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DYNAMICS ANALYSIS OF THE WEIBULL MODEL

CHABANE BEDJGUELEL, HACENE GHAROUT, and BAKIR FARHI

Abstract. In this work, we study the dynamics of the Weibull model in dimension one, represented by the Weibull function with three parameters. The positive fixed points have been studied and implicitly expressed in terms of the Lambert W function as well as the existence and stability conditions. We deduce that this Weibull function defines an Allee function for certain parameter values. Numerical simulations have been presented to illustrate the theoretical results.

MSC 2020. 03C45, 37H20.

Key words. Weibull function, fixed point, stability, Allee function.

1. INTRODUCTION

Weibull's law covers a whole family of laws (exponential law, normal law, etc.); it is used in several scientific fields in pure and applied mathematics [10], in the fields of life and survival analysis, and in the reliability of mechatronic systems [7], in the measurement of the life span of electronic components [6], and in wind turbines to estimate the theoretical amount of wind energy available on a given site [1, 4]. In this work, we study the Weibull function defined by

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad f(x) = rx^{p-1}e^{-qx^p}. \tag{1}$$

The aim of this paper is to investigate and compare the stability analysis of fixed points. We first give the following property, which can be easily verified:

Property 1.1. *If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the function defined by* (1)*, then the following holds:*

(i) *f is continuous for $p > 1$, and $f(x) > 0$ for all $x > 0$.*
(ii) *If $p > 1$, then f is unimodal, with a unique critical point*

$$c = \sqrt[p]{\frac{p-1}{pq}},$$

where f attains its global maximum, $\lim_{x\to 0} f(x) = 0$, and $\lim_{x\to +\infty} f(x) = 0$.

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(iii) *The Schwarz derivative of f is negative for all $x \in \mathbb{R}_+ \backslash \{c\}$ and $p > 2$.*
(iv) *If $p = 1$, then f is strictly decreasing,*

$$\lim_{x\to 0} f(x) = r, \text{ and } \lim_{x\to +\infty} f(x) = 0.$$

2. FIXED POINT OF THE WEIBULL FUNCTION

In this section, we study the positive fixed points of the Weibull function defined by (1).

We are only interested in real values with $x \in \mathbb{R}_+$ and the parameters r, p, q under the following conditions:

$$\Omega_0 = \{(r, p, q) \in \mathbf{R^3} : r, p, q > 0 \text{ and } p \geq 1\}.$$

The fixed points of f (i.e., the roots of the equation $f(x) = x$) satisfy $x = 0$ or $rx^{p-2}e^{-qx^p} - 1 = 0$.

At $x = 0$ (the trivial fixed point), the derivative of the Weibull function is discontinuous and verifies:

$$\begin{cases} \lim\limits_{x\to 0} f'(x) = \infty, & \text{if } 1 < p < 2, \\ \lim\limits_{x\to 0} f'(x) = r, & \text{if } p = 2, \\ \lim\limits_{x\to 0} f'(x) = 0, & \text{if } p > 2. \end{cases} \tag{2}$$

It is clear that $x = 0$ is always a fixed point of f. In the following proposition, we study non-trivial fixed points. The existence and the number of fixed points always depend on the parameter p. The fixed points are implicitly expressed in terms of the Lambert W function.

Recall that the Lambert W function is defined as the multivalued inverse of the function $w \longrightarrow we^w$. Thus, we have for any complex numbers z and w:

$$z = we^w, \text{ i.e, } w = W(z).$$

For $x, y \in \mathbb{R}$, the equation $ye^y = x$ can be solved in y only if $x > -\frac{1}{e}$. Precisely, for $x \geq 0$, we have a unique solution $y = W_0(x)$, and for $-\frac{1}{e} < x < 0$, we have two: $y = W_0(x)$ and $y = W_{-1}(x)$ (see Figure 2.1).

Throughout this paper, we will use the expression $X(r, p, q)$ $(r, p, q > 0)$ to denote the expression $X(r, p, q) = \frac{pq}{2-p} r^{\frac{p}{2-p}}$; x_f to denote the first positive fixed point and x_s the second fixed point.

PROPOSITION 2.1. *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the Weibull function, defined by (1) in the parameter space Ω_0.*

(I) *If $1 < p < 2$, then the Weibull function has one non-zero fixed point, given by:*

$$x_f = \sqrt[p]{\frac{2-p}{pq} W_0(X(r, p, q))}.$$

(II) *If $p > 2$, then the number of fixed point of f depends on the values of the parameter r. We have precisely:*

(i) *If* $X(r,p,q) > -\frac{1}{e}$, *then* f *has two positive fixed points given by*

$$x_f = \sqrt[p]{\frac{2-p}{pq}W_{-1}(X(r,p,q))},\ x_s = \sqrt[p]{\frac{2-p}{pq}W_0(X(r,p,q))}.$$

(ii) *If* $X(r,p,q) = -\frac{1}{e}$, *then* f *has one positive fixed point given by*

$$x_f = \sqrt[p]{\frac{2-p}{pq}W_{-1}(X(r,p,q))}.$$

(iii) *If* $X(r,p,q) < -\frac{1}{e}$, *then* f *has no positive fixed points.*

Proof. For $x > 0$, the equation of the fixed point $f(x) = x$, corresponds to:

$$rx^{p-1}e^{-qx^p} = x.$$

This allows us to have

$$xe^{\dfrac{q}{2-p}x^p} = r^{\dfrac{1}{2-p}}.$$

By raising both sides to the power p and multiplying by $\frac{pq}{2-p}$, we get the equation as:

$$ye^y = z,$$

where $y = \frac{pq}{2-p}x^p$ and $z = \frac{pq}{2-p}r^{\frac{p}{2-p}}$. Since the solution of the last equation is $y = W(z) = W\left(\frac{pq}{2-p}r^{\frac{p}{2-p}}\right)$, we finally obtain:

$$x = \sqrt[p]{\frac{2-p}{pq}W\left(\frac{pq}{2-p}r^{\frac{p}{2-p}}\right)}.$$

(I) If $1 < p < 2$, we have $X(r,p,q) > 0$, then the equation has only one solution as follows

$$x_f = \sqrt[p]{\frac{2-p}{pq}W_0(X(r,p,q))}.$$

(II) If $p > 2$, we have $X(r,p,q) < 0$. In this case, we distinguish three cases. In the first case, if $X(r,p,q) > -\frac{1}{e}$, the equation has two solutions as follows

$$x_f = \sqrt[p]{\frac{2-p}{pq}W_{-1}(X(r,p,q))} \text{ and } x_s = \sqrt[p]{\frac{2-p}{pq}W_0(X(r,p,q))}.$$

In the second case, if $X(r,p,q) = -\frac{1}{e}$, the equation has one solution, as follows

$$x_f = \sqrt[p]{\frac{2-p}{pq}W_{-1}\left(X(r,p,q)\right)}.$$

In the third case, if $X(r,p,q) < -\frac{1}{e}$, the equation has no solutions.

$\square$

REMARK 2.2. When $p > 2$, the comparison between $X(r,p,q)$ and r is equivalent to the comparison between r and $\phi(p,q)$. A graphical analysis of the function $f_1(x) = rx^{p-2}e^{-qx^p}$ reveals that the Weibull function has a single positive fixed point when $r = \phi(p,q)$, two positive fixed points when $r > \phi(p,q)$, and no positive fixed points when $r < \phi(p,q)$, where

$$\phi(p,q) = \sqrt[p]{\left(\frac{p-2}{pq}\right)^{2-p} e^{p-2}}.$$

Indeed, the function f_1 is differentiable and has only one critical point,

$$c = \sqrt[p]{\frac{p-2}{pq}}.$$

It is strictly increasing on $[0, c[$ and strictly decreasing on $]c, +\infty]$. Moreover, when x tends to 0 or $+\infty$, f_1 tends to 0. Thus, the critical point c represents the maximum of f_1.

The condition $f_1(c) < 1$ is equivalent to $r < \phi(p,q)$ and implies that f_1 can never have the value 1. Therefore, $x = 0$ is the only fixed point of f.

If $f_1(c) = 1$, then f_1 passes through the value 1 only once, so the Weibull function has a single positive fixed point.

For $r > \phi(p,q)$, the condition $f_1(c) > 1$ implies that f_1 passes through the value 1 twice, so the Weibull function has two positive fixed points.

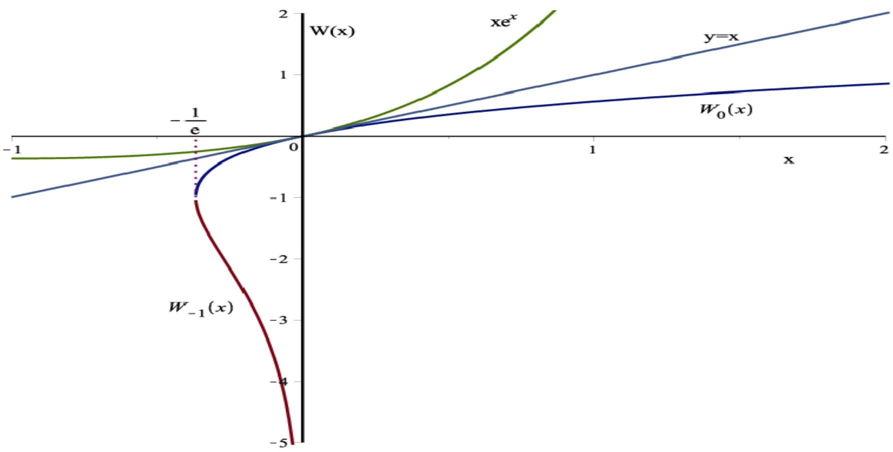

Fig. 2.1 – The Lambert W function graph is composed of two branches: the principal branch W_0 (colored blue) and the secondary branch W_1 (colored red). The curve is obtained through the symmetry of the graph of the function xe^x (colored green) with respect to the $y = x$ axis.

3. STABILITY OF THE FIXED POINT

We recall that if a fixed point x_p satisfies: $|f'(x_p)| < 1$, then it is asymptotically stable, and if it satisfies $|f'(x_p)| > 1$, then it is unstable.

PROPOSITION 3.1. *Assume that* $p \in]1,2[$. *Then the Weibull function has one positive fixed point:*

$$x_f = \sqrt[p]{\frac{2-p}{pq} W_0\left(X(r,p,q)\right)},$$

which is globally asymptotically stable if only if $r < r^*$, *with*

$$r^* = e\sqrt[p]{q^{p-2}}.$$

Proof. Let $p \in]1,2[$ from Proposition 2.1, which tells us that f has a unique positive fixed point $x_f = \sqrt[p]{\frac{2-p}{pq} W_0\left(X(r,p,q)\right)}$. Recall that the fixed point equation at a positive fixed point x_f satisfies the following condition:

$$rx^{p-2}e^{-qx^p} = 1 \text{ or equivalently } rx^{p-2} = e^{qx^p}. \tag{3}$$

On the other hand, consider the expression of the first derivative of the Weibull function

$$f'(x) = rx^{p-2}e^{-qx^p}\left(p - 1 - pqx^p\right). \tag{4}$$

From (3) and (4) it follows that $f'(x) = p - 1 - pqx^p$, and thus $f''(x) = -p^2qx^{p-1} < 0$, for any $x > 0$. By the monotonicity of f', we find that $f'(x_f) < f'(c) = 0$. Clearly, we have $f'(x_f) < 1$. To show that x_f is stable, it is sufficient to solve the following system:

$$\begin{cases} f(x_f) = x_f, \\ f'(x_f) > -1. \end{cases}$$

This is equivalent to

$$\begin{cases} r = x_f^{2-p}e^{qx^p}, \\ x_f < \sqrt[p]{\dfrac{1}{q}}. \end{cases} \tag{5}$$

As the function $\Psi(x) = x^{2-p}e^{qx^p}$ is increasing for $p \in]1,2[$, then

$$\Psi(x_f) < \Psi\left(\sqrt[p]{\frac{1}{q}}\right), \text{ i.e., } r < \Psi\left(\sqrt[p]{\frac{1}{q}}\right) = r^* \text{ (see Figure 3.2).}$$

To prove that x_f is attractive, we use Theorem 3.2, and we find that the equation $f^2(x) = x$ has no solution different than x_f. We define $g(x) = f^2(x)$, and we proceed to a graphical analysis of this function. $\square$

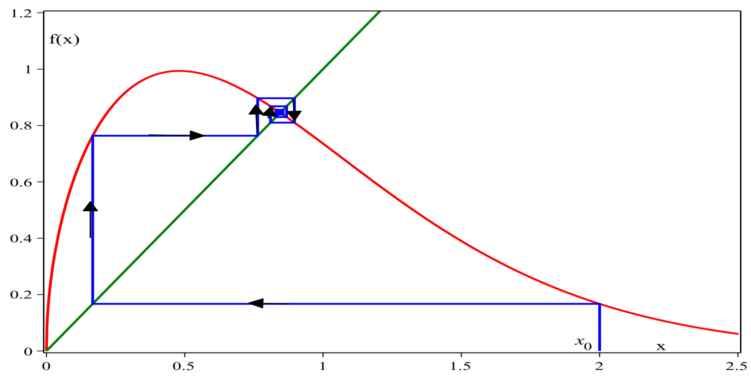

Fig. 3.2 – The graph of f when $p = 3, q = 0.5$, and $r = 1.58$ shows that for any initial point in $\mathbb{R}_+$, one approaches the positive fixed point, which illustrates the result obtained in Proposition 3.1

THEOREM 3.2 ([2, Theorem 2.6, page 48]). *Assume that $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous, has a unique fixed point x_p, and is bounded on the interval $]0, x_p[$. Furthermore, suppose that there exist x_1, x_2, with $0 < x_1 < x_p < x_2$, such that $f(x_1) > x_1$ and $f(x_2) < x_2$. Then, x_p is a global attractor of f in $\mathbb{R}_+$ if and only if there is no fixed point of f^2 distinct from x_p.*

PROPOSITION 3.3. *Let $p > 2$, then the number and stability of fixed points of f depend on the values of $X(r, p, q)$. We have precisely:*

(i) *If $X(r, p, q) < -\frac{1}{e}$, then the Weibull function has $x = 0$, a single fixed point that represents a global attractor, i.e.,*

$$\lim_{n\to+\infty} f^n(x) = 0, \text{ for all } x \in \mathbb{R}_+.$$

(ii) *If $X(r, p, q) = -\frac{1}{e}$, then the Weibull function has exactly two fixed points, the trivial fixed point $x = 0$ and*

$$x_f = \sqrt[p]{\frac{2-p}{pq} W_{-1}(X(r, p, q))},$$

which is 0, is stable with a basin of attraction $[0, x_f[$, and it is verified that

$$\lim_{n\to+\infty} f^n(x) = 0, \text{ for all } x \in \mathbb{R}_+ \backslash [x_f, x_f^*],$$

and

$$\lim_{n\to+\infty} f^n(x) = x_f, \text{ for all } x \in [x_f, x_f^*].$$

(iii) *If $X(r, p, q) < -\frac{1}{e}$, then the Weibull function has three fixed points: $x = 0$, x_s, which are asymptotically stable, and x_f is unstable. Moreover, the basin of attraction of x_s is $]x_f, +\infty)[$, and the basin of attraction of 0 is $]0, x_f[$.*

Proof. Let $p > 2$ and $X(r,p,q) < -\frac{1}{e}$. From Proposition 2.1, it follows that the Weibull function f has a single fixed point, $x = 0$. According to (2), the trivial fixed point is stable. Furthermore, we have $f(x) < x$ for all $x \in \mathbb{R}_+$ (see Figure 3.3). Thus, the sequence $f^n(x)$ is decreasing and bounded below by 0. As f is continuous, $f^n(x)$ converges to the only fixed point $x = 0$. Therefore, we conclude that $\lim_{n\to+\infty} f^n(x) = 0$ for all $x \in \mathbb{R}_+$.

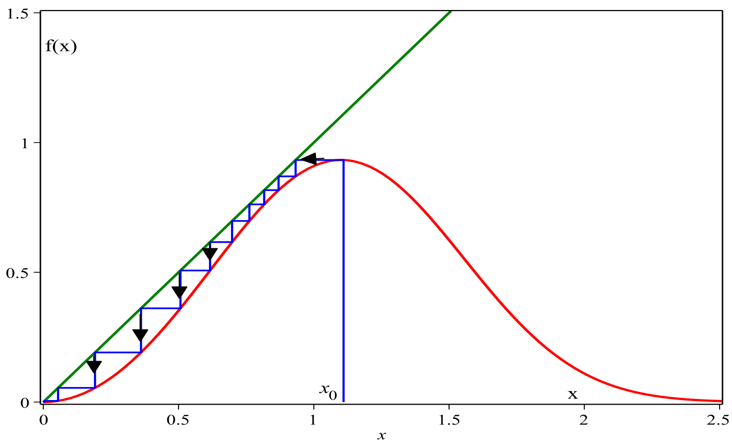

Fig. 3.3 – The graph of f when $p = 3$, $q = 0.5$, and $r = 1.58$ corresponds to f with one fixed point, all orbits converging to zero.

If $p > 2$ and $X(r,p,q) < -\frac{1}{e}$, by Proposition 2.1, f has two fixed points $x = 0$ and x_f. From (2), $x = 0$ is stable. Moreover, from the unimodality and monotonicity of the function f, we conclude the following inequalities:

$$0 < x_f < c < x_f^*, \quad f'(x) > 0, \text{ for all } x \in [0, x_f[$$

and

$$f'(x) < 0 \text{ for all } x \in [x_f^*, +\infty[,$$

(see Figure 3.4), i.e., $f(x) > x_f$, for all $x \in [x_f, x_f^*]$. Hence, $f^n(x)$ is a decreasing sequence that converges to the attractor $x = 0$ for any $x \in [0, +\infty[\backslash[x_f, x_f^*]$. Consequently, $\lim\limits_{n\to+\infty} f^n(x) = 0$ for all $x \in \mathbb{R}_+\backslash[x_f, x_f^*]$.

Let $x \in [x_f, x_f^*]$. From the monotonicity and unimodality of f, we have:

$$x_f < c < x_f^*, \quad f'(x) > 0, \text{ for all } x \in [x_f, c[$$

and

$$f'(x) < 0, \text{ for all } x \in [c, x_f^*[.$$

In addition,

$$x_f < f(x) < f(c), \text{ for all } x \in [x_f, x_f^*].$$

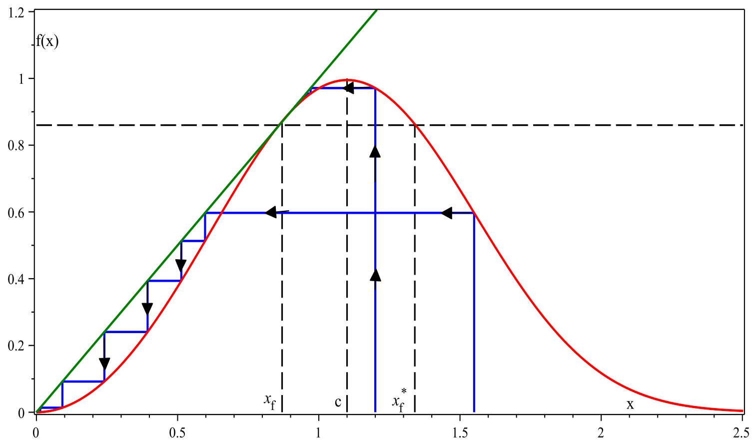

Fig. 3.4 – The graph of f when $p = 3$, $q = 0.5$, and $r = 1.6$ corresponds to f with a fixed point; all orbits in $\mathbb{R}_+ \setminus [x_f, x_f^*]$ converge to the fixed point zero, and all orbits in the interval $[x_f, x_f^*]$ converge to the fixed point x_f.

Consequently, x_f is the only fixed point in the interval $[x_f, x_f^*]$. As $f(x) < x$, for all $x \in [x_f, x_f^*]$, then we conclude that $f^n(x)$ is a monotonically decreasing sequence and is bounded below by x_f. Continuity of the function f implies that the sequence $f^n(x)$ converges to the only fixed point x_f. Thus, we conclude for x_f that

$$\lim_{n\to+\infty} f^n(x) = x_f, \text{ for all } x \in [x_f, x_f^*].$$

We recall that the equation at the non-trivial fixed point $f(x) = x$ corresponds to:

$$rx^{p-1}e^{-qx^p} = x.$$

For $x > 0$, the equation of the fixed point becomes $rx^{p-2}e^{-qx^p} = 1$. Let $h(x) = rx^{p-2}$ and $g(x) = e^{-qx^p}$. The equation has one positive fixed point if and only if the fixed point satisfies the following systems:

$$\begin{cases} g(x) = h(x), \\ g'(x) = h'(x). \end{cases}$$

This is equivalent to

$$\begin{cases} rx^{p-2} = e^{qx^p}, \\ r(p-2)x^{p-3} = pqx^{p-1}e^{qx^p}. \end{cases} \tag{6}$$

Solving the system (6) gives

$$x^* = \sqrt[p]{\frac{p-2}{pq}} \text{ and } r_1 = \sqrt[p]{\left(\frac{p-2}{pq}\right)^{2-p}} e^{p-2}.$$

Thus, we conclude the following inequality

$$x_f < x^* < x_s.$$

From (4) we get

$$f'(x_f) = p - 1 - pqx_f^p > p - 1 - pqx^{*p} = 1 > p - 1 - pqx_s^p = f'(x_s).$$

Hence, the first fixed point x_f is unstable, and the second fixed point x_s is stable only if $f(x_s) > -1$ (see Figure 3.5). According to (5), we have

$$r < r^* = e\sqrt[p]{q^{p-2}}.$$

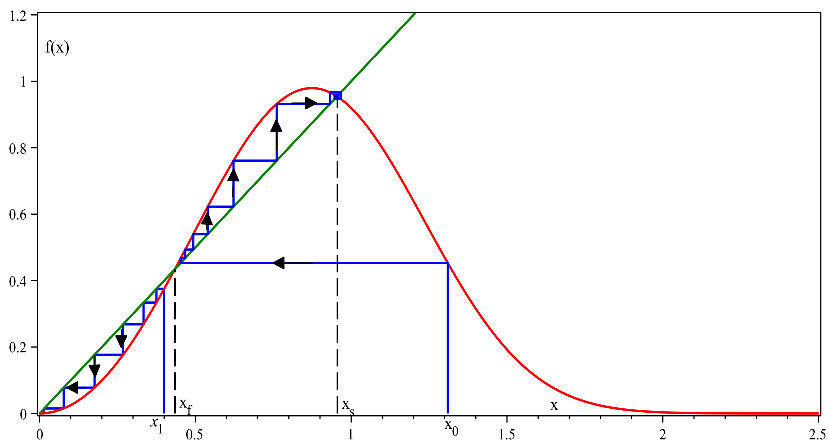

Fig. 3.5 – The graph of f when $p = 3, q = 0.5$, and $r = 3.3$ illustrates the result obtained in Proposition 3.3 when f admits three fixed points. All orbits in the interval $[0, x_f[$ approach zero, and all orbits in the interval $]x_f, +\infty[$ approach the second fixed point x_s.

□

Corollary 3.4. *If $p = 1$, then the Weibull function has a single positive fixed point given by $W(qr)$, which is stable for $r, q \in \Lambda_{r,q}$ with*

$$\Lambda_{r,q} = \{r, q \in \mathbb{R}_+ : W(qr) > \ln(r) + \ln(q)\}.$$

When $p = 2$, the existence and stability of the fixed points of the Weibull function depend on the parameter r:

(i) *If $0 < r < 1$, then the trivial fixed point $x = 0$ is stable, while if $r > 1$, it is unstable.*

(ii) *For $1 < r < e$, the second fixed point $x = \sqrt{\frac{\log(r)}{q}}$, which corresponds to the case $r \geq 1$, is stable, and if $r > e$, it is unstable.*

Proof. Let $p = 1$ and consider the fixed point equation $f(x) = x$, which means that $re^{-qx} = x$, which is equivalent to $qxe^{qx} = qr$. So $x = W(qr)$. Let's show the stability of the fixed point. We know that the derivative of f at the fixed point is given by

$$f'(W(qr)) = -rqe^{-W(qr)}.$$

It is clear that $f(-W(qr)) < 1$.

To demonstrate the stability of the fixed point, all we need to do is check that $f'(-W(qr)) > -1$. To do this, assume that $-rqe^{-W(qr)} > -1$. This implies that

$$W(qr) > \ln(r) + \ln(q), \text{ (see Figure 3.6)}.$$

For $p = 2$, we have

$$f(x) = x \text{ is equivalent to } \begin{cases} x = 0 \\ \text{or} \\ x = \sqrt{\frac{\log(r)}{q}}, \text{ for } r \geq 1 \end{cases}$$

To prove the stability of the fixed point, it suffices to study the first derivative of f at the fixed points.

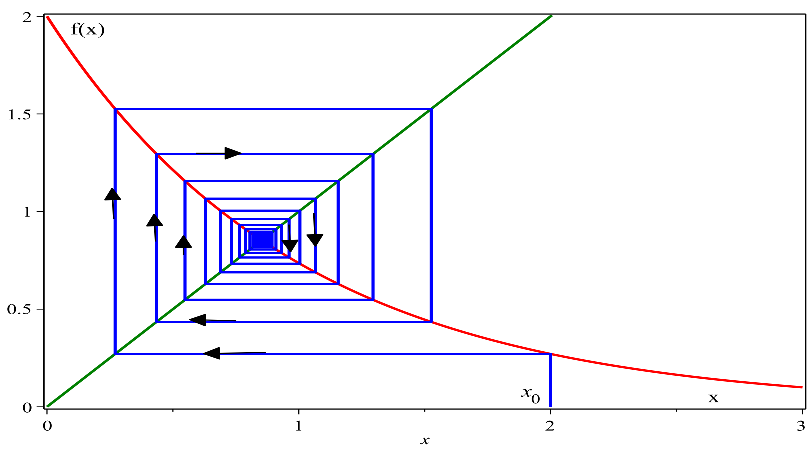

Fig. 3.6 – The graph of f when $p = 1, q = 1$, and $r = 2$ shows that the one fixed point is asymptotically stable.

□

4. ALLEE FUNCTION

In this section, we give the conditions for the Weibull function to belong to the class of Allee functions defined in [8,9]. We first define the Allee functions as follows:

DEFINITION 4.1. A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an Allee function if it has three fixed points: $x = 0$, x_1, and x_2, where $x = 0$ and x_2 are asymptotically stable but x_1 is unstable and satisfies the following systems:

(i) $0 < x_1 < c < x_2 < +\infty$, where c is the only critical point of f.
(ii) For all $x \in [0, x_1[\cup]x_2, +\infty[$ we have $f(x) < x$.
(iii) For all $x \in]x_1, x_2[$ we have $f(x) > x$.

REMARK 4.2. Taking into account Proposition 2.1 and Remark 2.2, we can affirm that the Weibull function is an Allee function when $p > 2$ and $\phi(p, q) < r$. Then, we define

$$\Gamma_0 = \{r, p, q, \in \mathbb{R}_+ : p > 2 \text{ and } \phi(p, q) < r < r^*\},$$

the set on which f defines an Allee function.

DEFINITION 4.3. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the Weibull function in the Γ_0 parameter space. The Allee effect region, denoted by R_{AE}, is defined as follows:

$$R_{AE} = \left\{(r, p, q) \in \Gamma_0 : f^2(c) < x_f\right\},$$

where x_f is the first fixed point and $c = \sqrt[p]{\frac{p-1}{pq}}$ is the critical point of f.

PROPOSITION 4.4. *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the Weibull function in the parameter space Γ_0. Consider the values:*

$$a = \max\{f^{-1}(x_f^*)\}, \quad b = \min\{f^{-1}(x_f^*)\},$$

such that $I =]a, b[\subset]x_f^*, x_f[$. If $(r, p, q) \in R_{AE}$, then $\lim_{n\to+\infty} f^n(x) = 0$ for all $x \in I$.

Proof. Let $(r, p, q) \in R_{AE}$, then $f^2(c) < x_f$; note that this last inequality is equivalent to $f(c) > x_f^*$, where $x_f^* = \max\{f^{-1}(x_f)\}$. Using the unimodality of f, we deduce the following inequality:

$$x_f < a < c < b < x_f^*, \text{ (see Figure 4.7)}.$$

In addition, from the monotonicity of f, we have $f'(x) > 0$ for all $x \in [0, c[$ and $f'(x) < 0$ for all $x \in]c, +\infty[$. On the other hand, considering condition (c) of Definition 4.1, it follows that

$$f(x) > x \text{ for all } x \in]a, b[.$$

Moreover, we have $x_f^* < f(x)$. Using the monotonicity of f, we find that:

$$0 < f^2(x) < x_f \text{ for all } x \in]a, b[.$$

In this region, ($]0, x_f[$), $f^n(x)$ is a monotonous descending sequence that is delimited below by 0. By the continuity of f, we deduce that the sequence $f^n(x)$ converges to $\inf_{x\in\mathbb{R}+}\{f^n(x)\} = 0$. Therefore,

$$\lim_{x\to+\infty} f^n(x) = 0 \text{ for all } x \in]a, b[\text{ (see Fig.4.7).}$$

$\square$

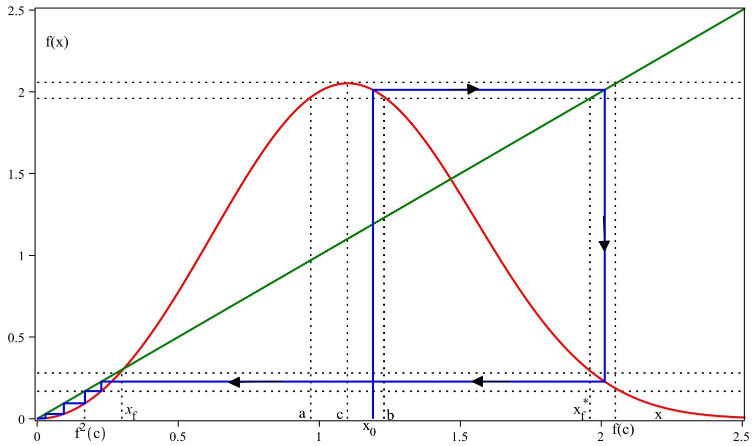

Fig. 4.7 – The graph of f when $f^2(c) < x_f$ ($p = 3, q = 0.5, r = 3.3$), shows that for any initial x_0 point located in the interval $]a, b[$, its orbit approaches the zero fixed point.

5. CONCLUSION

In this paper, the dynamics of a three-parameter Weibull function are studied. The fixed points are obtained in terms of the Lambert function for different parameter values. Stability analysis of the fixed points has been presented. The results are illustrated by numerical examples. Finally, we have defined a parameter space for the Weibull function to present an Allee function.

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