1}(t)-u_n(t)|\leq \int_a^b|\mathcal{S}_1(t,\tau)|.|f_{n+1}(\tau)-f_n(\tau)|\mathrm{d}\tau+\int_a^b|\mathcal{S}_2(t,\tau)|.|g_{n+1}(\tau)-g_n(\tau)|\mathrm{d}\tau\leq s_1\int_a^b|f_{n+1}(\tau)-f_n(\tau)|\mathrm{d}\tau+s_2\int_a^b|g_{n+1}(\tau)-g_n(\tau)|\mathrm{d}\tau\leq s_1\int_a^b L_1(\tau)(|u_n(\tau)-u_{n-1}(\tau)|+|v_n(\tau)-v_{n-1}(\tau)|)\mathrm{d}\tau+s_2\int_a^b L_2(\tau)(|u_n(\tau)-u_{n-1}(\tau)|+|v_n(\tau)-v_{n-1}(\tau)|)\mathrm{d}\tau\leq s(|L(.)|_1)^{n-1}(s_1\int_a^b L_1(\tau)\mathrm{d}\tau+s_2\int_a^b L_2(\tau)\mathrm{d}\tau).$$

Similarly, we obtain for almost all $t\in I$,

$$|v_{n+1}(t)-v_n(t)|\leq s(|L(.)|_1)^{n-1}(s_3\int_a^b L_1(\tau)\mathrm{d}\tau+s_4\int_a^b L_2(\tau)\mathrm{d}\tau).$$

Thus, (9) is true for n.

From (9) the sequences $\{u_n(.)\},\{v_n(.)\}$ are Cauchy in the space $C(I,\mathbf{R})$. Let $u(.)\in C(I,\mathbf{R})$ and $v(.)\in C(I,\mathbf{R})$ be their limits in $C(I,\mathbf{R})$. Also, from (8) we deduce that, for almost all $t\in I$, the sequences $\{f_n(t)\}$, $\{g_n(t)\}$ are Cauchy in $\mathbf{R}$. We consider $f(.)$, $g(.)$ their pointwise limit.

Also, from (9) and Hypothesis we deduce

$$\begin{array}{l}|u_n(t)-x(t)|+|v_n(t)-y(t)|\leq |u_1(t)-x(t)|+|v_1(t)-y(t)|\\ +\sum_{i=1}^{n-1}(|u_{i+1}(t)-u_i(t)|+|v_{i+1}(t)-v_i(t)|)\\ \leq s+\sum_{i=1}^{n}s(|L(.)|_1)^i\leq \frac{s}{1-|L(.)|_1}.\end{array}\tag{10}$$

and for almost all $t \in I$

$$|f_n(t) - (p(t)x'(t))'| + |g_n(t) - (q(t)y'(t))'| \leq |f_1(t) - (p(t)x'(t))'| + |g_1(t) - (q(t)y'(t))'| + \sum_{i=1}^{n-1}(|f_{i+1}(t) - f_i(t)| + |g_{i+1}(t) - g_i(t)|) \leq |f_1(t) - (p(t)x'(t))'| + |g_1(t) - (q(t)y'(t))'| + \sum_{i=1}^{n-1}(L_1(t) + L_2(t))(|u_i(t) - u_{i-1}(t)| + |v_i(t) - v_{i-1}(t)|) \leq r_1(t) + r_2(t) + (L_1(t) + L_2(t))\frac{s}{1-|L(.)|_1}.$$

In particular, from the last inequality it follows that the sequences $f_n(.)$, $g_n(.)$ are integrably bounded and therefore, their limits $f(.), g(.)$ belong to $L^1(I, \mathbf{R})$.

Finally, we realize the construction in (6)–(8). By induction, we suppose that for $L \geq 1$, $u_l(.), v_l(.) \in C(I, \mathbf{R})$ and $f_l(.), g_l(.) \in L^1(I, \mathbf{R})$, $l = 1, 2, ...L$ with (6) and (8) for $l = 1, 2, ...L$ and (7) for $l = 1, 2, ...L - 1$ are constructed.

Again with Hypothesis 3.1

$$F(t, u_L(t), v_L(t)) \cap \{f_L(t) + (L_1(t)|u_L(t) - u_{L-1}(t)| + L_1(t)|v_L(t) - v_{L-1}(t)|)[-1, 1]\} \neq \emptyset,$$
$$G(t, u_L(t), v_L(t)) \cap \{g_L(t) + (L_2(t)|u_L(t) - u_{L-1}(t)| + L_2(t)|v_L(t) - v_{L-1}(t)|)[-1, 1]\} \neq \emptyset$$

for almost all $t \in I$.

Lemma 2 gives the existence of measurable selections $f_{L+1}(.)$ of $F(., u_L(.), v_L(.))$ and $g_{L+1}(.)$ of $G(., u_L(.), v_L(.))$ such that

$$|f_{L+1}(t) - f_L(t)| \leq L_1(t)(|u_L(t) - u_{L-1}(t)| + |v_L(t) - v_{M-1}(t)|) \quad \text{a.e. } (I),$$
$$|g_{L+1}(t) - g_L(t)| \leq L_2(t)(|u_L(t) - u_{L-1}(t)| + |v_L(t) - v_{L-1}(t)|) \quad \text{a.e. } (I).$$

We define $u_{L+1}(.), v_{L+1}(.)$ as in (6) with $n = L + 1$.

We take $n \to \infty$ in (6) and (10) and the proof is complete. □

Corollary 3.2. *Assume that $M, K \neq 0$, Hypothesis is satisfied, $|L(.)|_1 < 1$, $d(0, F(t, 0, 0)) \leq L_1(t)$ a.e. $t \in I$ and $d(0, G(t, 0, 0)) \leq L_2(t)$ a.e. $t \in I$.*

Then there exists $(u(.), v(.)) \in C(I, \mathbf{R})^2$ a solution of problem (1)–(2) satisfying for all $t \in I$

$$|u(t)| + |v(t)| \leq \frac{(s_1 + s_3)|L_1(.)|_1 + (s_2 + s_4)|L_2(.)|_1}{1 - |L(.)|_1}. \tag{11}$$

Proof. We apply Theorem 1 with $x(.) = y(.) = 0$, $r_1(.) = L_1(.)$ and $r_2(.) = L_2(.)$. □

Remark 3.3. A similar result to the one in Corollary 1 may be found in [1]; namely, Theorem 4.3. The proof of Theorem 4.3 in [1] is done by using the set-valued contraction principle. Our approach improves the hypothesis concerning the set-valued map in [1]. More exactly, we do not require for the values of F and G to be compact as in [1] and we do not require that the Lipschitz constant of F and G to be mappings from $C(I, \mathbf{R})$ as in [1]. Moreover, Theorem 4.3 in [1] does not contains a priori bounds for solutions as in (11).

EXAMPLE 3.4. Let us consider the problem

$$(12)\qquad \begin{array}{l} (\frac{1}{t^2+2}u'(t))' \in [-\frac{1}{160}\cdot\frac{|u(t)|}{1+|u(t)|},0]\cup[0,\frac{1}{160}\cdot\frac{|v(t)|}{1+|v(t)|}], \quad \text{a.e. } ([0,2]),\\ (\frac{2}{t+6}v'(t))' \in [-\frac{1}{188}\cdot\frac{|\cos u(t)|}{1+|\cos u(t)|},0]\cup[0,\frac{1}{188}\cdot\frac{|\sin v(t)|}{1+|\sin v(t)|}], \quad \text{a.e. } ([0,2]),\end{array}$$

with boundary conditions as in [1]; namely,

$$(13)\qquad \begin{cases} \frac{1}{2}u(0)+u(2)=\frac{2}{3}\int_0^{\frac{1}{4}}v(s)\mathrm{d}s, \quad \frac{5}{8}u'(0)+\frac{4}{7}u'(2)=\int_0^{\frac{1}{4}}v'(s)\mathrm{d}s,\\ 2v(0)+\frac{1}{6}v(2)=\frac{3}{4}\int_1^2 u(s)\mathrm{d}s, \quad \frac{1}{5}v'(0)+\frac{3}{5}v'(2)=\frac{5}{3}\int_1^2 u'(s)\mathrm{d}s,\end{cases}$$

In this case, $p(t)=\frac{1}{t^2+2}$, $q(t)=\frac{2}{t+6}$, $a=0$, $b=2$, $\eta=\frac{1}{4}$, $\xi=1$, $\alpha_1=\frac{2}{3}$, $\alpha_2=1$, $\alpha_3=\frac{4}{3}$, $\alpha_4=\frac{5}{3}$, $a_1=\frac{1}{2}$, $a_2=1$, $a_3=\frac{5}{8}$, $a_4=\frac{4}{7}$, $b_1=2$, $b_2=\frac{1}{6}$, $b_3=\frac{1}{5}$, $b_4=\frac{3}{5}$.

Define $F(.,.),G(.,.):I\times\mathbf{R}\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ by

$$(14)\qquad \begin{array}{l} F(t,u,v)=[-\frac{1}{160}\cdot\frac{|u|}{1+|u|},0]\cup[0,\frac{1}{160}\cdot\frac{|v|}{1+|v|}], \quad \text{a.e. } ([0,2]),\\ G(t,u,v)=[-\frac{1}{188}\cdot\frac{|\cos u|}{1+|\cos u|},0]\cup[0,\frac{1}{188}\cdot\frac{|\sin v|}{1+|\sin v|}], \quad \text{a.e. } ([0,2]),\end{array}$$

Since

$$\begin{array}{l} \sup\{|z|; \quad z\in F(t,u,v)\}\le\frac{1}{160} \quad \forall\, t\in[0,2],\ u,v\in\mathbf{R},\\ \sup\{|z|; \quad z\in G(t,u,v)\}\le\frac{1}{188} \quad \forall\, t\in[0,2],\ u,v\in\mathbf{R}\end{array}$$

and

$$\begin{array}{l} d_H(F(t,u_1,v_1),F(t,u_2,v_2))\le\frac{1}{160}(|u_1-u_2|+|v_1-v_2|)\ \forall\, u_1,u_2,v_1,v_2\in\mathbf{R},\\ d_H(G(t,u_1,v_1),G(t,u_2,v_2))\le\frac{1}{188}(|u_1-u_2|+|v_1-v_2|)\ \forall\, u_1,u_2,v_1,v_2\in\mathbf{R}.\end{array}$$

in this situation $L_1(t)\equiv\frac{1}{160}$ and $L_2(t)\equiv\frac{1}{188}$. By standard computations (e.g., [1]) $s_1+s_3\approx 19,27$, $s_2+s_4\approx 3,97$; therefore, $(s_1+s_3)|L_1(.)|_1+(s_2+s_4)|L_2(.)|_1\approx 0,35<1$. So, we may apply Corollary 3.3 in order to obtain the existence of a solution for problem (12)–(13).

REFERENCES

[1] B. Ahmad, A. Almalki, S. K. Ntouyas and A. Alsaedi, *Existence results for a self-adjoint system of nonlinear second-order ordinary differential inclusions with nonlocal integral boundary conditions*, J. Inequal. Appl., **2022** (2022), 1–41.

[2] J. P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhäuser, Basel, 1990.

[3] A. Cernea, *A Filippov type existence theorem for a class of second-order differential inclusions*, JIPAM. J. Inequal. Pure Appl. Math., **9** (2008), 1–6.

[4] A. Cernea, *Continuous version of Filippov's theorem for a Sturm-Liouville type differential inclusion*, Electron. J. Differential Equations, **2008** (2008), 1–7.

[5] A. Cernea, *On a boundary value problem for a Sturm-Liouville differential inclusion*, J. Syst. Sci. Complex., **23** (2010), 390–394.

[6] A. Cernea, *On controllability for Sturm-Liouville type differential inclusions*, Filomat, **27** (2013), 1321–1327.

[7] A. Cernea, *On a Sturm-Liouville type functional differential inclusion with "maxima"*, Adv. Dyn. Syst. Appl., **13** (2018), 101–112.

[8] A. Cernea, *Existence of solutions for some coupled systems of fractional differential inclusions*, Mathematics, **8** (2020), 1–10.

[9] A. Cernea, *On some coupled systems of fractional differential inclusions*, Fract. Differ. Calc., **11** (2021), 133–145.

[10] A. Cernea, *A note on a coupled system of Caputo-Fabrizio fractional differential inclusions*, Annals of Communications in Mathematics , **4** (2021), 190–195.

[11] A. Cernea, *On a self-adjoint coupled system of second-order differential inclusions*, J. Comput. Math., **6** (2022), 124–133.

[12] A. F. Filippov, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control, **5** (1967), 609–621.

[13] H. M. Srivastava, S. K. Ntouyas, M. Alsulami, A. Alsaedi and B. Ahmad, *A self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal multi-point boundary conditions on arbitrary domain*, Appl. Sci., **11** (2021), 1–14.

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