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THE ABSOLUTE FRATTINI AUTOMORPHISMS

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Abstract. Let G be a finite non-abelian p-group, where p is a prime number, and $\mathrm{Aut}(G)$ be the group of all automorphisms of G. An automorphism α of G is an absolute central automorphism if $x^{-1}\alpha(x) \in L(G)$, where $L(G)$ is the absolute center of G. In addition, α is an absolute Frattini automorphism if $x^{-1}\alpha(x) \in \Phi(L(G))$, where $\Phi(L(G))$ is the Frattini subgroup of the absolute center of G, and let $LF(G)$ denote the group of all such automorphisms of G. Also, we denote by $C_{LF(G)}(Z(G))$ and $C_{LA(G)}(Z(G))$, respectively, the group of all absolute Frattini automorphisms and the group of all absolute central automorphisms of G fixing elementwise the center $Z(G)$ of G. Here, we give necessary and sufficient conditions on a finite non-abelian p-group G of class two such that $C_{LF(G)}(Z(G)) = C_{LA(G)}(Z(G))$ holds. Moreover, we investigate the conditions under which $LF(G)$ is a torsion-free abelian group.

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1. INTRODUCTION

In this paper, p denotes a prime number. Let G be a finite group. We assume that $G^{'}$, $Z(G)$, $\Phi(G)$, exp(G), $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$ and $\exp(G)$, are the commutator subgroup, the centre, Frattini subgroup, the exponent, the group of all automorphisms and the inner automorphisms of G, respectively. For any group G, Hegarty [4] defined the subgroups $L(G)$ and $Z(G)$ of G such that

$$L(G) = \{g \in G \mid \alpha(g) = g \ \forall \alpha \in \mathrm{Aut}(G)\},$$
$$Z(G) = \{g \in G \mid \alpha(g) = g \ \forall \alpha \in \mathrm{Inn}(G)\}$$

and called $L(G)$ the absolute center of G. Note that $L(G) \leq Z(G)$.

An automorphism α of G is called a central automorphism, if $[g, \alpha] = g^{-1}\alpha(g) \in Z(G)$, for each $g \in G$. The central automorphisms fix the commutator subgroup G' of G, elementwise and form a normal subgroup $\mathrm{Aut}^Z(G)$ of $\mathrm{Aut}(G)$, where $Z = Z(G)$. In a similar way, Hegarty [3] defined an absolute central automorphism of G as follows: an automorphism α of G is

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called an absolute central automorphism if it induces the identity automorphism on $G/L(G)$, or, equivalently, $[g, \alpha] = g^{-1}\alpha(g) \in L(G)$, for each $g \in G$. Let $LA(G) = \mathrm{Aut}^{L(G)}(G)$ denotes the group of all absolute central automorphisms of G. Clearly, $LA(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ contained in $\mathrm{Aut}^{Z}(G)$.

In addition, Moghaddam and Safa [5] have investigated some properties of absolute central automorphisms. Assume that $LF(G) = \mathrm{Aut}^{\Phi(L(G))}(G)$ denotes the group of all absolute Frattini automorphisms, α of G, such that $[g, \alpha] = g^{-1}\alpha(g) \in \Phi(L(G))$, for each $g \in G$. One can easily check that $LF(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and contained in $LA(G)$. Now, let $C_{LF(G)}(Z(G))$ and $C_{LA(G)}(Z(G))$, respectively, to be the group of all absolute Frattini automorphisms and the group of all absolute central automorphisms of G, fixing the center $Z(G)$ of G, elementwise. Shabani Attar [9] gave necessary and sufficient conditions for any finite non-abelian p-group such that $\mathrm{Aut}^{Z}(G) = C_{\mathrm{Aut}^{Z}(G)}(Z(G))$, where $C_{\mathrm{Aut}^{Z}(G)}(Z(G))$ is the group of all central automorphisms of G fixing $Z(G)$, elementwise. On the other hand, Rai [6] obtained necessary and sufficient conditions on a finite p-group G under which $\mathrm{Aut}^{Z}(G) = C_{\mathrm{IA}(G)}(Z(G))$, where $\mathrm{IA}(G)$ and $C_{\mathrm{IA}(G)}(Z(G))$, respectively, the group of all derived automorphisms and the group of all derived automorphisms of G fixing $Z(G)$, elementwise. In another study, Singh and Gumber [10], gave necessary and sufficient conditions on a finite non-abelian p-group G, such that $C_{LA(G)}(Z(G)) = \mathrm{Aut}^{Z}(G)$.

In this article we present neccessary and sufficient conditions on a finite non-abelian p-group G of class 2, in which $C_{LF(G)}(Z(G)) = C_{LA(G)}(Z(G))$. We also find the conditions on the group G, so that $LF(G)$ is a torsion-free abelian group.

Throughout this paper, we utilize the following well-known lemmas and theorem:

Lemma 1.1 ([1, Lemma 3]). *Let G be any group, and let Y be a central subgroup of G contained in a normal subgroup X of G. Then the group of all automorphisms of G that induce the identity on both X and G/Y is isomorphic to* $\mathrm{Hom}(G/X, Y)$.

Lemma 1.2 ([2, Lemma E]). *Suppose H is an abelian p-group of exponent p^c, and K is cyclic group of order divisible by p^c. Then $Hom(H, K)$ is isomorphic to H.*

Lemma 1.3 ([4, Lemma 3.1]). *Let G be a finite non-abelian p-group. Then $L(G) \leq \Phi(G)$.*

Remark 1.4 ([7]). Let G be a finite group, then $\Phi(G) = G$ if and only if G is the trivial group.

Theorem 1.5 ([8, Theorem 2.6]). *Let G be a purely non-abelian group satisfying maximal and minimal conditions on normal subgroups. Let M be a* *central subgroup of G and let $\mathrm{Aut}^M(G)$ denote the group of all those automorphisms of G which induce the identity on G/M. Then:*

i) *there is a one-one correspondence between $\mathrm{Aut}^M(G)$ and $\mathrm{Hom}(G, M)$, and*

ii) *if M is contained in $L(G)$, then $\mathrm{Aut}^M(G) \cong \mathrm{Hom}(G, M)$.*

Now, we find necessary and sufficient conditions on a finite non-abelian p-group G of class 2, such that $C_{LA(G)}(Z(G)) = C_{LF(G)}(Z(G))$. Let

$$G/Z(G) = C_{p^{a_1}} \times C_{p^{a_2}} \times ... \times C_{p^{a_k}}$$

where $C_{p^{a_i}}$ is a cyclic group of order p^{a_i}, $1 \leq i \leq k$, and $a_1 \geq a_2 \geq ... \geq a_k \geq 1$. Since $\Phi(L(G)) \leq L(G) \leq Z(G)$, so we can write

$$\Phi(L(G)) = C_{p^{b_1}} \times C_{p^{b_2}} \times ... \times C_{p^{b_l}}$$

and let

$$L(G) = C_{p^{c_1}} \times C_{p^{c_2}} \times ... \times C_{p^{c_m}}$$

be the cyclic decompositions of the corresponding abelian group, where $b_i \geq b_{i+1} \geq 1$ and $c_i \geq c_{i+1} \geq 1$. Since $\Phi(L(G))$ is a subgroup of $L(G)$, we have $l \leq m$ and $b_j \leq c_j$, for all j, $1 \leq j \leq l$. Considering the notation above, we prove the main theorem of this paper.

2. MAIN RESULTS

THEOREM 2.1. *Let G be a finite non-abelian p-group of class two. Then $C_{LA(G)}(Z(G)) = C_{LF(G)}Z(G)$, if and only if $L(G)$ is the trivial subgroup of G ($\Phi(L(G)) = L(G)$), or $\Phi(L(G)) < L(G)$, $l = m$, and $a_1 \leq b_s$, where s is the largest integer between 1 and l such that $b_s < c_s$.*

Proof. Suppose that $C_{LA(G)}(Z(G)) = C_{LF(G)}(Z(G))$ and $\Phi(L(G)) \neq L(G)$. So we have $\Phi(L(G)) < L(G)$. We claim that $l = m$, and $a_1 \leq b_s$, where s is the largest integer between 1 and l such that $b_s < c_s$. As $\Phi(L(G)) < L(G) \leq Z(G)$, using Lemma 1.1, we can see that $C_{LA(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), L(G))$ and $C_{FA(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), \Phi(G))$. So,

$$|C_{LA(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), L(G))|$$

and

$$|C_{LF(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), \Phi(L(G)))|.$$

Thus, we have $|\mathrm{Hom}(G/Z(G), \Phi(L(G)))| = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{min\{a_i, b_j\}}$ and

$$|\mathrm{Hom}(G/Z(G), L(G))| = \prod_{1\leq i\leq k, 1\leq j\leq m} p^{min\{a_i, c_j\}}.$$

As, $C_{LF(G)}(Z(G)) = C_{LA(G)}(Z(G))$, hence

$$|\mathrm{Hom}(G/Z(G), \Phi(L(G)))| = |\mathrm{Hom}(G/Z(G), L(G))|.$$

Therefore, $\prod_{1\le i\le k, 1\le j\le l} p^{min\{a_i,b_j\}} = \prod_{1\le i\le k, 1\le j\le m} p^{min\{a_i,c_j\}}$. Since $\Phi(L(G)) < L(G)$, we have $l \le m$, and $b_j \le c_j$, for every j, $1 \le j \le l$, therefore $\min\{a_i, b_j\} \le \min\{a_i, c_j\}$ for all i, $1 \le i \le k$ and for all j, $1 \le j \le l$. If $l < m$, then $|\mathrm{Hom}(G/Z(G), \Phi(L(G)))| < |\mathrm{Hom}(G/Z(G), L(G)|$, which is not true. Thus $l = m$ and $\min\{a_i, b_j\} = \min\{a_i, c_j\}$, for all i, $1 \le i \le k$ and for all j, $1 \le j \le l$. Since $\Phi(L(G)) < L(G)$, there exists some j between 1 and l such that $b_j < c_j$. Let s be the largest integer between 1 and l such that $b_s < c_s$. We show that $a_1 \le b_s$. Suppose for a contradiction that $b_s < a_1$. Thus $b_s = \min\{a_1, b_s\} = \min\{a_1, c_s\}$, which is impossible. Therefore, $a_1 \le b_s$.

Conversely, if $\Phi(L(G)) = L(G)$, then $C_{LA(G)}(Z(G)) = C_{LF(G)}(Z(G))$. Suppose that $\Phi(L(G)) < L(G)$, $l = m$, and $a_1 \le b_s$, where s is the largest integer between 1 and l such that $b_s < c_s$. Now

$$|C_{LF(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), \Phi(L(G)))| = \prod_{1\le i\le k, 1\le j\le l} p^{min\{a_i,b_j\}}$$

and $|C_{LA(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), L(G))| = \prod_{1\le i\le k, 1\le j\le m} p^{min\{a_i,c_j\}}$. Note that $a_i \le b_j \le c_j$, for all $1 \le i \le k$ and $1 \le j \le s$, whence $\min\{a_i, b_j\} = a_i = \min\{a_i, c_j\}$, for all $1 \le i \le k$ and $1 \le j \le s$. On the other hand $c_j = b_j$, for all $j > s$, so we have $\min\{a_i, b_j\} = a_i = \min\{a_i, c_j\}$, for all $1 \le i \le k$ and $s+1 \le j \le m$. Therefore $|C_{LF(G)}(Z(G))| = |C_{LA(G)}(Z(G))|$. Because $C_{LF(G)}(Z(G))$ is a subgroup of $C_{LA(G)}(Z(G))$. We have $C_{LF(G)}(Z(G)) = C_{LA(G)}(Z(G))$. The proof of the theorem is complete. □

The following corollary is the consequence of the above theorem.

Corollary 2.2. *Let G be a finite non-abelian p-group of class two and $\exp(L(G)) = p$. Then $\Phi(L(G)) = 1$ and $C_{LF(G)}(Z(G)) = 1$. Hence*

$$C_{LA(G)}(Z(G)) \ne C_{LF(G)}(Z(G)).$$

Example 2.3. Let $G = M_2(4, 1) = \langle a, b;\ a^{16} = b^2 = 1,\ [a, b] = a^8\rangle$, where $p = 2$, $n = 4$ and $m = 1$. G is a minimal non-abelian finite 2-group, we have $L(G) = Z(G) = \Phi(G) = \mathbb{Z}_8$ and $|G'| = 2$, $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. One can easily check that $\Phi(L(G)) = \mathbb{Z}_4$. Appling Theorem 2.1, we can see that $C_{LA(G)}(Z(G)) = C_{LF(G)}(Z(G)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 2.4. *Let G be a non-abelian finitely generated group, in which $\Phi(L(G))$ is indecomposable and torsion free and $G/\Phi(L(G))$ is torsion free abelian. Suppose that $f \in \mathrm{Hom}(G, \Phi(L(G))$. Then $\Phi(L(G)) \le \mathrm{Ker}(f)$.*

Proof. Since $\Phi(L(G))$ and $G/\Phi(L(G))$ are abelian, so $G' \le \Phi(L(G)) \cap \mathrm{Ker}(f)$ and hence $G/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is abelian. The map $\sigma : x(\Phi(L(G)) \cap \mathrm{Ker}(f)) \longmapsto x\Phi(L(G))$ defines an epimorphism from $G/\Phi(L(G)) \cap \mathrm{Ker}(f)$ onto $G/\Phi(L(G))$. Since $G/\Phi(L(G))$ is a free abelian group, by ([7, Theorem 4.2.4]), there exists a homomorphism $\delta : G/\Phi(L(G) \longrightarrow G/\Phi(L(G)) \cap \mathrm{Ker}(f)$ such that $\sigma \circ \delta$ is an identity on $G/\Phi(L(G))$. As $\mathrm{Im}(\delta)$ is a subgroup of

$G/\Phi(L(G)) \cap \mathrm{Ker}(f)$, there exists a subgroup H of G containing $\Phi(L(G)) \cap \mathrm{Ker}(f)$ such that $\mathrm{Im}(\delta) = H/\Phi(L(G) \cap \mathrm{Ker}(f)$. Because $G/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is abelian, $H/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is a normal subgroup of $G/\Phi(L(G)) \cap \mathrm{Ker}(f)$. So H is a normal subgroup of G.

Since δ is an injective homomorphism from $G/\Phi(L(G))$ to $G/\Phi(L(G)) \cap \mathrm{Ker}(f)$ and its image is $H/\Phi(L(G)) \cap \mathrm{Ker}(f)$. This means, if we pull back H via the inverse of δ, then we get G. But the inverse image is $H\Phi(L(G))$. Hence $G = H\Phi(L(G))$. Also, $\Phi(L(G)) \cap H = \Phi(L(G)) \cap \mathrm{Ker}(f)$ because $\sigma \circ \delta$ is an identity on $G/\Phi(L(G))$. Here $H \neq 1$, otherwise if $H = 1$, then $G = \Phi(L(G))$, and so G is abelian, which is a contradiction as G is non abelian. From the fact that G is finitely generated, it follows that $H/\Phi(L(G)) \cap Ker(f)$, and so $\Phi(L(G))/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is a finitely generated abelian group. Furthermore, $\Phi(L(G))/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is torsion free. Let $t \in \Phi(L(G))$ and $k \in \mathbb{N}$, such that $(t(\Phi(L(G)) \cap \mathrm{Ker}(f)))^k = 1$. Since $\Phi(L(G))$ is torsion free, $f(t) = 1$, and so $t \in \mathrm{Ker}(f)$. Therefore, $t \in \Phi(L(G)) \cap \mathrm{Ker}(f)$. This shows that $\Phi(L(G))/\Phi(L(G)) \cap \mathrm{Ker}(f)$ is free abelian. Hence, by ([7, Theorem 4.2.5]), we have $\Phi(L(G)) = (\Phi(L(G)) \cap \mathrm{Ker}(f)) \times A$, for some $A \leq \Phi(L(G))$. Since $\Phi(L(G))$ is indecomposable, thus $\Phi(L(G)) \cap Ker(f) = 1$ or $A = 1$. If $\Phi(L(G)) \cap \mathrm{Ker}(f) = 1$, then $G' = 1$, and so G is abelian. This is a contradiction. Therefore, we must have $A = 1$, and this means $\Phi(L(G)) = \Phi(L(G)) \cap \mathrm{Ker}(f)$. Thus, $\Phi(L(G))$ is contained in $\mathrm{Ker}(f)$ as desired. □

Theorem 2.5. *Let G be a non-abelian finitely generated group, in which $\Phi(L(G))$ is indecomposable and torsion free and $G/\Phi(L(G))$ is torsion free abelian, then* $\mathrm{Hom}(G, \Phi(L(G)))$ *is a torsion free abelian group.*

Proof. Let$f \in \mathrm{Hom}(G, \Phi(L(G))$. Let $\sigma_f : G/\Phi(L(G)) \longrightarrow \Phi(L(G))$ as $\sigma_f(x) = f(x)$. Now, we show that σ_f a homomorphism from $G/\Phi(L(G))$ to $\Phi(L(G))$. First, we prove σ_f is well defined. Suppose that $x_1, x_2 \in G$ and $x_1\Phi(L(G)) = x_2\Phi(L(G))$, then $x_1x_2^{-1} \in \Phi(L(G))$ implies $f(x_1x_2^{-1}) = 1$ by Lemma 2.4, and this means $f(x_1) = f(x_2)$. Clearly, σ_f is a homomorphism. Thus, $\sigma_f \in \mathrm{Hom}(G/\Phi(L(G)), \Phi(L(G)))$. Now, it is easy to see that the map $\phi : f \longmapsto \sigma_f$ is an isomorphism from $\mathrm{Hom}(G, \Phi(L(G)))$ to Hom $(G/\Phi(L(G)), \Phi(L(G)))$, So, $\mathrm{Hom}(G, \Phi(L(G))) \cong \mathrm{Hom}(G/\Phi(L(G)), \Phi(L(G)))$. Since $G/\Phi(L(G))$ is a free abelian group, there exists $n \in \mathbb{N}$, such that $G/\Phi(L(G)) = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{n-times}$. Therefore,

$$\begin{aligned}
\mathrm{Hom}(G, \Phi(L(G))) &\cong \mathrm{Hom}(G/\Phi(L(G)), \Phi(L(G))) \\
&\cong \mathrm{Hom}(\underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{n-times}, \Phi(L(G))) \\
&\cong \underbrace{\mathrm{Hom}(\mathbb{Z}, \Phi(L(G))) \times \ldots \times \mathrm{Hom}(\mathbb{Z}, \Phi(L(G)))}_{n-times}
\end{aligned}$$

$$\cong \underbrace{\Phi(L(G))) \times \Phi(L(G))) \times ...\Phi(L(G)))}_{n-times},$$

by ([7, Theorem 4.7 and 4.9]). Since $\Phi(L(G))$ is a torsion-free abelian group, $\mathrm{Hom}(G, \Phi(L(G)))$ is a torsion free abelian group. □

Immediate from Theorems 1.5 and 2.5, we get the following corollary.

COROLLARY 2.6. *Let G be a purely non-abelian and finitely generated group satisfying maximal and minimal conditions on normal subgroups, in which $\Phi(L(G))$ is indecomposable and torsion free and $G/\Phi(L(G))$ is torsion free abelian, then $LF(G)$ is a torsion free abelian group.*

REFERENCES

[1] J. L. Alperin, *Groups with finitely many automorphisms*, Pacific J. Math., **12** (1962), 1–5.
[2] M. J. Curran and D. J. McCaughan, *Central automorphisms that are almost inner*, Comm. Algebra, **29** (2001), 2081–2087.
[3] P. V. Hegarty, *The absolute center of a group*, J. Algebra, **169** (1994), 929–935.
[4] H. Meng and X. Guo, *The absolute center of finite groups*, J. Group Theory, **18** (2015), 887–904.
[5] M. R. R. Moghaddam and H. Safa, *Some properties of autocentral automorphisms of a group*, Ric. Mat., **59** (2010), 257–264.
[6] P. K. Rai, *On class-preserving automorphisms of groups*, Proc. Indian Acad. Sci. Sect. A, **124** (2014), 169–173.
[7] D. J. S. Robinson, *Course in the Theory of Groups*, Springer-Verlag, New York, 1980.
[8] S. S. Chahal, D. Gumber and H. Kalra, *On Autocentral Automorphisms of Finite p-Groups*, Results Math., **76** (2021), 1–8.
[9] M. Shabani Attar, *Finite p-groups in which each central automorphism fixes centre elementwis*, Comm. Algebra, **40** (2012), 1096–1102.
[10] M. Singh and D. Gumber, *On the coincidence of the central and absolute central automorphism groups of finite p-groups*, Math. Notes, **107** (2020), 863–866.

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