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COMPATIBLE STRUCTURE IN IDEAL ČECH CLOSURE SPACES

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Abstract. In Al-Omari et al., *Touch points in ideal Čech closure spaces*, Mathematica, **64** (2022), $(\mathcal{C}, f, \mathcal{I})$ is a Čech closure space with an ideal $\mathcal{I}$. For $H \subseteq \mathcal{C}$, the set $\widetilde{f}(H)$, called Čech touch points, is defined by $\widetilde{f}(H) = \{x \in \mathcal{C} : H \cap N \notin \mathcal{I}$ for every $N \in \mathcal{N}(x)\}$. Several characterizations of these sets will also be discussed through this paper. Moreover, we obtain characterizations of $\widetilde{f}$-operator in an ideal Čech closure space $(\mathcal{C}, f, \mathcal{I})$, we investigate the notion of f-compatibility with an ideal $\mathcal{I}$ and obtain several characterizations of the compatibility.
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1. INTRODUCTION AND PRELIMINARIES

A non-empty collection of subsets of $\mathcal{C}$ is called an ideal $\mathcal{I}$ on a space $\mathcal{C}$ if the following properties are satisfies:

(1) If $H \in \mathcal{I}$ and $K \subseteq H$ then $K \in \mathcal{I}$.
(2) If $H \in \mathcal{I}$ and $K \in \mathcal{I}$ then $H \cup K \in \mathcal{I}$.

An ideal topological space is topological space $(\mathcal{C}, \tau)$ with an ideal $\mathcal{I}$ on $\mathcal{C}$ and is denoted by $(\mathcal{C}, \tau, \mathcal{I})$ (see [9, 10]). First we recall several definitions.

An operator $f : \mathcal{P}(\mathcal{C}) \longrightarrow \mathcal{P}(\mathcal{C})$ defined on the power set $\mathcal{P}(\mathcal{C})$ of a set $\mathcal{C}$ such that the following holds:

(1) $f(\emptyset) = \emptyset$;
(2) $H \subseteq f(H)$ for all $H \subseteq \mathcal{C}$;
(3) $f(H \cup K) = f(H) \cup f(K)$ for every A, $B \in \mathcal{P}(\mathcal{C})$.

is called a Čech closure operator (see [7,8]) and the pair $(\mathcal{C}, f)$ is a Čech closure space. A subset H of $\mathcal{C}$ is said to be f-closed in $(\mathcal{C}, f)$ if $f(H) = H$ holds. And H is f-open if $\mathcal{C} - H$ is f-closed.

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By the closure operator we defined the interior operator $f^* : \mathcal{P}(\mathcal{C}) \longrightarrow \mathcal{P}(\mathcal{C})$ in the usual way: $f^*(H) = \mathcal{C} - f(\mathcal{C} - H)$.

Let $(\mathcal{C}, f, \mathcal{I})$ be a Čech closure space with an ideal $\mathcal{I}$. For a subset H of $\mathcal{C}$, the set $\widetilde{f}(H)$ called touch points is defined as: $\widetilde{f}(H) = \{x \in \mathcal{C} : H \cap N \notin \mathcal{I}$ for every $N \in \mathcal{N}(x)\}$. We investigate the properties of touch points and construct a topology on X from the touch points. Moreover, in an ideal Čech closure space $(\mathcal{C}, f, \mathcal{I})$, we define f-compatibility with the ideal $\mathcal{I}$ and obtain several characterizations of the compatibility. Also the papers [2–6] have introduced some property related to compatible structure in ideal Čech closure spaces.

Remark 1.1. Let $(\mathcal{C}, f)$ be a Čech closure space.
(1) $f^*(\emptyset) = \emptyset$.
(2) $f^*(\mathcal{C}) = \mathcal{C}$.
(3) $f^*(H) \subseteq H$ for every $H \subseteq \mathcal{C}$.
(4) $f^*(H \cap K) = f^*(H) \cap f^*(K)$ for all H, $K \in \mathcal{P}(\mathcal{C})$.

A subset N is a neighborhood of a point x (respectively, subset H) in $\mathcal{C}$ if $x \in f^*(N)$ (respectively, $H \subseteq f^*(N)$) holds. The collection of all neighborhoods of x will be denoted by $\mathcal{N}_x$ or $\mathcal{N}(x)$.
In $(\mathcal{C}, f)$, a point $x \in f(H)$ if and only if for each neighborhood N of x, $N \cap A \neq \emptyset$ holds.

We set $f(H) = \cap\{N : A \subseteq N, \mathcal{C} - N \in \mathcal{N}(x)\}$ and $f^*(H) = \cup\{U : U \subseteq H, U \in \mathcal{N}(x)\}$.

Definition 1.2 ([16]). Given f and f^* be the be closure map and its dual map on $\mathcal{C}$. Then the neighborhood map $\mathcal{N} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ and the convergent map $\mathcal{N}^* : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ assign to each $x \in \mathcal{C}$ the collections

$$\mathcal{N}(x) = \{N \in \mathcal{P}(\mathcal{C}) : x \in f^*(N)\}$$

$$\mathcal{N}^*(x) = \{Q \in \mathcal{P}(\mathcal{C}) : x \in f(Q)\}$$

of its neighborhoods and convergents, respectively.

Lemma 1.3. *Given $(\mathcal{C}, f)$ be a Čech closure space. Then the properties holds*
(1) $Q \in \mathcal{N}^*(x)$ *if and only if* $\mathcal{C} - Q \notin \mathcal{N}(x)$.
(2) $x \in f(A)$ *if and only if* $\mathcal{C} - A \notin \mathcal{N}(x)$.
(3) $x \in f^*(A)$ *if and only if* $\mathcal{C} - A \notin \mathcal{N}^*(x)$.

Lemma 1.4 ([1]). *Given $(\mathcal{C}, f)$ be a Čech closure space, then*
(1) $\mathcal{C} \in \mathcal{N}(x)$ *for every* $x \in \mathcal{C}$.

(2) $\emptyset \notin \mathcal{N}(x)$ *for every* $x \in \mathcal{C}$.
(3) *If* $N \in \mathcal{N}(x)$, *then* $x \in f^*(N) \subseteq N$.
(4) *If* $N, M \in \mathcal{N}(x)$, *then we have* $N \cap M \in \mathcal{N}(x)$.
(5) *If* $N \cup M \in \mathcal{N}^*(x)$, *then* $N \in \mathcal{N}^*(x)$ *or* $M \in \mathcal{N}^*(x)$.

Definition 1.5 ([1]). Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. For $H \subseteq \mathcal{C}$, we define the set: $\widetilde{f}(H) = \{x \in \mathcal{C} : H \cap N \notin \mathcal{I}$ for all $N \in \mathcal{N}(x)\}$. And $\widetilde{f}(H)$ it is called touch points of H with respect to f and $\mathcal{I}$.

Lemma 1.6 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *and* $(\mathcal{C}, g, \mathcal{J})$ *be ideal Čech-spaces,* $\mathcal{I}$ *and* $\mathcal{J}$ *be ideals on* $\mathcal{C}$, *and let* H *and* K *be subsets of* $\mathcal{C}$. *Then we have:*

(1) *For* $H \subseteq K$, *we have* $\widetilde{f}(H) \subseteq \widetilde{f}(K)$.
(2) *For* $\mathcal{I} \subseteq \mathcal{J}$, *we have* $\widetilde{f}(H) \supseteq \widetilde{g}(H)$.
(3) $\widetilde{f}(H) = f(\widetilde{f}(H)) \subseteq f(H)$ *and* $\widetilde{f}(H)$ *is* f*-closed.*
(4) *For* $H \subseteq \widetilde{f}(H)$, *we have* $\widetilde{f}(H) = f(\widetilde{f}(H)) = f(H)$.
(5) *For* $H \in \mathcal{I}$, *we have* $\widetilde{f}(H) = \emptyset$.

Lemma 1.7 ([1]). *Given* $(C, f, \mathcal{I})$ *be an ideal Čech-space and* $x \in \mathcal{C}$. *If* $N \in \mathcal{N}(x)$, *then* $N \cap \widetilde{f}(H) = N \cap \widetilde{f}(N \cap A) \subseteq \widetilde{f}(N \cap H)$ *for any subset* H *of* $\mathcal{C}$.

Theorem 1.8 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space and* H, $K \subseteq \mathcal{C}$. *Then the following hold:*

(1) $\widetilde{f}(\emptyset) = \emptyset$.
(2) $\widetilde{f}(\widetilde{f}(H)) \subseteq \widetilde{f}(H)$.
(3) $\widetilde{f}(H) \cup \widetilde{f}(K) = \widetilde{f}(H \cup K)$.

Theorem 1.9 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space,* $\overline{A} = \widetilde{f}(H) \cup H$ *and* H, K *be subsets of* $\mathcal{C}$. *Then*

(1) $\overline{\emptyset} = \emptyset$.
(2) $H \subseteq \overline{H}$.
(3) $\overline{H \cup K} = \overline{H} \cup \overline{K}$.
(4) $\overline{H} = \overline{\overline{H}}$.

By Theorem 1.9, we obtain that $\overline{H} = H \cup \widetilde{f}(H)$ is a Kuratowski closure operator. We will denote by $\tau_{cl}(x)$ the topology generated by $\overline{H}$, that is, $\tau_{cl}(x) = \{U \subseteq X : \overline{\mathcal{C} - U} = \mathcal{C} - U\}$. A subset H of $\mathcal{C}$ is said to be $\tau_{cl}(x)$-closed if and only if $\widetilde{f}(H) \subseteq H$. It is said to be $\tau_{cl}(x)$-open if the complement is $\tau_{cl}(x)$-closed.

THEOREM 1.10 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space. Then* $\beta(f, \mathcal{I}) = \{V - I : V \in \mathcal{N}(x), I \in \mathcal{I}, x \in \mathcal{C}\}$ *is a basis for* $\tau_{cl}(x)$ *and* $\mathcal{N}(x) \subseteq \tau_{cl}(x)$.

THEOREM 1.11 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space and* $x \in \mathcal{C}$, *then the following are equivalent:*

(1) $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$;
(2) $\mathcal{C} = \widetilde{f}(\mathcal{C})$;
(3) *For all* $N \in \mathcal{N}(x)$, $N \subseteq \widetilde{f}(N)$;
(4) *If* $I \in \mathcal{I}$, *then* $f^*(I) = \emptyset$.

LEMMA 1.12 ([1]). *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space and* H, K *be subsets of* $\mathcal{C}$. *Then*

(1) $\widetilde{f}(H) - \widetilde{f}(K) = \widetilde{f}(H - K) - \widetilde{f}(K)$.
(2) $\widetilde{f}(H \cup K) = \widetilde{f}(H) = \widetilde{f}(H - K)$ *if* $K \in \mathcal{I}$.

2. f_ψ-OPERATOR IN IDEAL ČECH-SPACE

DEFINITION 2.1. Let $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. An operator $f_\psi : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ is defined as follows for every $A \in X$, $f_\psi(H) = \{x \in \mathcal{C}$: there exists $U \in \mathcal{N}(x)$ such that $U - H \in \mathcal{I}\}$ and observes that $f_\psi(H) = \mathcal{C} - \widetilde{f}(\mathcal{C} - H)$.

Several basic behavior of the operator f_ψ are included in the below theorem.

THEOREM 2.2. *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space. Then the following hold:*

(1) *For* $H \subseteq \mathcal{C}$, *we have* $f_\psi(H)$ *is* f*-open*
(2) *For* $H \subseteq K$, *we have* $f_\psi(H) \subseteq f_\psi(K)$.
(3) *For* $H, K \in \mathcal{P}(\mathcal{C})$, *we have* $f_\psi(H \cap K) = f_\psi(H) \cap f_\psi(K)$.
(4) *For* $U \in \tau_{cl}(x)$, *we have* $U \subseteq f_\psi(U)$.
(5) *For* $H \subseteq \mathcal{C}$, *we have* $f_\psi(H) \subseteq f_\psi(f_\psi(H))$.
(6) *For* $H \subseteq \mathcal{C}$, *we have* $f_\psi(H) = f_\psi(f_\psi(H))$ *if and only if* $\widetilde{f}(X - H) = \widetilde{f}(\widetilde{f}(\mathcal{C} - H))$.
(7) *For* $H \in \mathcal{I}$, *we have* $f_\psi(H) = \mathcal{C} - \widetilde{f}(\mathcal{C})$.
(8) *For* $H \subseteq \mathcal{C}$, *we have* $H \cap f_\psi(H) = int(H)$.
(9) *For* $H \subseteq \mathcal{C}$, $I \in \mathcal{I}$, *we have* $f_\psi(H - I) = f_\psi(H)$.
(10) *For* $H \subseteq \mathcal{C}$, $I \in \mathcal{I}$, *we have* $f_\psi(H \cup I) = f_\psi(H)$.
(11) *For* $(H - K) \cup (K - H) \in \mathcal{I}$, *we have* $f_\psi(H) = f_\psi(K)$.

Proof. (1) Follows by Lemma 1.6 (3).
(2) Follows by Lemma 1.6 (1).

(3) By (2) that $f_\psi(H \cap K) \subseteq f_\psi(H)$ and $f_\psi(H \cap K) \subseteq f_\psi(K)$. Hence $f_\psi(H \cap K) \subseteq f_\psi(H) \cap f_\psi(K)$. Now let $x \in f_\psi(H) \cap f_\psi(K)$. There exist $U, V \in \mathcal{N}(x)$ such that $U - H \in \mathcal{I}$ and $V - K \in \mathcal{I}$. Let $G = U \cap V \in \mathcal{N}(x)$ and we have $G - H \in \mathcal{I}$ and $G - K \in \mathcal{I}$ by heredity. Thus $G - (H \cap K) = (G - H) \cup (G - K) \in \mathcal{I}$ by additivity, and hence $x \in f_\psi(H \cap K)$. We have $f_\psi(H) \cap f_\psi(K) \subseteq f_\psi(H \cap K)$ and the proof is complete.

(4) If $U \in \tau_{cl}(x)$, then $X - U$ is $\tau_{cl}(x)$-closed then we have $\widetilde{f}(\mathcal{C} - U) \subseteq \mathcal{C} - U$ and hence $U \subseteq \mathcal{C} - \widetilde{f}(\mathcal{C} - U) = f_\psi(U)$.

(5) Follows by (4).

(6) Follows by the facts:

(i) $f_\psi(H) = \mathcal{C} - \widetilde{f}(\mathcal{C} - H)$.
(ii) $f_\psi(f_\psi(H)) = \mathcal{C} - \widetilde{f}[\mathcal{C} - (\mathcal{C} - \widetilde{f}(\mathcal{C} - H))] = \mathcal{C} - \widetilde{f}(\widetilde{f}(\mathcal{C} - H))$.

(7) By Lemma 1.12 we obtain that $\widetilde{f}(\mathcal{C} - H) = \widetilde{f}(X)$ if $H \in \mathcal{I}$.

(8) If $x \in H \cap f_\psi(H)$, then $x \in H$ and there exists a $U_x \in \mathcal{N}(x)$ such that $U_x - H \in \mathcal{I}$. Then by Theorem 1.10, $U_x - (U_x - H)$ is an $\tau_{cl}(x)$-open of x and $x \in int(H)$. On the other hand, if $x \in int(H)$, there exists a basic $\tau_{cl}(x)$-open $V_x - I$ of x, where $V_x \in \mathcal{N}(x)$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq H$ which implies $V_x - H \subseteq I$ and hence $V_x - H \in \mathcal{I}$. Hence $x \in H \cap f_\psi(\mathcal{C})$.

(9) By Lemma 1.12 and $f_\psi(H - I) = \mathcal{C} - \widetilde{f}[\mathcal{C} - (H - I)] = \mathcal{C} - \widetilde{f}[(\mathcal{C} - H) \cup I] = \mathcal{C} - \widetilde{f}(\mathcal{C} - H) = f_\psi(H)$.

(10) By Lemma 1.12 and $f_\psi(H \cup I) = X - \widetilde{f}[\mathcal{C} - (H \cup I)] = \mathcal{C} - \widetilde{f}[(\mathcal{C} - H) - I] = \mathcal{C} - \widetilde{f}(\mathcal{C} - H) = f_\psi(H)$.

(11) Let $(H - K) \cup (K - H) \in \mathcal{I}$. Let $H - K = I$ and $K - H = J$. Observe that $I, J \in \mathcal{I}$ by heredity. Also clear that $K = (H - I) \cup J$. Hence $f_\psi(H) = f_\psi(H - I) = f_\psi[(H - I) \cup J] = f_\psi(K)$ by (9) and (10). □

Corollary 2.3. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. We have $U \subseteq f_\psi(U)$ for all $U \in \mathcal{N}(x)$.*

Proof. Since $f_\psi(U) = \mathcal{C} - \widetilde{f}(\mathcal{C} - U)$ is true. Now $\widetilde{f}(\mathcal{C} - U) \subseteq \overline{\mathcal{C} - U} = \mathcal{C} - U$, since $U \in \mathcal{N}(x)$. Hence, $U = \mathcal{C} - (\mathcal{C} - U) \subseteq \mathcal{C} - \widetilde{f}(\mathcal{C} - U) = f_\psi(U)$. □

Theorem 2.4. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space and $H \subseteq \mathcal{C}$. Then the following hold:*

(1) $f_\psi(H) = \cup\{U \in \mathcal{N}(x) : U - H \in \mathcal{I}\}$.
(2) $f_\psi(H) \supseteq \cup\{U \in \mathcal{N}(x) : (U - H) \cup (H - U) \in \mathcal{I}\}$.

Proof. (1) This follows by the definition of f_ψ-operator.
(2) Since $\mathcal{I}$ is heredity, it is clear that $\cup\{U \in \mathcal{N}(x) : (U - H) \cup (A - U) \in \mathcal{I}\} \subseteq \cup\{U \in \mathcal{N}(x) : U - H \in \mathcal{I}\} = f_\psi(H)$ for every $H \subseteq \mathcal{C}$. □

THEOREM 2.5. *Assume that $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. If $\varrho = \{H \subseteq X : H \subseteq f_\psi(H)\}$. Then ϱ is a topology for $\mathcal{C}$ and $\varrho = \tau_{cl}(x)$.*

Proof. Assume that $\varrho = \{H \subseteq \mathcal{C} : H \subseteq f_\psi(H)\}$. First, we prove that ϱ is a topology. Clear that $\emptyset \subseteq f_\psi(\emptyset)$ and $\mathcal{C} \subseteq f_\psi(\mathcal{C}) = \mathcal{C}$, and thus $\emptyset$ and $\mathcal{C} \in \varrho$. Now if $H, K \in \varrho$, then $H \cap K \subseteq f_\psi(H) \cap f_\psi(K) = f_\psi(H \cap K)$ then $H \cap K \in \varrho$. If $\{H_\alpha : \alpha \in \Delta\} \subseteq \varrho$, then $H_\alpha \subseteq f_\psi(H_\alpha) \subseteq f_\psi(\cup H_\alpha)$ for all α and hence $\cup H_\alpha \subseteq f_\psi(\cup H_\alpha)$. This shows ϱ is a topology. Now if $U \in \tau_{cl}(x)$ and $x \in U$, then by Theorem 1.10 there exist $V \in \mathcal{N}(x)$ and $I \in \mathcal{I}$ such that $x \in V - I \subseteq U$. Clearly $V - U \subseteq I$ so that $V - U \in \mathcal{I}$ by heredity and then $x \in f_\psi(U)$. Thus $U \subseteq f_\psi(U)$ and we shown $\tau_{cl}(x) \subseteq \varrho$. Now let $H \in \varrho$, we have $H \subseteq f_\psi(H)$, that is, $H \subseteq \mathcal{C} - \widetilde{f}(\mathcal{C} - H)$ and $\widetilde{f}(\mathcal{C} - H) \subseteq \mathcal{C} - H$. This shows that $\mathcal{C} - H$ is $\tau_{cl}(x)$-closed and then $H \in \tau_{cl}(x)$. Thus $\varrho \subseteq \tau_{cl}(x)$ and hence $\varrho = \tau_{cl}(x)$. □

3. SOME PROPERTIES OF *f*-COMPATIBLE IN IDEAL ČECH-SPACES

DEFINITION 3.1 ([1]). Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. Then f is f-compatible with respect to ideal $\mathcal{I}$, denoted $f \cong \mathcal{I}$, if the following holds for all $H \subseteq \mathcal{C}$: For all $x \in H$ and $U \in \mathcal{N}(x)$, and if $U \cap H \in \mathcal{I}$, then $H \in \mathcal{I}$.

THEOREM 3.2. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space, f be f-compatible with respect to $\mathcal{I}$ such that $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Let G be a $\tau_{cl}(x)$-open set such that $G = U - H$, where $U \in \mathcal{N}(x)$ and $H \in \mathcal{I}$. Then $f(\widetilde{f}(G)) = f(G) = \widetilde{f}(G) = \widetilde{f}(U) = f(U) = f(\widetilde{f}(U))$.*

Proof. (1) Let $G = U - H$, where $U \in \mathcal{N}(x)$ and $H \in \mathcal{I}$. Since $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$, by Theorem 1.11 we have $U \subseteq \widetilde{f}(U)$. Hence by Lemma 1.6, $\widetilde{f}(U) = f(\widetilde{f}(U)) = f(U)$.
(2) Since G is $\tau_{cl}(x)$-open, $\mathcal{C} - G = \overline{\mathcal{C} - G}$ and hence $\widetilde{f}(\mathcal{C} - G) \subseteq \mathcal{C} - G$. By Lemma 1.12, $\widetilde{f}(\mathcal{C}) - \widetilde{f}(G) \subseteq \widetilde{f}(\mathcal{C} - G)$. But $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$ and by Theorem 1.11, $\widetilde{f}(\mathcal{C}) = \mathcal{C}$ and hence $\mathcal{C} - \widetilde{f}(G) \subseteq \widetilde{f}(\mathcal{C} - G) \subseteq \mathcal{C} - G$. Therefore, $G \subseteq \widetilde{f}(G)$. Hence, $f(G) \subseteq f(\widetilde{f}(G))$. Hence by Lemma 1.6, $\widetilde{f}(G) = f(G) = f(\widetilde{f}(G))$.
(3) Again, $G \subseteq U$ implies that $\widetilde{f}(G) \subseteq \widetilde{f}(U)$. By Lemma 1.12, $\widetilde{f}(G) = \widetilde{f}(U - H) \supseteq \widetilde{f}(U) - \widetilde{f}(H) = \widetilde{f}(U)$ since $H \in \mathcal{I}$. Thus $\widetilde{f}(U) = \widetilde{f}(G)$.
By (1), (2) and (3), we obtain the result. □

Lemma 3.3 ([1]). *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space, then* $f \cong \mathcal{I}$ *iff* $H - \widetilde{f}(A) \in \mathcal{I}$ *for all* $H \subseteq \mathcal{C}$.

Theorem 3.4. *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space. Then* $f \cong \mathcal{I}$ *if and only if* $f_{\psi}(H) - H \in \mathcal{I}$ *for all* $H \subseteq \mathcal{C}$.

Proof. Necessity. Let $f \cong \mathcal{I}$ and let $H \subseteq \mathcal{C}$. clearly that $x \in f_{\psi}(H) - H \in \mathcal{I}$ if and only if $x \notin H$ and $x \notin \widetilde{f}(\mathcal{C} - H)$ if and only if $x \notin H$ and there exists $U_x \in \mathcal{N}(x)$ such that $U_x - H \in \mathcal{I}$ if and only if there exists $U_x \in \mathcal{N}(x)$ such that $x \in U_x - H \in \mathcal{I}$. Now, for each $x \in f_{\psi}(H) - H$ and $U_x \in \mathcal{N}(x)$, $U_x \cap (f_{\psi}(H) - H) \in \mathcal{I}$ by heredity and hence $f_{\psi}(H) - H \in \mathcal{I}$ by assumption that $f \cong \mathcal{I}$.

Sufficiency. Let $H \subseteq \mathcal{C}$ and assume that for each $x \in H$ there exists $U_x \in \mathcal{N}(x)$ such that $U_x \cap H \in \mathcal{I}$. Observe that $f_{\psi}(\mathcal{C} - H) - (\mathcal{C} - H) = \{x :$ there exists $U_x \in \mathcal{N}(x)$ such that $x \in U_x \cap H \in \mathcal{I}\}$. Thus we have $A \subseteq f_{\psi}(\mathcal{C} - H) - (\mathcal{C} - H) \in \mathcal{I}$ and hence $H \in \mathcal{I}$ by heredity of $\mathcal{I}$. □

Lemma 3.5. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$ *and* $H \subseteq \mathcal{C}$, *then* H *is a* $\tau_{cl}(x)$*-closed iff* $H = K \cup I$ *such that* K *is* f*-closed and* $I \in \mathcal{I}$.

Proof. If H is a $\tau_{cl}(x)$-closed set, then $\widetilde{f}(H) \subseteq H$. Hence $H = H \cup \widetilde{f}(H) = (H - \widetilde{f}(H)) \cup \widetilde{f}(H)$. Then by Lemma 1.6 $\widetilde{f}(H)$ is f-closed set and by Lemma 3.3 $A - \widetilde{f}(H) \in \mathcal{I}$. Conversely, if $H = K \cup I$ such that K is f-closed set and $I \in \mathcal{I}$, then by Lemma 1.12 we get that $\widetilde{f}(H) = \widetilde{f}(K \cup I) = \widetilde{f}(K) \cup \widetilde{f}(I) = \widetilde{f}(K) \subseteq f(K) = K \subseteq H$. Implies that H is a $\tau_{cl}(x)$-closed. □

Corollary 3.6. *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$. *Then* $\beta(f, \mathcal{I})$ *is a topology on* $\mathcal{C}$ *and hence* $\beta(f, \mathcal{I}) = \tau_{cl}(x)$.

Proof. Let $H \in \tau_{cl}(x)$. Then by Lemma 3.5, $\mathcal{C} - H = F \cup I$, where F is f-closed and $I \in \mathcal{I}$. Then $H = \mathcal{C} - (F \cup I) = (\mathcal{C} - F) \cap (\mathcal{C} - I) = (\mathcal{C} - F) - I = V - I$, where $V = \mathcal{C} - F \in \mathcal{N}(x)$. Thus every $\tau_{cl}(x)$-open set is form of the $V - I$, where $V \in \mathcal{N}(x)$ and $I \in \mathcal{I}$. Hence by by Theorem 1.10 the result follows. □

Proposition 3.7. *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$, $H \subseteq \mathcal{C}$. *If* $N \in \mathcal{N}(x)$ *and* $N \subseteq \widetilde{f}(A) \cap f_{\psi}(H)$, *then* $N - H \in \mathcal{I}$ *and* $N \cap H \notin \mathcal{I}$.

Proof. If $N \subseteq \widetilde{f}(H) \cap f_{\psi}(H)$, then $N - H \subseteq f_{\psi}(H) - H \in \mathcal{I}$ by Theorem 3.4 and hence $N - H \in \mathcal{I}$ by heredity. Since $N \in \mathcal{N}(x)$ and $N \subseteq \widetilde{f}(H)$, we get $N \cap H \notin \mathcal{I}$ by the definition of $\widetilde{f}(H)$. □

As a consequence of proposition, we have.

Corollary 3.8. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$. *Then* $f_\psi(f_\psi(H)) = f_\psi(H)$ *for all* $H \subseteq \mathcal{C}$.

Proof. $f_\psi(H) \subseteq f_\psi(f_\psi(H))$ follows from Theorem 2.2 (5). Since $f \cong \mathcal{I}$, it follows from Theorem 3.4 that $f_\psi(H) \subseteq H \cup I$ for some $I \in \mathcal{I}$ and hence $f_\psi(f_\psi(H)) = f_\psi(H)$ by Theorem 2.2 (10). □

Theorem 3.9. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$. *Then* $f_\psi(H) = \cup\{f_\psi(U) : U \in \mathcal{N}(x), f_\psi(U) - H \in \mathcal{I}\}$.

Proof. Let $\Phi(H) = \cup\{f_\psi(U) : U \in \mathcal{N}(x), f_\psi(U) - A \in \mathcal{I}\}$. Clearly, $\Phi(H) \subseteq f_\psi(H)$. Now let $x \in f_\psi(H)$. Then there exists $U \in \mathcal{N}(x)$ such that $U - H \in \mathcal{I}$. By Corollary 2.3, $U \subseteq f_\psi(U)$ and $f_\psi(U) - H \subseteq [f_\psi(U) - U] \cup [U - H]$. By Theorem 3.4, $f_\psi(U) - U \in \mathcal{I}$ and hence $f_\psi(U) - H \in \mathcal{I}$. Hence $x \in \Phi(H)$ and $\Phi(H) \supseteq f_\psi(H)$. Consequently, we obtain $\Phi(H) = f_\psi(H)$. □

In [12], Newcomb defines $H = K$ [mod $\mathcal{I}$] if $(H - K) \cup (K - H) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 2.2 (11), we have that if $H = K$ [mod $\mathcal{I}$], then $f_\psi(H) = f_\psi(K)$.

Definition 3.10. Let $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. A subset H of X is called a Baire set with respect to $\mathcal{N}(x)$ and $\mathcal{I}$, denoted $H \in \mathcal{W}_r(X, f, \mathcal{I})$, if there exists $U \in \mathcal{N}(x)$ such that $H = U$ [mod $\mathcal{I}$].

Lemma 3.11. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$. *If* U, $V \in \mathcal{N}(x)$ *and* $f_\psi(U) = f_\psi(V)$, *then* $U = V$ *[mod* $\mathcal{I}$*].*

Proof. Since $U \in \mathcal{N}(x)$, we have $U \subseteq f_\psi(U)$ and hence $U - V \subseteq f_\psi(U) - V = f_\psi(V) - V \in \mathcal{I}$ by Theorem 3.4. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V$ [mod $\mathcal{I}$]. □

Theorem 3.12. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space such that* $f \cong \mathcal{I}$. *If* H, $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I})$, *and* $f_\psi(H) = f_\psi(K)$, *then* $H = K$ *[mod* $\mathcal{I}$*].*

Proof. Let $U, V \in \mathcal{N}(x)$ such that $H = U$ [mod $\mathcal{I}$] and $K = V$ [mod $\mathcal{I}$]. Now $f_\psi(H) = f_\psi(U)$ and $f_\psi(K) = f_\psi(V)$ by Theorem 2.2(11). Since $f_\psi(H) = f_\psi(U)$ implies that $f_\psi(U) = f_\psi(V)$ and hence $U = V$ [mod $\mathcal{I}$] by Lemma 1.11. Hence $H = K$ [mod $\mathcal{I}$] by transitivity. □

4. MORE PROPERTIES OF AN IDEAL ČECH-SPACES

Lemma 4.1. *Given* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space. If* $A \in \mathcal{N}(x)$ *then* $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$ *if and only if* $\widetilde{f}(H) = f(H)$.

Proof. Let $H \in \mathcal{N}(x)$ then by Lemma 1.6 we have $\widetilde{f}(H) \subseteq f(H)$. Let $x \in f(H)$, then for all $U_x \in \mathcal{N}(x)$ containing x we have $U_x \cap H \neq \phi$. Again $U_x \cap A \in \mathcal{N}(x)$, so $U_x \cap H \notin \mathcal{I}$, since $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Hence $x \in \widetilde{f}(H)$. Therefore, $\widetilde{f}(H) = f(H)$. Conversely, for any $A \in \mathcal{N}(x)$ we have $\widetilde{f}(H) = f(H)$. Then $\mathcal{C} = \widetilde{f}(\mathcal{C})$ and then $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$ by Theorem 1.11. □

PROPOSITION 4.2. *Let $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space.*

1. *If $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$, then there exists $H \in \mathcal{N}(x)$ such that $K = H$ [mod $\mathcal{I}$].*
2. *If $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$, then $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$ if and only if there exists $H \in \mathcal{N}(x)$ such that $K = H$ [mod $\mathcal{I}$].*

Proof. (1) If $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$, then $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I})$. Now if there does not exist $H \in \mathcal{N}(x)$ such that $K = H$ [mod $\mathcal{I}$], we have $K = \emptyset$ [mod $\mathcal{I}$]. Then $K \in \mathcal{I}$ which is a contradiction.
(2) If there exists $H \in \mathcal{N}(x)$ such that $K = H$ [mod $\mathcal{I}$]. Then $H = (K - J) \cup I$, where $J = K - H, I = H - K \in \mathcal{I}$. If $K \in \mathcal{I}$, then $H \in \mathcal{I}$ by heredity and additivity, which contradicts that $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. □

PROPOSITION 4.3. *Let $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space with $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. If $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$, then $f_\psi(K) \cap f(\widetilde{f}(K)) \neq \emptyset$.*

Proof. Let $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$, then by Proposition 4.2(1), there exists $H \in \mathcal{N}(x)$ such that $K = H$ [mod $\mathcal{I}$]. This implies that $\emptyset \neq H \subseteq \widetilde{f}(H) = \widetilde{f}((K - J) \cup I) = \widetilde{f}(K) = f(\widetilde{f}(K))$, where $J = K - H, I = H - K \in \mathcal{I}$ by Theorem 1.8 and Lemma 1.12. Also $\emptyset \neq H \subseteq f_\psi(H) = f_\psi(K)$ by Theorem 2.2 (11), so that $H \subseteq f_\psi(K) \cap f(\widetilde{f}(K))$. □

Given an ideal Čech-space $(\mathcal{C}, f, \mathcal{I})$, let $\mathcal{U}(\mathcal{C}, f, \mathcal{I})$ denote $\{H \subseteq \mathcal{C}$: there exists $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$ such that $K \subseteq H\}$.

PROPOSITION 4.4. *Let $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space with $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. The following are equivalent:*

1. *$H \in \mathcal{U}(\mathcal{C}, f, \mathcal{I})$;*
2. *$f_\psi(H) \cap f(\widetilde{f}(H)) \neq \emptyset$;*
3. *$f_\psi(H) \cap \widetilde{f}(H) \neq \emptyset$;*
4. *$f_\psi(H) \neq \emptyset$;*
5. *$f(H) \neq \emptyset$;*
6. *There exists $N \in \mathcal{N}(x)$ such that $N - H \in \mathcal{I}$ and $N \cap H \notin \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): Let $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$ such that $K \subseteq H$. Then $f(\widetilde{f}(K)) \subseteq f(\widetilde{f}(H))$ and $f_\psi(K) \subseteq f_\psi(H)$ and hence $f(\widetilde{f}(K)) \cap f_\psi(K) \subseteq f(\widetilde{f}(H)) \cap f_\psi(H)$. By Proposition 4.3, we have $f_\psi(H) \cap f(\widetilde{f}(H)) \neq \emptyset$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (5): If $f_\psi(H) \neq \emptyset$, then there exists $U \in \mathcal{N}(x)$ such that $U - H \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - H) \cup (U \cap H)$, we have $U \cap H \notin \mathcal{I}$. By Theorem 2.2, $\emptyset \neq (U \cap H) \subseteq f_\psi(U) \cap H = f_\psi((U - H) \cup (U \cap H)) \cap H = f_\psi(U \cap H) \cap H \subseteq f_\psi(H) \cap H = f(H)$. Hence $f(H) \neq \emptyset$.

(5) $\Rightarrow$ (6): If $f(H) \neq \emptyset$, then by Theorem 1.10 there exists $N \in \mathcal{N}(x)$ and $I \in \mathcal{I}$ such that $\emptyset \neq N - I \subseteq H$. We have $N - H \in \mathcal{I}$, $N = (N - H) \cup (N \cap H)$ and $N \notin \mathcal{I}$. Hence $N \cap H \notin \mathcal{I}$.

(6) $\Rightarrow$ (1): Let $K = N \cap H \notin \mathcal{I}$ with $N \in \mathcal{N}(x)$ and $N - H \in \mathcal{I}$. Then $K \in \mathcal{W}_r(\mathcal{C}, f, \mathcal{I}) - \mathcal{I}$ since $K \notin \mathcal{I}$ and $(K - N) \cup (N - K) = N - H \in \mathcal{I}$. □

THEOREM 4.5. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space, where $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Then for $H \subseteq \mathcal{C}$, $f_\psi(H) \subseteq \widetilde{f}(H)$.*

Proof. Let $x \in f_\psi(H)$ and $x \notin \widetilde{f}(H)$. Then there exists a nonempty $U_x \in \mathcal{N}(x)$ such that $U_x \cap H \in \mathcal{I}$. Since $x \in f_\psi(H)$, by Theorem 2.4, $x \in \cup\{U \in \mathcal{N}(x) : U - H \in \mathcal{I}\}$ and there exists $V \in \mathcal{N}(x)$ such that $x \in V$ and $V - H \in \mathcal{I}$. Now we have $U_x \cap V \in \mathcal{N}(x)$, $U_x \cap V \cap H \in \mathcal{I}$ and $(U_x \cap V) - H \in \mathcal{I}$ by heredity. Then by finite additivity we get $(U_x \cap V \cap H) \cup (U_x \cap V - H) = (U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in \mathcal{N}(x)$, this is contrary to $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Therefore, $x \in \widetilde{f}(H)$. Hence $f_\psi(H) \subseteq \widetilde{f}(H)$. □

COROLLARY 4.6. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space, where $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Then for $H \subseteq \mathcal{C}$, $f_\psi(H) \subseteq f(\widetilde{f}(H))$.*

THEOREM 4.7. *Given $(\mathcal{C}, f, \mathcal{I})$ be an ideal Čech-space. Then the following are equivalent:*

(1) $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$;
(2) $f_\psi(\emptyset) = \emptyset$;
(3) *If $H \subseteq \mathcal{C}$ is $\tau_{cl}(x)$-closed, then* $f_\psi(H) - H = \emptyset$;
(4) *If $I \in \mathcal{I}$, then* $f_\psi(I) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Since $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$, by Theorem 2.4 we obtain $f_\psi(\emptyset) = \cup\{U \in \mathcal{N}(x) : U \in \mathcal{I}\} = \emptyset$.

(2) $\Rightarrow$ (3): Suppose $x \in f_\psi(H) - H$, then there exists a $U_x \in \mathcal{N}(x)$ such that $x \in U_x - H \in \mathcal{I}$ and $U_x - H \in \mathcal{N}(x)$. But $U_x - H \in \{U \in \mathcal{N}(x) : U \in \mathcal{I}\} = f_\psi(\emptyset)$ which implies that $f_\psi(\emptyset) = \emptyset$. Hence $f_\psi(H) - H = \emptyset$.

(3) $\Rightarrow$ (4): Let $I \in \mathcal{I}$ then $f_\psi(I) = f_\psi(I \cup \emptyset) = f_\psi(\emptyset) = \emptyset$.

(4) $\Rightarrow$ (1): Let $H \in \mathcal{N}(x) \cap \mathcal{I}$, then $H \in \mathcal{I}$ and by (4) $f_\psi(H) = \emptyset$. Since $H \in \mathcal{N}(x)$, by Corollary 2.3 we get $H \subseteq f_\psi(H) = \emptyset$. Hence $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. □

THEOREM 4.8. *Let* $(\mathcal{C}, f, \mathcal{I})$ *be an ideal Čech-space. Then* $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$ *if and only if* $\widetilde{f}[f_\psi(H)] = f[f_\psi(H)]$ *for all* $H \subseteq \mathcal{C}$.

Proof. Let $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. It is clear that $\widetilde{f}[f_\psi(H)] \subseteq f[f_\psi(H)]$. For the reverse inclusion, let $x \in f[f_\psi(H)]$. Then for every $U_x \in \mathcal{N}(x)$, $U_x \cap f_\psi(H) \neq \emptyset$ and $U_x \cap f_\psi(H) \in \mathcal{N}(x)$ implies that $U_x \cap f_\psi(H) \notin \mathcal{I}$, since $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. Hence $x \in \widetilde{f}[f_\psi(H)]$. Hence $\widetilde{f}[f_\psi(H)] = f[f_\psi(H)]$. Conversely, suppose that $\widetilde{f}[f_\psi(H)] = f[f_\psi(H)]$, for every $H \subseteq \mathcal{C}$. Then for $\mathcal{C} \subseteq \mathcal{C}$, $\widetilde{f}[f_\psi(\mathcal{C})] = f[f_\psi(\mathcal{C})]$. Hence $\widetilde{f}[\mathcal{C} - \widetilde{f}(\mathcal{C} - \mathcal{C})] = f[\mathcal{C} - \widetilde{f}(\mathcal{C} - \mathcal{C})]$, implies that $\widetilde{f}(\mathcal{C}) = f(\mathcal{C}) = \mathcal{C}$. Hence $\mathcal{N}(x) \cap \mathcal{I} = \emptyset$. □

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