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EXISTENCE OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. The aim of this study is to prove the existence of solutions for Caputo boundary value problems of nonlinear fractional integro-differential equations with integral boundary conditions, by using the measure of non compactness combined with Mönch's fixed point theorem. Two examples are offered to demonstrate our outcomes.

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1. INTRODUCTION

In recent years, fractional differential equations have attracted the interest of many authors, due their applications in various fields of sciences, as in physics, chemistry, hydrology, biophysics, thermodynamics, blood flow problems, statistical mechanics, and control theory. That is, most phenomena can be represented by fractional differential equations, and some results were given in this way, for example, see [11, 18, 19] and the references therein.

Particularly, differential equations with integral boundary conditions have different applications in applied science such as in underground water flow, thermo-elasticity, population dynamics, and some results are given in this way, for instance see [2, 5, 8, 12].

By using Krasnoselskii's fixed point theorem and the Banach principle, Ahmad and Sivasundaram [3] investigated the existence of solutions for the following boundary value problem

$$\begin{cases} {}^{c}\mathrm{D}^{\alpha}x(t) = f(t,x(t)) + \displaystyle\int_0^t k(t,s,x(s))\mathrm{d}s, \ 0 \le t \le T, \ 0 < \alpha < 1, \\ x(0) = u_0 - g(x), \end{cases}$$

where ${}^{c}D^{\alpha}$, $0 < \alpha \le 1$ is the Caputo fractional derivative and f, g, k are given continuous functions.

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Abdo et al. [1] used the method of the upper and lower solutions, Schauder and Banach fixed point theorems to prove the existence and uniqueness of the solution for the following boundary problem:

$$\begin{cases} {}^c\mathrm{D}^\alpha x(t) = f(t, x(t)),\ 0 \le t \le 1,\ 1 < \alpha \le 2, \\ x(0) = \displaystyle\int_0^1 h(s)g(s, x(s))\mathrm{d}s + \lambda, \end{cases}$$

where ${}^c\mathrm{D}^\alpha$, $0 < \alpha < 1$ is the Caputo fractional derivative, $\lambda \ge 0$ and f is a given function.

In [10], Hamrouni and Beloul employed Mönch's fixed point theorem to discuss the existence of solutions of the following boundary value problem:

$$\begin{cases} {}^c\mathrm{D}^\alpha x(t) + f(t, x(t), {}^c D^\alpha x(t)),\ 0 \le t \le 1,\ 1 < \alpha \le 2, \\ ax(0) - bx'(0) = 0 \\ x(1) = \displaystyle\int_0^1 h(s)g(s, x(s))\mathrm{d}s + \lambda, \end{cases}$$

where ${}^c\mathrm{D}^\alpha$, $1 < \alpha \le 2$ is the Caputo fractional derivative, f, g, and h are given by: $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $g \in [0,1] \times \mathbb{R} \to \mathbb{R}$, $h \in L^1([0,1], \mathbb{R})$, $a, b, \lambda \in \mathbb{R}_+$, $a + b > 0$ and $\frac{a}{a+b} < \alpha - 1$.

Motivated by the above works, in this paper we study the existence of solutions to the following boundary value problem:

$$(1) \begin{cases} {}^c\mathrm{D}^\alpha x(t) = f(t, x(t)) + \displaystyle\int_0^t k(t, s, x(s))\mathrm{d}s,\ 0 \le t \le 1,\ 1 < \alpha \le 2, \\ ax(0) - bx'(0) = 0, \\ x(1) = \displaystyle\int_0^1 h(s)g(s, x(s))\mathrm{d}s + \lambda, \end{cases}$$

where ${}^c\mathrm{D}^\alpha$, $1 < \alpha \le 2$ is the Caputo fractional derivative, f, g, and h are given functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $g \in [0,1] \times \mathbb{R} \to \mathbb{R}$, $h \in L^1([0,1], \mathbb{R})$, $a, b, \lambda \in \mathbb{R}_+$, $a + b > 0$ and $\frac{a}{a+b} < \alpha - 1$.

2. PRELIMINARIES

Denote by $X = C([0,1], \mathbb{R})$ the Banach space of all continuous functions $x : [0,1] \longrightarrow \mathbb{R}$, with the usual supremum norm

$$\| x \|_\infty = \sup\{|x(t)|,\ \ t \in [0;1]\}.$$

Let $L^1([0,1])$ be the Banach space of measurable functions $x : [0,1] \longrightarrow \mathbb{R}$ which are Bochner integrable, equipped with the norm

$$\| x \|_{L^1} = \int_0^1 |x(t)|\mathrm{d}t.$$

Let us recall some fundamental facts about the Kuratowski measure of non compactness.

Definition 2.1 ([4]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of non compactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by

$$\mu(B) = \liminf\{\epsilon > 0 : \ B \subseteq \cup_{i=1}^{n} B_i, \ \operatorname{diam}(B_i) \leq \epsilon\}, \ \ B \in \Omega_E,$$

where

$$\operatorname{diam}(B_i) = \sup\{\|u - v\|_E : u, v \in B_i\}.$$

Lemma 2.2 ([4]). *Let A and B be two bounded sets.*

(1) $\mu(B) = 0 \Rightarrow B$ *is relatively compact.*
(2) $\mu(B) = \mu(\overline{B})$.
(3) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(4) $\mu(A + B) \leq \mu(A) + \mu(B)$.
(5) $\mu(cB) \leq |c|\mu(B), \ c \in \mathbb{R}$.
(6) $\mu(\operatorname{conv} B) = \mu(B)$.

Lemma 2.3 ([6]). *Let X be a Banach space, If $V \subseteq C([[a, b], X)$ is equicontinuous and bounded, then $\mu(V(t))$ is continuous and*

$$\mu\left(\int_a^t V(s)\mathrm{d}s\right) \leq \int_a^t \mu(V(s))\mathrm{d}s,$$

where $\displaystyle\int_a^t V(s)\mathrm{d}s = \left\{\int_a^t x(s)\mathrm{d}s, x \in V\right\}$.

Now, we give some definitions and properties of the Riemann-Liouville and Caputo derivatives of fractional order.

Definition 2.4 ([13]). The fractional integral of order $\alpha > 0$ of the function $h \in L^1([a, b], \mathbb{R}_+)$ is defined by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t - s)^{1-\alpha}}\mathrm{d}s,$$

where $\Gamma(\alpha)$ denotes the classical gamma function, provided that the right hand side is defined on $(0, \infty)$.

Definition 2.5 ([13]). For a given function $f : (0, \infty) \to \mathbb{R}$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of x is defined by

$$\mathrm{D}_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{x^{(n)}(s)}{(t - s)^{\alpha-n+1}}\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Definition 2.6 ([13]). The Caputo fractional derivative of order $\alpha > 0$ of a function $x : (0, \infty) \to \mathbb{R}$ is defined by

$${}^{c}\mathrm{D}^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} x^{(n)}(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

LEMMA 2.7 ([13]). *Let α be a non negative real number, then the differential equation*

$$^{c}\mathrm{D}^{\alpha}x(t)=0$$

has a solution given by:

$$x(t)=\sum_{i=0}^{n-1}c_i t^i,\quad c_i\in\mathbb{R},\ i=0,1,...,n-1,\ n=[\alpha]+1.$$

LEMMA 2.8 ([7]). *Let $\alpha>0$, then*

$$I^{\alpha}(^{c}\mathrm{D}^{\alpha}x(t))=x(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1},$$

$c_i\in\mathbb{R}$, $i=0,1,...,n-1$, $n=[\alpha]+1$.

THEOREM 2.9 ([17]). *Let C be a bounded, closed and convex subset of a Banach space such that $0\in C$, and let $T: C\to C$ be a continuous mapping. If the implication*

$$V=\overline{\mathrm{conv}}T(V)\quad or\quad V=T(V)\cup 0\qquad\Rightarrow\qquad \mu(V)=0$$

holds for every subset V of C, then T has a fixed point.

3. MAIN RESULTS

LEMMA 3.1. *A function x is a solution of (1) if and only if it is a solution of the following integral equation:*

$$\begin{aligned}x(t)\ &=\frac{1}{\Gamma(\alpha)}\left[\int_0^t(t-s)^{\alpha-1}\left(f(s,x(s))+\int_0^s k(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]\\&+\frac{at+b}{a+b}\Big[\int_0^1 h(s)g(s,x(s))\mathrm{d}s+\lambda-\frac{1}{\Gamma(\alpha)}\Big(\int_0^1(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s\\&+\int_0^1(1-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\Big].\end{aligned}$$

Proof. Assume x satisfies (1), then from Lemma 2.8 we have

$$\begin{aligned}I^{\alpha}(^{c}\mathrm{D}^{\alpha}x(t))\ &=x(t)-c_0-c_1t\\&=\frac{1}{\Gamma(\alpha)}\left[\int_0^t f(s,x(s))\mathrm{d}s+\int_0^t\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\right],\end{aligned}$$

which implies that

$$x(t)=c_0+c_1t+\frac{1}{\Gamma(\alpha)}\left[\int_0^t f(s,x(s))\mathrm{d}s+\int_0^t\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\right].$$

Applying the boundary conditions, we find

$$ax(0)-bx'(0)=ac_0-bc_1=0$$

and

$$x(1) = c_0 + c_1$$
$$+ \frac{1}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-1} f(s, x(s))\mathrm{d}s + \int_0^1 (1-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\right]$$
$$= \int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda,$$

then

$$c_0 = \frac{b}{a+b}\Big[\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda$$
$$- \frac{1}{\Gamma(\alpha)}\Big(\int_0^1 (1-s)^{\alpha-1} f(s, x(s))\mathrm{d}s + \int_0^1 (1-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\Big)$$

and

$$c_1 = \frac{a}{a+b}\Big[\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda$$
$$- \frac{1}{\Gamma(\alpha)}\Big(\int_0^1 (1-s)^{\alpha-1} f(s, x(s))\mathrm{d}s + \int_0^1 (1-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\Big).$$

Consequently, we obtain

$$x(t) = \frac{1}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1} f(s, x(s))\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\Big)$$
$$+ \frac{at+b}{a+b}\Big[\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda$$
$$- \frac{1}{\Gamma(\alpha)}\Big(\int_0^1 (1-s)^{\alpha-1} f(s, x(s))\mathrm{d}s + \int_0^1 (1-s)^{\alpha-1}\int_0^s k(s,\tau,x(\tau)\mathrm{d}\tau\mathrm{d}s\Big).$$

Then we can write

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\Big(f(s,x(s)) + \int_0^s k(s,\tau,x(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &+ \frac{at+b}{a+b}\Big(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda\Big), \end{aligned}$$

where G is the Green function given by:

$$G(t,s) = \frac{1}{(a+b)\Gamma(\alpha)} \tag{2}$$
$$\cdot \begin{cases} (a+b)(t-s)^{\alpha-1} + (at+b)(1-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (at+b)(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

$\square$

LEMMA 3.2. *The function G defined by (2) satisfies*

$$G(t,s) \leq \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof. If $0 \leq s \leq t \leq 1$, then

$$\begin{aligned} G(t,s) &= \frac{1}{(a+b)\Gamma(\alpha)}\left[(a+b)(t-s)^{\alpha-1} + (at+b)(1-s)^{\alpha-1}\right] \\ &\leq \frac{1}{(a+b)\Gamma(\alpha)}\left[(a+b)(t-s)^{\alpha-1} + (a+b)(1-s)^{\alpha-1}\right] \\ &\leq \frac{1}{\Gamma(\alpha)}\left[(t-s)^{\alpha-1} + (1-s)^{\alpha-1}\right] \\ &\leq \frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} + (1-s)^{\alpha-1}\right] \\ &= \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

If $0 \leq t \leq s \leq 1$, then

$$\begin{aligned} G(t,s) &= \frac{1}{(a+b)\Gamma(\alpha)}\left[(at+b)(1-s)^{\alpha-1}\right] \\ &\leq \frac{1}{(a+b)\Gamma(\alpha)}\left[(a+b)(1-s)^{\alpha-1}\right] \\ &\leq \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &\leq \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

□

Now, we assume that:

(A_1) There exists $\varphi \in L^\infty([0,1])$, such that

$$|f(t,x(t)) - f(t,y(t)| \leq \varphi(t)|x-y|$$

and for each bounded subset A of X, we have

$$\gamma(f(t,A)) \leq \varphi_0\gamma(A).$$

(A_2) There exists $\phi \in L^\infty([0,1])$, such that

$$|k(t,s,u(s)) - k(t,s,v(s)| \leq \phi(t)|u-v|$$

and for each bounded subset B of X, we have

$$\gamma(k(t,s,B)) \leq \phi_0\gamma(B).$$

(A_3) There exists $\psi \in L^\infty([0,1])$, such that

$$|g(t,x(t)) - g(t,y(t)| \leq \psi(t)|x-y|$$

and for each bounded subset B of X, we have

$$\gamma(g(t,C) \leq \psi_0\gamma(C),$$

where $\varphi_0 = \sup\limits_{t\in[0,1]} |\varphi(t)|$, $\psi_0 = \sup_{t\in[0,1]} |\psi(t)|$ and $\phi_0 = \sup\limits_{t\in[0,1]} |\phi(t)|$.

Theorem 3.3. *Under the assumptions* $(A_1) - (A_3)$ *the problem* (1) *has a solution provided*

$$\frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0) + \|h\|_{L^1}\psi_0 + \lambda < 1.$$

Proof. Define an operator $T : X \longrightarrow X$ by

$$\begin{aligned} Tx(t) = {} & \int_0^1 G(t,s)\left(f(s,x(s)) + \int_0^s k(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ & + \frac{at+b}{a+b}\left(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda\right). \end{aligned}$$

Clearly, the fixed points of the operator T are solutions of problem (1). We show that T satisfies the assumptions of Theorem 3.3. The proof will be given in three steps.

Step 1. T is continuous.

Let (x_n) be a sequence such that $x_n \longrightarrow x$ in $C([0,1],\mathbb{R})$. Then for each $t \in [0,1]$ we have

$$\begin{aligned} |T(x_n)(t) - T(x)(t)| \leq {} & \int_0^1 |G(t,s)||f(s,x_n(s)) - f(s,x(s))|\mathrm{d}s \\ & + \int_0^1 |G(t,s)| \int_0^s |k(s,\tau,x_n(\tau)) - k(s,\tau,x(\tau))|\mathrm{d}\tau \\ & + \int_0^1 |h(s)||g(s,x_n(s)) - g(s,x(s))|\mathrm{d}s. \end{aligned}$$

Assumptions (A_1), (A_2) and (A_3) give:

$$\begin{aligned} |T(x_n)(t) - T(x)(t)| \leq {} & \int_0^1 |G(t,s)|\varphi_0|x_n(s) - x(s)|\mathrm{d}s \\ & + \int_0^s \phi_0|x_n(s) - x(s)|\mathrm{d}\tau + \int_0^1 |h(s)|\psi_0\big|x_n(s) - x(s)\big|\mathrm{d}s \\ \leq {} & \left[\frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0) + \|h\|_{L^1}\psi_0\right]|x_n(s) - x(s)|. \end{aligned}$$

Then

$$\|T(x_n)(t) - T(x)(t)\|_\infty \leq \left[\frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0) + |h\|_{L^1}\psi_0\right]\|x_n(s) - x(s)\|_\infty.$$

Since $x_n \longrightarrow x$, for each $t \in [0,1]$, we get

$$\|T(x_n)(t) - T(x)(t)\| \longrightarrow 0, \quad \text{when} \quad n \longrightarrow \infty.$$

Therefore, T is continuous.

Let r be a constant such that

$$\frac{2}{\Gamma(\alpha+1)}[(\varphi_0+\phi_0)r + f^* + k^*] + \|h\|_{L^1}\psi_0 r + g^* + \lambda \le r,$$

where

$$f^* = \sup_{t\in[0,1]} |f(t,0)|, \quad g^* = \sup_{s\in[0,1]} |g(s,0)|, \text{ and } k^* = \sup_{t\in[0,1]} |k(t,0,0)|.$$

Let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$. It is clear that B_r is a bounded subset, closed and convex of X.

Step 2. $T(B_r) \subseteq B_r$.

Let x be an element of B_r, we show that $Tx \in B_r$. In fact, for each $t \in [0,1]$ we have

$$|T(x)(t)| = \Big| \int_0^1 G(t,s) f(s,x(s))\mathrm{d}s$$
$$+ \int_0^1 \int_0^s G(t,s)k(s,\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s + \frac{at+b}{a+b}\left(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda\right)\Big|.$$

By (A_1), (A_2) and (A_3), for each $t \in [0,1]$ we have

$$\begin{aligned} |f(t,x(t))| &= |f(t,x(t)) - f(t,0) + f(t,0)| \\ &\le |f(t,x(t)) - f(t,0)| + |f(t,0)| \\ &\le \varphi_0|x| + f^*, \end{aligned} \tag{3}$$

$$\begin{aligned} |k(t,s,x(s))| &= |k(t,s,x(s)) - k(t,0,0) + k(t,0,0)| \\ &\le |k(t,s,x(s)) - k(t,0,0)| + |k(t,0,0)| \\ &\le \phi_0|x| + k^* \end{aligned} \tag{4}$$

and

$$\begin{aligned} |g(t,x(t))| &= |g(t,x(t)) - g(t,0) + g(t,0)| \\ &\le |g(t,x(t)) - g(t,0)| + |g(t,0)| \\ &\le \psi_0|x| + g^*. \end{aligned} \tag{5}$$

Then we have

$$\|Tx(t)\| \le \frac{2}{\Gamma(\alpha)}((\varphi_0+\phi_0))r + f^* + k^*) + (\|h\|_{L^1}\psi_0 r + g_*) + \lambda.$$

It follows that for each $t \in [0,1]$ we have $\|Tx(t)\| \le r$, which implies

$$T(B_r) \subset B_r.$$

Step 3. TB_r is bounded and equicontinuous.
According to **Step 2**, we have

$$TB_r = \{Tx : x \in B_r\} \subset B_r.$$

So for every $x \in B_r$ we have $\|T(x)\|_\infty \leq r$, which implies that TB_r is bounded. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in TB_r$, so we have

$$\begin{aligned}
&|T(x)(t_2) - T(x)(t_1)| \\
&= \Big| \int_0^1 G(t_2,s) \left[f(s,x(s)) + \int_0^s k(s,\tau,x(\tau))\mathrm{d}\tau \right] \mathrm{d}s \\
&+ \frac{at_2+b}{a+b} \left(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda \right) - \int_0^1 G(t_1,s) \Big[f(s,x(s)) \\
&+ \int_0^s k(s,\tau,x(\tau))\mathrm{d}\tau \Big] \mathrm{d}s + \frac{at_1+b}{a+b} \left(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda \right) \Big| \\
&\leq \Big| \int_0^1 |G(t_2,s) - G(t_1,s)| \left| f(s,x(s)) + \int_0^s |k(s,\tau,x(\tau))\mathrm{d}\tau \right| \mathrm{d}s \\
&+ \frac{a}{a+b} \int_0^1 |h(s)g(s,x(s))(t_2-t_1)|\,\mathrm{d}s + \lambda \frac{a}{a+b} \Big| t_2 - t_1) \Big| \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)|\,|f(s)|\mathrm{d}s \\
&+ \frac{a|t_2-t_1|}{a+b} \int_0^1 |h(s)||g(s,x(s))|\mathrm{d}s + \lambda \frac{a}{a+b} |t_2 - t_1|.
\end{aligned}$$

According to (3), (4) and (5), we have

$$\begin{aligned}
\|T(x)(t_2) - T(x)(t_1)\| \leq ((\varphi_0 + \phi_0)r + f^* + k^*) \int_0^1 [|G(t_2,s) - G(t_1,s)|]\,\mathrm{d}s \\
+ \frac{a(\psi_0 r + g_*)}{a+b} \|h(s)\|_{L^1} |t_2 - t_1|\mathrm{d}s + \lambda \frac{a}{a+b} |t_2 - t_1|.
\end{aligned}$$

As $t_2 \longrightarrow t_1$, the right hand side of the above inequality tends to zero, since G is uniformly continuous (a continuous on a compact) and $|t_2 - t_1| \to 0$. Then TB_r is equicontinuous.

Let $V \subset TB_r$, such that $V = \{Tx, x \in B_r\}$ so $V \subset \overline{\mathrm{conv}}(T(V) \cup \{0\})$. The subset V is bounded and equicontinuous, then from Lemma 2.3 the function $v : t \longmapsto \mu(V(t)) \in \mathbb{R}$ is continuous on $[0,1]$. From Lemma 2.3 and the properties of the measure μ, we get

$$v(t) = \mu(V(t)) \leq \mu\big(TV(t) \cup \{0\}\big) \leq \mu(TV(t)).$$

So, we have

$$
\begin{aligned}
\mu(TV(t)) &= \mu\big(Tx(t),\, x \in V\big) \\
&= \mu\Big(\int_0^1 G(t,s)\left[f(s,x(s)) + \int_0^s |k(s,\tau,x(\tau))\mathrm{d}\tau\right]\mathrm{d}s \\
&+ \frac{at+b}{a+b}\left(\int_0^1 h(s)g(s,x(s))\mathrm{d}s + \lambda\right),\, x \in V\Big) \\
&\leq \mu\left(\int_0^1 G(t,s)\left[f(t,x(t)) + \int_0^s |k(s,\tau,x(\tau))\mathrm{d}\tau\right]\mathrm{d}s, x \in V\right) \\
&+ \|h\|_{L^1}\mu\left(\int_0^1 g(s,x(s))\mathrm{d}s, x \in V\right) \\
&\leq \int_0^1 |G(t,s)|\mu\left(f(t,x(t)) + \int_0^s |k(s,\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&+ \|h\|_{L^1}\int_0^1 \mu(g(s,x(s)))\mathrm{d}s.
\end{aligned}
$$

Then by (A_1), (A_2), (A_3) and by the properties of the measure of non-compactness, we get:

$$
\begin{aligned}
\mu\Big(f(t,x(t)) + \int_0^s |k(s,\tau,x(\tau))\mathrm{d}\tau\Big) &\leq \varphi_0\mu(x(t)) + \phi_0\mu(x(t)) \\
&\leq (\varphi_0 + \phi_0)\mu(x(t))
\end{aligned}
$$

and

$$
\mu\Big(g(x(s))\Big) \leq \psi_0\mu(x(s)),
$$

Then

$$
\begin{aligned}
&\mu\Big(T(V)(t)\Big) \\
&\leq (\varphi_0+\phi_0)\int_0^1 G(t,s)\mu(x(s))\mathrm{d}s + \frac{at+b}{a+b}\left(\psi_0\int_0^1 h(s)\mu(x(s))\mathrm{d}s + \lambda\right) \\
&\leq \frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0)\int_0^1 (1-s)^{\alpha-1}v(s)\mathrm{d}s + \psi_0\|h\|_{L^1}\int_0^1 v(s)\mathrm{d}s) \\
&\leq \left(\frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0) + \psi_0\|h\|_{L^1}\right)\|v\|_\infty,
\end{aligned}
$$

which implies

$$
\|v\|_\infty \leq \left(\frac{2}{\Gamma(\alpha+1)}(\varphi_0+\phi_0) + \psi_0\|h\|_{L^1}\right)\|v\|_\infty.
$$

Hence $\|v\|_\infty = 0$, thus $v(t) = 0$ for each $t \in [0,1]$, which implies $V(t)$ is relatively compact in X. From the theorem of Ascoli-Arzela, V is relatively compact in B_r.

Applying now Mönch's theorem, we conclude that T has a fixed point, which is a solution of the problem (1). $\square$

Finally, we present the two following examples to illustrate our main results.

EXAMPLE 3.4. Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^c\mathrm{D}^{\frac{3}{2}}x(t) = \dfrac{1}{10}\dfrac{t}{1+t^2}x(t) + \dfrac{1}{10}\displaystyle\int_0^t e^{-t}\dfrac{sx(s)}{1+|x(s)|}\mathrm{d}s, \quad t \in [0,1] \\ x(0) - 5x'(0) = 0 \\ x(1) = \dfrac{1}{10}\displaystyle\int_0^1 sx(s)\mathrm{d}s \end{cases}$$

where

$$f(t,x) = \frac{1}{10}\frac{t}{1+t^2}x(t), \quad k(t,s,x) = \frac{1}{10}e^{-t}\frac{sx(s)}{1+|x(s)|},$$

$t \in [0,1]$, $x \in C([0,1],X)$,

$$h(s) = s, \quad g(s,x) = \frac{sx(s)}{10}, \quad g \in C(X,X), \alpha = 1.5, a = 1, b = 5.$$

Clearly $\dfrac{a}{a+b} < \alpha - 1$, hence

$$\begin{aligned} |f(t,x) - f(t,y)| &\le \frac{1}{10}\frac{t}{1+t^2}\left|\frac{t}{1+t^2}x - \frac{1}{10}\frac{t}{1+t^2}y\right| \\ &\le \frac{1}{10}\frac{t}{1+t^2}|x-y|, \end{aligned}$$

$$\begin{aligned} |k(t,s,x) - k(t,s,y)| &= \frac{1}{10}e^{-t}\left|\frac{sx}{1+|x|} - \frac{sy}{1+|y|}\right| \\ &\le \frac{1}{10}e^{-t}\frac{|x-y|}{(1+|x|)(1+|y|)} \\ &\le \frac{1}{10}e^{-t}|x-y|, \end{aligned}$$

$$|g(t,x)| = \left|\frac{tx}{10} - \frac{ty}{10}\right| \le \frac{t}{10}|x-y|.$$

Then for

$$\varphi(t) = \frac{1}{10}\frac{t}{1+t^2}, \quad \phi(t) = \frac{1}{10}e^{-t}, \quad \psi(t) = \frac{t}{10}$$

and

$$\varphi_0 = \frac{1}{20}, \quad \phi_0 = \frac{1}{10}, \quad \psi_0 = \frac{1}{10},$$

all the conditions (A_1), (A_2) and (A_3) are satisfied and

$$\frac{2}{\Gamma(\alpha)}(\varphi_0 + \phi_0) + \|h\|_{L^1}\psi_0 + \lambda \cong 0.308 < 1.$$

Hence, problem (1) admits at least one solution on $[0,1]$.

Example 3.5. Consider the following fractional boundary value problem:

$$\begin{cases} {}^c\mathrm{D}^{\frac{3}{2}}x(t) = \frac{1}{10}\frac{t}{1+t^2}x(t) + \frac{2\sqrt[4]{3}}{30}\int_0^t e^{-t}\arctan(t+\sin t + x(s))\mathrm{d}s, \quad t\in[0,1] \\ x(0) - 5x'(0) = 0 \\ x(1) = \frac{2}{5}\int_0^1 \frac{1}{(s+1)^2}\left(x(s) + \sqrt{1+x^2(s)}\right)\mathrm{d}s, \end{cases}$$

where

$$f(t,x) = \frac{1}{10}\frac{t}{1+t^2}x(t), \quad k(t,s,x) = \frac{2\sqrt[4]{3}}{30}e^{-t}\arctan(t+\sin t + x(s)),$$

$t\in[0,1]$, $x\in C([0,1],X)$,

$$h(s) = s, \quad g(s,x) = \frac{2}{5}\frac{1}{(s+1)^2}\left(x(s) + \sqrt{1+x^2(s)}\right), \quad g\in C(X,X),$$

$\alpha = 1.5$, $a=1$, $b=5$. Clearly $\dfrac{a}{a+b} < \alpha - 1$, hence

$$\begin{aligned} |f(t,x) - f(t,y)| &\le \frac{1}{10}\left|\frac{t}{1+t^2}x - \frac{1}{10}\frac{t}{1+t^2}y\right| \\ &\le \frac{1}{10}\frac{t}{1+t^2}|x-y|. \end{aligned}$$

In fact, observe first that using standard tools of differential calculus we can easily show that

$$|k(t,s,x) - k(t,s,y)| \le \frac{1}{10}e^{-t}|x-y|$$

$$\begin{aligned} |g(t,x) - g(t,y)| &= \frac{2}{5}\frac{1}{(t+1)^2}\left|\frac{1}{2}(x - y + \sqrt{1+x^2} - \sqrt{1+y^2}\right| \\ &= \frac{2}{5}\frac{1}{(t+1)^2}\left|\frac{1}{2}(x-y)\left(1 + \frac{x+y}{\sqrt{1+x^2}+\sqrt{1+y^2}}\right)\right| \\ &\le \frac{1}{10}\frac{1}{(t+1)^2}|x-y|. \end{aligned}$$

By considering

$$\varphi(t) = \frac{1}{10}\frac{t}{1+t^2}, \quad \phi(t) = \frac{1}{10}e^{-t}, \quad \psi(t) = \frac{1}{10(t+1)^2},$$

$$\varphi_0 = \frac{1}{20}, \quad \phi_0 = \frac{1}{10}, \quad \psi_0 = \frac{1}{10}$$

and

$$\frac{2}{\Gamma(\alpha+1)}(\varphi_0 + \phi_0) + \|h\|_{L^1}\psi_0 + \lambda \cong 0.308 < 1.$$

Then all the conditions of Theorem 3.3 are satisfied, and such, problem (1) has a solution in X.

4. CONCLUSIONS

In this paper we have proved an existence theorem of the solution for a boundary valued problem of nonlinear fractional integro-differential equations with integral boundary conditions. The technique is based on the measure of non compactness combined with Mönch's fixed point theorem. We also gave two numerical examples to illustrate the validity of our findings. Our research encourages the use of this method in the investigation of the existence problems and the stability of the solution for some problems.

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