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ON PRIME BANACH ALGEBRAS WITH CONTINUOUS DERIVATIONS

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Abstract. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ with center $Z(\mathcal{A})$. In this paper, we show that, if a non-injective continuous derivation of $\mathcal{A}$ satisfies some local differential identities, then $\mathcal{A}$ must be commutative. We give several applications, and we provide examples to show that some hypotheses of our theorems are necessary.

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1. INTRODUCTION

Throughout this article $\mathcal{R}$ will be an associative ring with center $Z(\mathcal{R})$ and usually, $\mathcal{R}$ is 2-torsion free, if whenever $2x = 0$ with $x \in \mathcal{R}$, we have necessarily $x = 0$. The ring $\mathcal{R}$ is said to be prime if for any $x, y \in \mathcal{R}$, $x\mathcal{R}y = \{0\}$ implies either $x = 0$ or $y = 0$. The Lie product and Jordan product of $x, y \in \mathcal{R}$ are denoted by $[x, y]$ and $x \circ y$ respectively, where $[x, y] = xy - yx$ and $x \circ y = xy + yx$. By derivation, we mean an additive mapping d on $\mathcal{R}$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$ and is called an inner derivation if $d(x) = [a, x]$ for all $x \in \mathcal{R}$ and for some fixed element $a \in \mathcal{R}$. An additive subgroup L of $\mathcal{R}$ is known to be a Lie ideal of $\mathcal{R}$ if $[l, r] \in L$, for all $l \in L$ and $r \in \mathcal{R}$. An additive subgroup J of $\mathcal{R}$ is a Jordan ideal if $j \circ r \in J$ for all $j \in J$ and $r \in \mathcal{R}$. If S is a nonvoid subset of $\mathcal{R}$, a mapping f on $\mathcal{R}$ is called a centralizing function (resp commuting function) on S if $[f(s), s] \in Z(R)$ (resp $[f(s), s] = 0$) for all $s \in S$.

A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. A famous result showed by Posner in 1957 states that: If a prime ring has a non-zero derivation which is centralizing on the entire ring, then the ring must be commutative. Inspired by this result, analogous results was obtained for automorphisms by J. Mayne [7]. This work has also been extended in various directions, for example Shakir Ali, Basudeb Dhara, Brahim Fahid and Mohd Arif Raza [5] showed that if $\mathcal{R}$ is a prime ring of characteristic not two and m, n are fixed positive integers and ξ is a

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automorphism of $\mathcal{R}$ satisfying $\xi([x^m, y^n]) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. For more results see [5,6].

In Banach algebras, Yood [8] proved that if a semiprime Banach algebra $\mathcal{A}$ having two nonvoid open subsets $\mathcal{G}_1$ and $\mathcal{G}_2$ verify for all $(x, y) \in \mathcal{G}_1 \times \mathcal{G}_2$ there is $(n, m) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $[x^n, y^m] = 0$, then $\mathcal{A}$ is commutative.

Our results on commutativity for Banach algebras involving derivations take a different direction.

Shakir Ali and Abdul Nadim Khan [12] proved that if $\mathcal{A}$ is a unitary prime Banach algebra and $\mathcal{A}$ has a nonzero continuous linear derivation $d : \mathcal{A} \to \mathcal{A}$ such that either $d((xy)^m) - x^m y^m \in Z(\mathcal{A})$ or $d((xy)^m) - y^m x^m \in Z(\mathcal{A})$ for an integer $m = m(x, y)$ and sufficiently many $x, y \in \mathcal{A}$ then $\mathcal{A}$ is commutative (for more examples, see [1,5,8]).

Motivated by these results, in the present article we have shown some results with a similar conclusion, but with other identities. In particular, we have proved that if a real or complex Banach algebra $\mathcal{A}$ has a non-injective continuous derivation d and there are two nonvoid open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfying: for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there is $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $d(x^p.y^q) + x^p \circ y^q \in Z(\mathcal{A})$, then $\mathcal{A}$ is commutative. In this context, other similar results have been found.

We started this paper by presenting some results, from which we have been inspired. Next, we introduced the proof of main results, by using Baire's theorem and some properties of functional analysis. To each result, we gave an application and we mentioned some immediate results of it. We conclude this work, with a set of examples confirming that some hypothesis in the main results are necessary.

Now, we state the results which present the motivation of this article.

Lemma 1.1 ([8, Theorem 2]). *Let $\mathcal{R}$ be a prime ring of characteristic not two and J a nonzero Jordan ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a nonzero derivation d such that $d([x, y]) \in Z(\mathcal{R})$ for all $x, y \in J$ then $\mathcal{R}$ is commutative.*

Lemma 1.2 ([2, Lemma 6]). *Let $\mathcal{R}$ be a prime ring with char $\neq 2$, d be a nonzero derivation of $\mathcal{R}$ and L a Lie ideal of $\mathcal{R}$. If $d(L) \subset Z(\mathcal{R})$, then $L \subset Z(\mathcal{R})$.*

Lemma 1.3 ([8, Lemma 4]). *Let $\mathcal{R}$ be a prime ring. For a nonzero element $a \in Z(\mathcal{R})$, if $ab \in Z(\mathcal{R})$ then $b \in Z(\mathcal{R})$.*

Fact 1. Any 2-torsion free nontrivial ring is of characteristic different from two but the converse is false as shown in the following example.

Example 1.4. In $\mathbb{F}_8 = \mathbb{Z}/8\mathbb{Z}$ we have $2.\overline{4} = \overline{0}$ and $\overline{4} \neq \overline{0}$, therefore $\mathbb{F}_8$ is not 2-torsion free but of charcteristic different from two (char($\mathbb{F}_8$)=8$\neq$ 2).

Fact 2. Any Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ is 2-torsion free.

Lemma 1.5. *Let $\mathcal{A}$ be a real or complex Banach algebra and let $P(t) = \sum_{k=0}^{n} A_k t^k$ is a polynomial of real variable t and coefficients A_k in $\mathcal{A}$. If*

$P(t) \in Z(\mathcal{A})$ for all t in an interior interval containing zero and $A_0 \in Z(\mathcal{A})$, then $A_k \in Z(\mathcal{A})$ for all $0 \leq k \leq n$.

Proof. For $t \neq 0$, we can write

$$P(t) = A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n.$$

While $A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n \in Z(\mathcal{A})$ and $A_0 \in Z(\mathcal{A})$, we conclude that

$$A_1 t + A_2 t^2 + \ldots + A_n t^n \in Z(\mathcal{A}).$$

We can simplify by t (*because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$*), we obtain

$$Q(t) = A_1 + A_2 t + \ldots + A_n t^{n-1} \in Z(\mathcal{A}),$$

and take the limit to zero (Q is continuous at 0 and $Z(\mathcal{A})$ is closed), we get $A_1 \in Z(\mathcal{A})$, we conclude that

$$A_2 t^2 + \ldots + A_n t^n \in Z(\mathcal{A}).$$

We can further simplify by t^2 and take limit to zero, we get $A_2 \in Z(\mathcal{A})$. And so on we get $A_k \in Z(\mathcal{A})$ for all $0 \leq k \leq n$. □

2. MAIN RESULTS

In the following results, $\mathcal{A}$ is a real or complex Banach algebra with center $Z(\mathcal{A})$ and d is a non-injective derivation on $\mathcal{A}$.

THEOREM 2.1. *Let $\mathcal{A}$ be a prime Banach algebra, $\mathcal{H}_1$ and $\mathcal{H}_2$ nonvoid open subsets of $\mathcal{A}$ and d be a continuous non-zero derivation of $\mathcal{A}$. If*

$$d(x^p.y^q) + x^p \circ y^q \in Z(\mathcal{A})$$

for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ where p, q are not necessarily fixed but they depend on the pair of elements x and y, then $\mathcal{A}$ is commutative.

Proof. For all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ we define the following sets :

$$O_{p,q} = \{(x, y) \in \mathcal{A}^2 \mid d(x^p.y^q) + x^p \circ y^q \notin Z(\mathcal{A})\} \text{ and}$$

$$F_{p,q} = \{(x, y) \in \mathcal{A}^2 \mid d(x^p.y^q) + x^p \circ y^q \in Z(\mathcal{A})\}.$$

We observe that $(\cap O_{p,q}) \cap (\mathcal{H}_1 \times \mathcal{H}_2) = \emptyset$, indeed: If they exist $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ such that $(x, y) \in O_{p,q}$ for all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, then $d(x^p.y^q) + x^p \circ y^q \notin Z(\mathcal{A})$ for all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, which is absurd with the hypothesis of the theorem.

Now we claim that each $O_{p,q}$ is open in $\mathcal{A} \times \mathcal{A}$. That is, we have to show that $F_{p,q}$ the complement of $O_{p,q}$ is closed. For this, we consider a sequence $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{p,q}$ converge to $(x, y) \in \mathcal{A} \times \mathcal{A}$. Since $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{p,q}$, so

$$d((x_k)^p.(y_k)^q) + (x_k)^p \circ (y_k)^q \in Z(\mathcal{A}) \text{ for all } k \in \mathbb{N}.$$

Since d is a continuous derivation, the sequence $(d((x_k)^p.(y_k)^q) + (x_k)^p \circ (y_k)^q)_{k\in\mathbb{N}}$ converges to $d(x^p.y^q) + x^p \circ y^q$, knowing that $Z(\mathcal{A})$ is closed, we obtain $d(x^p.y^q) + x^p \circ y^q \in Z(\mathcal{A})$. Therefore $(x, y) \in F_{p,q}$ and $F_{p,q}$ is closed, then $O_{p,q}$ is open.

If every $O_{p,q}$ is dense, we know that their intersection is also dense by the Baire category theorem, which contradicts with $(\cap O_{p,q}) \cap (\mathcal{H}_1 \times \mathcal{H}_2) = \emptyset$. Hence, there is $(n, m) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $O_{n,m}$ is not a dense set and there exists a nonvoid open subset $\mathcal{O} \times \mathcal{O}'$ in $F_{n,m}$ such that:

$$d(x^n y^m) + x^n \circ y^m \in Z(\mathcal{X}) \text{ for all } x \in \mathcal{O}, y \in \mathcal{O}',$$

Fix $y \in \mathcal{O}'$. Let $x \in \mathcal{O}$ and $z \in \mathcal{A}$, then $x + tz \in \mathcal{O}$ for all sufficiently small real t. Therefore $P(t) = d((x + tz)^n y^m) + (x + tz)^n \circ y^m \in Z(\mathcal{A})$. We can write

$$P(t) = A_{n,0}(x, z, y) + A_{n-1,1}(x, z, y)t + A_{n-2,2}(x, z, y)t^2 + \ldots + A_{0,n}(x, z, y)t^n.$$

The first term in this polynomial is $A_{n,0}(x, z, y) = d(x^n y^m) + x^n \circ y^m$ who belongs to $Z(\mathcal{A})$, by Lemma 1.5, we conclude that $A_{0,n}(x, z, y) = d(z^n y^m) + z^n \circ y^m \in Z(\mathcal{A})$. Consequently, given $y \in \mathcal{O}'$ we have $d(x^n y^m) + x^n \circ y^m \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. We reverse the roles of $\mathcal{O}$ and $\mathcal{O}'$ in the above settings with x is fixed in $\mathcal{A}$, we find that $d(x^n y^m) + x^n \circ y^m \in Z(\mathcal{A})$ for all $(x, y) \in \mathcal{A}^2$.

Replacing x by x^m and y by y^n, we find that

$$d(x^{nm}.y^{mn}) + x^{nm} \circ y^{mn} \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

By permuting x and y, we obtain

$$d(y^{nm}.x^{mn}) + y^{nm} \circ x^{mn} \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

Then

$$(d(x^{nm}.y^{mn}) + x^{nm} \circ y^{mn}) - (d(y^{nm}.x^{mn}) + y^{nm} \circ x^{mn}) \in Z(\mathcal{A}), \forall x, y \in \mathcal{A}.$$

Since

$$y^{nm} \circ x^{mn} = x^{nm} \circ y^{mn} \text{ for all } x, y \in \mathcal{A},$$

we obtain

$$d([x^{nm}, y^{mn}]) \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

Let $x \in \mathcal{A}$. If we put $a = x^{nm}$, we obtain $P(t) = d([a, (y + ta)^{nm}]) \in Z(\mathcal{A})$ for all $y \in \mathcal{A}$ and for all real $t \in \mathbb{R}$. It can be written as

$$P(t) = d([a, (y + ta)^{mn}]) = \sum_{k=0}^{mn} d([a, A_{nm-k,k}])t^k,$$

where $A_{nm-k,k}$ denotes the sum of all terms in which y appears exactly $nm - k$ times and a appears exactly k times. The first term in this polynomial is $d([a, A_{nm,0}]) = d([x^{nm}, y^{nm}]) \in Z(\mathcal{A})$, by Lemma 1.5, we conclude that

$d([a, A_{nm-k,k}(x,y)]) \in Z(\mathcal{A})$ for all $0 \leq k \leq nm$. The coefficient of t in this polynomial is $d([a, A_{nm-1,1}])$ and $A_{nm-1,1} = \sum_{k=0}^{nm-1} a^{nm-1-k} y a^k$. Since

$$\begin{aligned}
\sum_{k=0}^{nm-1} [a, a^{nm-1-k} y a^k] &= [a, a^{nm-1} y] + [a, a^{nm-2} y a] \\
&+ [a, a^{nm-3} y a^2] + \ldots + [a, y a^{nm-1}] \\
&= (a a^{nm-1} y - a^{nm-1} y a) + (a a^{nm-2} y a - a^{nm-2} y a a) \\
&+ \ldots + (a y a^{nm-1} - y a^{nm-1} a) \\
&= (a^{nm} y - a^{nm-1} y.a) + (a^{nm-1} y a - a^{nm-2} y a^2) + \ldots \\
&+ (a.y a^{nm-1} - y a^{nm}) \\
&= a^{nm} y - y a^{nm} = [a^{nm}, y].
\end{aligned}$$

Then

$$d([a, A_{nm-1,1}(x,y)]) = \sum_{k=0}^{nm-1} d([a, x^{nm-1-k} y x^k]) = d([a^{nm}, y]) \in Z(\mathcal{A}).$$

Thus

$$d([x^{n^2 m^2}, y]) \in Z(\mathcal{A}) \text{ for all } (x,y) \in \mathcal{A}^2.$$

Now, we claim that the restriction of d on $Z(\mathcal{A})$ is non-injective.
For this, we consider $a \in \mathcal{A}$ such that $d(a) = 0$ and b be a non-zero element of $Z(\mathcal{A})$. For all $t \in \mathbb{R}$, we have

$$P(t) = d[(a+tb)^p, y^p]) \in Z(\mathcal{A}) \text{ for all } y \in \mathcal{A},$$

where $p = n^2 m^2$. By reason of $b^m \in Z(\mathcal{A})$ $(\forall m \in \mathbb{N}^*)$, we can write

$$P(t) = \sum_{k=0}^{p} \binom{k}{p} d([a^{p-k} b^k, y^p]) t^k = \sum_{k=0}^{p} A_k(a,b,y) t^k \in Z(\mathcal{A}) \text{ for all } y \in \mathcal{A}$$

where $A_k(a,b,y) = \binom{k}{p} d([a^{p-k} b^k, y^p])$ and $A_0(a,b,y) = d([a^p, y^p]) \in Z(\mathcal{A})$, according to Lemma 1.5, we conclude that $A_k(a,b,y) \in Z(\mathcal{A})$ for all $0 \leq k \leq p$.
In particular for $k = p-1$, we get

$$pd([ab^{p-1}, y^p]) \in Z(\mathcal{A}),$$

we can simplify by p (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$) and we obtain

$$d([ab^{p-1}, y^p]) \in Z(\mathcal{A}),$$

therefore $d(b^{p-1}[a, y^p]) \in Z(\mathcal{A})$ (because b^{p-1} is belongs to $Z(\mathcal{A})$). Since

$$d(b^{p-1}[a, y^p]) = d(b^{p-1})[a, y^p] + b^{p-1} d([a, y^p]) \text{ for all } y \in \mathcal{A}$$

and $b^{p-1} d([a, y^p])$ is an element of $Z(\mathcal{A})$, then $d(b^{p-1})[a, y^p] \in Z(\mathcal{A})$ for all $y \in \mathcal{A}$.
As $d(b^{p-1})$ is a non-zero element of $Z(\mathcal{A})$, by Lemma 1.3, we conclude that

$$[a, y^p] \in Z(\mathcal{A}) \text{ for all } y \in \mathcal{A}.$$

Therefore, for all $t \in \mathbb{R}$ we have

$$Q(t) = [a, (a + ty)^p] \in Z(\mathcal{A}) \text{ for all } y \in \mathcal{A}.$$

That is

$$Q(t) = \sum_{k=0}^{p} [a, B_{p-k,k}(a, y)] t^k \text{ for all } y \in \mathcal{A}$$

where $B_{p-k,k}(a, y)$ denotes the sum of all terms in which a appears exactly $p - k$ times and y appears exactly k times. By Lemma 1.5, we deduce that $[a, B_{p-k,k}(a, y)] \in Z(\mathcal{A})$ for all $k \leq p$.

The coefficient of t in this polynomial is $[a, B_{p-1,1}(a, y)]$ where $B_{p-1,1}(a, y) = \sum_{k=0}^{p-1} a^{p-1-k} y a^k$, therefore $[a, B_{p-1,1}(a, y)] = \sum_{k=0}^{p-1} [a, a^{p-1-k} y a^k] \in Z(\mathcal{A})$, we observe that

$$\begin{aligned}
\sum_{k=0}^{p-1} [a, a^{p-1-k} y a^k] &= [a, a^{p-1} y] + [a, a^{p-2} y a] + [a, a^{p-3} y a^2] + \ldots + [a, y a^{p-1}] \\
&= a.a^{p-1} y - a^{p-1} y.a + a.a^{p-2} y a - a^{p-2} y a.a \\
&+ \ldots + a.y a^{p-1} - y a^{p-1}.a \\
&= a^p y - a^{p-1} y.a + a^{p-1} y a - a^{p-2} y a^2 + \ldots + a.y a^{p-1} - y a^p \\
&= a^p y - y a^p = [a^p, y].
\end{aligned}$$

Consequently,

$$[a^p, y] \in Z(\mathcal{A}) \text{ for all } y \in \mathcal{A}.$$

There are two cases:

Case 1. If $a^p \in Z(\mathcal{A})$, since d is injective in $Z(\mathcal{A})$ and $d(a^p) = 0$ then $a^p = 0$, therefore $a = 0$ and d is injective.

Case 2. If $a^p \notin Z(\mathcal{A})$ then $d(\mathcal{A}) \subset \mathcal{A}$ where d is an inner derivation associated to a^p, by Lemma 1.2 we have $\mathcal{A} \subset Z(\mathcal{A})$, so d is injective.

Lastly, we conclude that the restriction of d on $Z(\mathcal{A})$ is non-injective, then there is a nonzero element a of $Z(\mathcal{A})$ such that $d(a) = 0$, we have $d([(x + ta)^p, y] \in Z(\mathcal{A})$ for all $(x, y) \in \mathcal{A}$ and $t \in \mathbb{R}$. Since $a^k \in Z(\mathcal{A})$ for all $k \in \mathbb{N}^*$, we can write

$$d([(x + ta)^p, y]) = \sum_{k=0}^{p} \binom{p}{k} d([a^{p-k} x^k, y]) t^k \in Z(\mathcal{A}),$$

the first term in this polynomial is $d([a^p, y])$ who belongs to $Z(\mathcal{A})$, according to Lemma 1.5, we conclude that $\binom{p}{k} d([a^{p-k} x^k, y]) \in Z(\mathcal{A})$ for all $0 \leq k \leq p$. In particular for $k = 1$, we have $pd([a^{p-1} x, y]) \in Z(\mathcal{A})$ we can simplify by p (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$), therefore $d([a^{p-1} x, y]) \in Z(\mathcal{A})$, since $a^{p-1} \in Z(\mathcal{A})$ we have $d(a^{p-1} [x, y]) \in Z(\mathcal{A})$, then $a^{p-1} d([x, y]) \in Z(\mathcal{A})$ (because $d(a^{p-1}) = 0$). So $a^{p-1} \in Z(\mathcal{A}) \setminus \{0\}$, according to Lemma 1.3 we have $d([x, y]) \in Z(\mathcal{A})$, by Lemma 1.1 $\mathcal{A}$ is commutative. $\square$

THEOREM 2.2. *Let $\mathcal{A}$ be a prime Banach algebra, $\mathcal{H}_1$ and $\mathcal{H}_2$ nonvoid open subsets of $\mathcal{A}$. Let d be a continuous non-zero derivation of $\mathcal{A}$, satisfying the following condition.*

For all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$d(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{A}),$$

then $\mathcal{A}$ is commutative.

Proof. Following the same steps as the proof of Theorem 2.1, we arrive to show there is $(n,m) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$d(x^n \circ y^m) + [x^n, y^m] \in Z(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}.$$

If $y = x$, then $2d(x^{n+m}) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$, since $\mathcal{A}$ is 2-torsion free (see **Fact 2**) therefore $d(x^{n+m}) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$.

Now, we claim that restriction of d on $Z(\mathcal{A})$ is non-injective.

For this, we consider $a \in \mathcal{A}$ such that $d(a) = 0$ and b be a non-zero element of $Z(\mathcal{A})$, we have $d((a + tb)^{n+m}) \in Z(\mathcal{A})$ for all $t \in \mathbb{R}$. Since $b^k \in Z(\mathcal{A})$ for all $k \in \mathbb{N}^*$, we can write $d((a + tb)^{n+m}) = \sum_{k=0}^{n+m} \binom{k}{n+m} d(a^{n+m-k}b^k)t^k = \sum_{k=0}^{n+m} A_k t^k$ where $A_k = \binom{k}{n+m} d(a^{n+m-k}b^k)$ and $A_0 = d(a^{n+m}) \in Z(\mathcal{A})$, by Lemma 1.5 we conclude that $A_k \in Z(\mathcal{A})$ for all $k \leq n + m$. In particular, for $k = n + m - 1$ we have $(n + m)d(ab^{n+m-1}) \in Z(\mathcal{A})$, we can simplify by $n + m$ (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$) and we obtain $d(ab^{n+m-1}) \in Z(\mathcal{A})$, therefore $ad(b^{n+m-1}) \in Z(\mathcal{A})$ (because $d(a) = 0$).

Suppose that $d(b^{n+m-1}) = 0$, then $b^{n+m-1} = 0$ (because $b^{n+m-1} \in Z(\mathcal{A})$ and d is injective in $Z(\mathcal{A})$), therefore $b = 0$ this is a contradiction. Then $d(b^{n-1}) \neq 0$, by Lemma 1.3 we conclude that $a \in Z(\mathcal{A})$. Since the restriction of d on $Z(\mathcal{A})$ is injective, we conclude that $a = 0$, that is d is injective. Lastly, we conclude that the restriction of d on $Z(\mathcal{A})$ is non-injective, then there is a nonzero element a of $Z(\mathcal{A})$ such that $d(a) = 0$, we have $d((x+ta)^{n+m}) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$ and $t \in \mathbb{R}$. Since $a^k \in Z(\mathcal{A})$ $(\forall k \in \mathbb{N}^*)$, we can write

$$d((x + ta)^{n+m}) = \sum_{k=0}^{n+m} \binom{n+m}{k} d(a^{n+m-k}x^k)t^k \in Z(\mathcal{A}).$$

The first term in this polynomial is $d(a^{n+m})$ who belongs to $Z(\mathcal{A})$, by Lemma 1.5 we conclude that $\binom{n}{k} d(a^{n+m-k}x^k) \in Z(\mathcal{A})$ for all $0 \leq k \leq n + m$, in particular, for $k = 1$, we have $(n+m)d(a^{n+m-1}x) \in Z(\mathcal{A})$, we can simplify by $n + m$ (because $Z(\mathcal{A})$ is a subspace of $\mathcal{A}$), therefore $d(a^{n+m-1}x) \in Z(\mathcal{A})$ and hence $d(a^{n+m-1})x + a^{n+m-1}d(x) \in Z(\mathcal{A})$, then $a^{n+m-1}d(x) \in Z(\mathcal{A})$ (because $d(a^{n+m-1}) = 0$), by Lemma 1.3 we have $d(x) \in Z(\mathcal{A})$, that is $d(\mathcal{A}) \subset Z(\mathcal{A})$. According to Lemma 1.2, $\mathcal{A}$ is commutative. □

APPLICATION 2.3. Let $\mathbb{R}$ be the field of real numbers. $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ endowed with usuals matrix addition and multiplication and the norm $\|.\|_1$ defined by

$\|A\|_1 = \sum_{1\leq i,j\leq 2} |a_{i,j}|$ for all $A = (a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathcal{A}$ Let d be the inner derivation induced by the element $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we have $d(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$, d is non-injective continuous derivation. Let $\mathcal{H}$ be a nonempty open subset of $\mathcal{A}$ included in $Z(\mathcal{A})$. For all $(A,B) \in \mathcal{H} \times \mathcal{H}$ and for all $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$ we have $A^p \in Z(\mathcal{A})$ and $B^q \in Z(\mathcal{A})$, then $d(A^p.B^q) + A^p \circ B^q \in Z(\mathcal{A})$. By Theorem 2.1, we conclude that $\mathcal{A}$ is commutative. Consequently $\mathcal{H} = \emptyset$.

We conclude that the only open subset included in $Z(\mathcal{X})$ is the empty set.

Application 2.4. Let E be a normed space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The space $\mathcal{A} = \mathcal{L}_c(E)$ of continuous linear applications from E to E, endowed with usual application addition and composition and the norm defined by $\| T \| = \sup_{\|x\|\leq 1} \| T(x) \|$ for all $T \in \mathcal{A}$, is a normed algebra over $\mathbb{K}$.

Let G be the subspace of $\mathcal{A}$ defined by $G = \{\lambda I_E \mid \lambda \in \mathbb{K}\}$, where I_E is the identity of E, we observe that $G \subset Z(\mathcal{A})$, according to Application 2.3 we conclude that the interior of G is empty because $\mathcal{A}$ is not commutative and $Int(G) \subset Int(Z(\mathcal{A})) = \emptyset$.

Corollary 2.5. *Let $\mathcal{A}$ be a prime Banach algebra, $\mathcal{H}_1$ and $\mathcal{H}_2$ two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous non-zero derivation d satisfying one of the following conditions*

i) *for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $d(x^p.y^q) - x^p \circ y^q \in Z(\mathcal{A})$,*

ii) *for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $d(x^p \circ y^q) - [x^p, y^q] \in Z(\mathcal{A})$,*

then $\mathcal{A}$ is commutative.

Proof. If d is a non-zero derivation satisfying $i)and\ ii)$, then d satisfies 1) for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $-((-d)(x^p.y^q) + x^p \circ y^q) \in Z(\mathcal{A})$,

2) for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $-((-d)(x^p \circ y^q) + [x^p, y^q]) \in Z(\mathcal{A})$. We multiply by -1 and we get the following results: for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $(-d)(x^p.y^q) + x^p \circ y^q \in Z(\mathcal{A})$, for all $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2)$, there exists $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*)$ such that $(-d)(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{A})$.

Since $(-d)$ is a continuous non-zero derivation, by Theorem 2.1 and Theorem 2.2, we deduce that $\mathcal{A}$ is commutative. □

We close this article with the following corollary.

Corollary 2.6. *Let $\mathcal{A}$ be a prime Banach algebra and D a dense part of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous non-zero derivation d such that*

$$\exists (p,q) \in \mathbb{N}^* \times \mathbb{N}^* : d(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{A}) \text{ for all } x, y \in D,$$

then $\mathcal{A}$ is commutative.

Proof. Let $x, y \in \mathcal{A}$, there exist two sequences $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ in D converging to x and y. Since $(x_k)_{k\in\mathbb{N}} \subset D$ and $(y_k)_{k\in\mathbb{N}} \subset D$, so

$$d((x_k)^p \circ (y_k)^q) + [(x_k)^p, (y_k)^q] \in Z(\mathcal{A}) \text{ for all } k \in \mathbb{N}.$$

Since d is continuous then the sequence $(d((x_k)^p \circ (y_k)^q) + [(x_k)^p, (y_k)^q])_{k\in\mathbb{N}}$ converges to $d(x^p \circ y^q) + [x^p, y^q]$, knowing that $Z(\mathcal{A})$ is closed, then $d(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{A})$.

We conclude that :

$$\exists (p,q) \in \mathbb{N}^* \times \mathbb{N}^* \ : d(x^p \circ y^q) + [x^p, y^q] \in Z(\mathcal{X}) \text{ for all } x, y \in \mathcal{A}.$$

By Theorem 2.2, we get the desired conclusion. □

Remark 2.7. We can conclude similar results as Corollary 2.6 if we replace one of the open sets in the preceding Theorems by a dense part of $\mathcal{A}$.

The following example shows that the hypothesis "$\mathcal{H}_1$ and $\mathcal{H}_2$ are open" in our Theorems is not superfluous.

Example 2.8. Let $\mathbb{R}$ be the field of real numbers. The ring $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ endowed with usuals matrix addition and multiplication and the norm defined by $\|A\|_1 = \sum_{1\leq i,j\leq 2} |a_{i,j}|$ for all $A = (a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathcal{A}$, is a real prime Banach algebra.

Let $\mathcal{F}_1 = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{R}\}$ and $\mathcal{F}_2 = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{R}^+\}$. Then $\mathcal{F}_1$ is not open in $\mathcal{A}$. Indeed, we have to show that the complement of $\mathcal{F}_1$ is not closed. For this, we consider the sequence $(\begin{pmatrix} 1+\dfrac{1}{n} & \dfrac{-1}{n} \\ \dfrac{1}{n} & 1+\dfrac{1}{n} \end{pmatrix})_{n\in\mathbb{N}^*}$ in $\mathcal{F}_1^c$ complement of $\mathcal{F}_1$ who converge to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \mathcal{F}_1^c$, then $\mathcal{F}_1^c$ is not closed, that is $\mathcal{F}_1$ is not open in $\mathcal{A}$.

Let d be the inner derivation induced by the element $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We have $d(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$, d is non-injective continuous derivation. For all $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{F}_1$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in \mathcal{F}_2$ and for all $(p,q) \in \mathbb{N}^2$, we have $A^p = \begin{pmatrix} a^p & 0 \\ 0 & a^p \end{pmatrix}$ *and* $B^q = \begin{pmatrix} b^q & 0 \\ 0 & b^q \end{pmatrix}$. So $A^pB^q = \begin{pmatrix} a^pb^q & 0 \\ 0 & a^pb^q \end{pmatrix}$, $A^p \circ B^q = \begin{pmatrix} 2a^pb^q & 0 \\ 0 & 2a^pb^q \end{pmatrix}$ *and* $[A^p, B^q] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We conclude that

$$d(A^p \circ B^q) = d([A^p, B^q]) = d(A^pB^q) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain:
1) $d(A^p.B^q) + A^p \circ B^q \in Z(\mathcal{A})$ and $d(A^p.B^q) - A^p \circ B^q \in Z(\mathcal{A})$,
2) $d(A^p \circ B^q) + [A^p, B^q] \in Z(\mathcal{A})$ and $d(A^p \circ B^q) - [A^p, B^q] \in Z(\mathcal{A})$.
But $\mathcal{A}$ is not commutative.

The following example shows that we cannot replace $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ in the hypothesis for all theorems.

Example 2.9. Let $\mathcal{A} = (\mathcal{M}_2(\mathbb{Z}/3\mathbb{Z}), +, \times, .)$ the 2-torsion free prime Banach algebra of square matrices of size 2 at coefficients in $\mathbb{Z}/3\mathbb{Z}$ with usual matrix addition and matrix multiplication. The norm is defined by $\|A\|_1 = \sum_{1\leq i,j\leq 2} |a_{i,j}|$ for all $A = (a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathcal{A}$ with $|.|$ is the norm defined on $\mathbb{Z}/3\mathbb{Z}$ by: $|\overline{0}| = 0$, $|\overline{1}| = 1$ and $|\overline{2}| = 2$.

Observe that $\mathcal{H} = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}/3\mathbb{Z}\}$ is open in $\mathcal{A}$. Indeed, let $A \in \mathcal{H}$ the open ball $B(A, 1) = \{X \in \mathcal{A}$ such that $\|A - X\|_\infty < 1\} = \{A\} \subset \mathcal{H}$, then $\mathcal{H}$ is a nonvoid open subset of $\mathcal{A}$.

Let d be the inner derivation induced by the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We have $d(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} c & d-a \\ c & -c \end{pmatrix}$, d is a non-injective continuous derivation. For all $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$ and for all $(A, B) \in \mathcal{H} \times \mathcal{H}$ we have :
1) $d(A^p.B^q) + A^p \circ B^q \in Z(\mathcal{A})$ and $d(A^p.B^q) - A^p \circ B^q \in Z(\mathcal{A})$.
2) $d(A^p \circ B^q) + [A^p, B^q] \in Z(\mathcal{A})$ and $d(A^p \circ B^q) - [A^p, B^q] \in Z(\mathcal{A})$.
So all the conditions are verified except $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ but $\mathcal{A}$ is not commutative.

Example 2.10. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ and d be the inner derivation induced by the element $M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathcal{A}$ is a non-commutative 2-torsion free prime complex Banach algebra. Then d defined by $d(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, is non-injective continuous derivation. Furthermore, we have $\mathcal{H} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$ such that $ad - bc \neq 0\}$ is open in $\mathcal{A}$.

Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{H}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$, then $A^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ and $B^q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ therefore $d(A^p \circ B^q) + [A^p, B^q] = \begin{pmatrix} 0 & -2p \\ 0 & 0 \end{pmatrix} \notin Z(\mathcal{A})$ for all $p, q \in \mathbb{N}^*$. This shows that the condition $d(A^p \circ B^q) + [A^p, B^q] \in Z(\mathcal{A})$ “for all $A \in \mathcal{H}$” ($\mathcal{H}$ is open of $\mathcal{A}$) in Theorem 2.2, is not superfluous.

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