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EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR A p-FRACTIONAL KIRCHHOFF EQUATION WITH CRITICAL GROWTH IN $\mathbb{R}^N$

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Abstract. This paper deals with the following p-fractional Kirchhoff equation

$$\left(a+b[u]_{s,p}^p\right)(-\Delta)_p^s u+|u|^{p-2}u=|u|^{p_s^*-2}u+f(u) \ \text{ in } \mathbb{R}^N,$$

where $a,b>0$, $s\in(0,1)$, $N>sp$, $p_s^*=\frac{Np}{N-sp}$ and f satisfies some hypotheses. By transforming this equation into an equivalent system, we establish the existence of at least one nontrivial solution or two nontrivial solutions without the well-known Ambrosetti-Rabinowitz (AR) condition. Furthermore, the nonexistence case is also treated. Our result extends and completes the recent works in the literature.

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1. INTRODUCTION AND MAIN RESULT

In this paper, we consider the p-fractional Kirchhoff equation

$$\left(a+b[u]_{s,p}^p\right)(-\Delta)_p^s u+|u|^{p-2}u=|u|^{p_s^*-2}u+f(u) \ \text{ in } \mathbb{R}^N, \tag{1}$$

where $[u]_{s,p}^p=\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y$, $a,b>0$, $s\in(0,1)$, $p>1$, $N>sp$, $p_s^*=\frac{Np}{N-sp}$, $(-\Delta)_p^s$ is the fractional p-Laplacian operator defined as

$$(-\Delta)_p^s u(x)=2\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\mathrm{d}y,\ x\in\mathbb{R}^N,$$

with $B_\varepsilon(x)$ is the ball with center at x and radius ε, and f is a function satisfying some conditions which will be specified later.

Because of the presence of the coefficient $(a+b[u]_{s,p}^p)$ and the operator $(-\Delta)_p^s$, equation (1) is nonlocal, namely it is no longer a pointwise identity. This phenomenon produces new mathematical complications and so makes the

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study of such type of problems specially interesting. Furthermore, equation (1) is a p-fractional version related to the following hyperbolic equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was initially introduced by Kirchhoff [3] as a generalization of the classical D'Alembert wave equation taking into consideration the change in length of the strings produced by transverse vibrations. For additional discussions and physical phenomena described by nonlinear vibration theory we mention [7].

A typical approach to solve such class of fractional Kirchhoff equations is to use the variational method, but due to the nonlocal term it seems hard to obtain the boundedness of any Palais-Smale or Cerami sequence without the Ambrosetti-Rabinowitz type condition:

(AR) there exists $\tau > 2p$ such that $tf(t) \geq \tau F(t) := \tau\int_0^t f(\varsigma)\mathrm{d}\varsigma$ for all $t \in \mathbb{R}$.

To our knowledge, there are a few works with weaker restriction or without (AR) assumption. When $p = 2$, in the low dimensions ($N \in \{2, 3\}$) and without (AR) type condition or monotonicity assumptions, by applying the monotonicity trick and the profile decomposition, Liu et al. [5] proved the existence of ground states in the critical case. In [2], Jin and Liu considered a general critical nonlinearity in the higher dimensions ($N > 3$). The authors obtained a nontrivial solution when b is small, via a perturbation technique. Recently, in the whole space $\mathbb{R}^3$, Zhang et al. [13] established the existence of ground state solution and sign-changing solution for a fractional Schrödinger-Kirchhoff equation with critical or supercritical nonlinearity, by adopting some arguments which are related to the Moser iterative method and cut-off function technique. For more related studies on the fractional Kirchhoff problems, we refer to [4, 9, 12, 15] and references therein.

The main purpose of this note is to discuss the existence of solutions for equation (1), where the nonlinearity f does not satisfy the classical (AR) type condition. We point out that Theorem 1.1 can be regarded as an extension of the main result in [6] obtained for the fractional Kirchhoff equation ($p = 2$) without any growth and the corresponding result of [14] for the Laplacian Kirchhoff problem with critical exponent. In order to get solutions, inspired by above facts, we propose to transform equation (1) into an equivalent system. Precisely, we will look for solutions of equation (1) by finding solutions of the following system in (u, λ)

$$(2)\qquad \begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = |u|^{p_s^*-2}u + f(u) & \text{in} \quad \mathbb{R}^N \\ a + b\lambda^{\frac{N-sp}{p}}T(u) = \lambda^s & \text{in} \quad (0, +\infty), \end{cases}$$

where $T(u) = [u]_{s,p}^p$. Before stating our main contribution, we give the following assumptions on f.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $C > 0$ and $q \in (p, p_s^*)$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}) \text{ for all } t \in \mathbb{R};$$

(f_2) $f(t) = o(t^{p-1})$ as $t \to 0$;

(f_3) there exist $r \in (p, p_s^*)$ and $\mu > 0$ such that $f(t) \geq \mu t^{r-1}$ for all $t \geq 0$. If $p < r \leq p_s^* - \frac{p}{p-1}$, μ is assumed to be large enough;

(f_4) $tf(t) - pF(t) \geq 0$ for all $t \in \mathbb{R}$.

We are now in position to state the main principal result which is formulated in the following theorem.

Theorem 1.1. *Suppose that $(f_1) - (f_4)$ hold. Then*

i) *If $N < 2sp$, for all $a, b > 0$, equation (1) has at least one nontrivial solution;*

ii) *If $N = 2sp$, there exist $b_0 \in (0, +\infty]$ such for all $a > 0$ and $b \in (0, b_0)$, equation (1) has at least one nontrivial solution and has no nontrivial solution if $b \geq b_0$ with $b_0 < \infty$;*

iii) *If $N > 2sp$, there exists $\sigma_0 > 0$ such that equation (1) has at least two nontrivial solutions if $ab^{\frac{sp}{N-2sp}} < \sigma_0$, has at least one nontrivial solution if $ab^{\frac{sp}{N-2sp}} = \sigma_0$ and has no nontrivial solution if $ab^{\frac{sp}{N-2sp}} > \gamma_0$, for some $\gamma_0 > 0$.*

2. PRELIMINARIES AND PROOF OF MAIN THEOREM

Let $W^{s,p}(\mathbb{R}^N)$ be the usual fractional Sobolev space:

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

with the norm

$$||u||_{s,p} = \left([u]_{s,p}^p + |u|_p^p\right)^{\frac{1}{p}}, \text{ where } |u|_p = \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

It is well known that the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded into $L^\nu(\mathbb{R}^N)$ for $\nu \in [p, p_s^*]$ and locally compact for $\nu \in (p, p_s^*)$. We denote by S_* the best fractional Sobolev constant:

$$S_* = \inf_{u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,p}^p}{|u|_{p_s^*}^p}.$$

We recall that $u \in W^{s,p}(\mathbb{R}^N)$ is a (weak) solution of equation (1) if

$$\left(a + b[u]_{s,p}^p\right) \langle u, \varphi \rangle_{s,p} + \int_{\mathbb{R}^N} |u|^{p-2} u \varphi \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{p_s^*-2} u \varphi \mathrm{d}x + \int_{\mathbb{R}^N} f(u) \varphi \mathrm{d}x,$$

for all $\varphi \in W^{s,p}(\mathbb{R}^N)$, where

$$\langle u, \varphi \rangle_{s,p} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \mathrm{d}x \mathrm{d}y.$$

Proposition 2.1. *Equation (1) has at least one solution* $u \in W^{s,p}(\mathbb{R}^N)$ *if and only if system (2) has at least one solution* $(v,\lambda) \in W^{s,p}(\mathbb{R}^N) \times (0,+\infty)$.

Proof. Assume that $u \in W^{s,p}(\mathbb{R}^N)$ is a solution of equation (1). Let the function $v : x \mapsto u(\lambda^{\frac{1}{p}} x)$ and

$$\lambda := \left(a + b\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{s}}.$$

Then

$$\lambda^s = a + b\lambda^{\frac{N-sp}{p}} \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y$$

and for all $\varphi \in W^{s,p}(\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |v|^{p-2} v\varphi \mathrm{d}x \\
= \;& \lambda^{s-\frac{N}{p}} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
& + \lambda^{-\frac{N}{p}} \int_{\mathbb{R}^N} |u|^{p-2} u\psi \mathrm{d}x \\
= \;& \lambda^{-\frac{N}{p}} \left(\int_{\mathbb{R}^N} |u|^{p_s^*-2} u\psi \mathrm{d}x + \int_{\mathbb{R}^N} f(u)\psi \mathrm{d}x \right) \\
= \;& \int_{\mathbb{R}^N} |v|^{p_s^*-2} v\varphi \mathrm{d}x + \int_{\mathbb{R}^N} f(v)\varphi \mathrm{d}x,
\end{aligned}$$

where $\psi : x \mapsto \varphi(\lambda^{-\frac{1}{p}} x)$. Hence (v,λ) is a solution of system (2).

Now, suppose that (v,λ) is a solution of system (2). Let the function $u : x \mapsto v(\lambda^{-\frac{1}{p}} x)$. Then

$$\lambda^s = a + b\lambda^{\frac{N-sp}{p}} \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = a + b\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y$$

and for all $\varphi \in W^{s,p}(\mathbb{R}^N)$,

$$\begin{aligned}
&\left(a + b\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y\right) \\
&\times \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^N} |u|^{p-2} u\varphi \mathrm{d}x \\
= \;& \lambda^s \lambda^{\frac{N}{p}-s} \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
& + \lambda^{\frac{N}{p}} \int_{\mathbb{R}^N} |v|^{p-2} v\psi \mathrm{d}x
\end{aligned}$$

$$= \lambda^{\frac{N}{p}}\left(\int_{\mathbb{R}^N}|v|^{p_s^*-2}v\psi \mathrm{d}x + \mu\int_{\mathbb{R}^N} f(v)\psi \mathrm{d}x\right)$$
$$= \int_{\mathbb{R}^N}|u|^{p_s^*-2}u\varphi + \int_{\mathbb{R}^N} f(u)\varphi \mathrm{d}x,$$

where $\psi : x \mapsto \varphi(\lambda^{\frac{1}{p}}x)$, which shows that u is a solution of equation (1). □

Now, we investigate the first equation of system (2):

$$(-\Delta)_p^s u + |u|^{p-2}u = |u|^{p_s^*-2}u + f(u) \text{ in } \mathbb{R}^N. \tag{3}$$

In light of the assumptions on f, it is standard to verify that the weak solutions of this equation correspond to the critical points of the energy functional $I : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}||u||_{s,p}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\mathrm{d}x - \int_{\mathbb{R}^N}F(u)\mathrm{d}x.$$

Under conditions of Theorem 1.1, we see that I has mountain pass geometric structure. So, in order to prove the existence of solutions of (3), we consider the mountain pass level

$$c_* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], W^{s,p}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.$$

Clearly $c_* > 0$ and by [10, Theorem 3], a $(Ce)_{c_*}$ sequence $\{u_n\} \subset W^{s,p}(\mathbb{R}^N)$ exists such that

$$I(u_n) \to c_* \text{ and } (1 + ||u_n||_{s,p})||I'(u_n)||_*.$$

Now we shall recall some useful estimates which can be founded in [8, Lemma 2.7]. According to [8], for any $\varepsilon > 0$ there is a nonnegative function $u_\varepsilon \in W^{s,p}(\mathbb{R}^N)$ satisfying the following inequalities.

Lemma 2.2 ([8]). *There exists $C = C(N,s,p)) > 0$ such that for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,*

$$[u_\varepsilon]_{s,p}^p \le S_*^{\frac{N}{sp}} + C\varepsilon^{\frac{N-sp}{p-1}} \text{ and } |u_\varepsilon|_{p_s^*}^{p_s^*} \ge S_*^{\frac{N}{sp}} - C\varepsilon^{\frac{N}{p-1}}.$$

Lemma 2.3 ([8]). *There exists $C = C(N,s,p)) > 0$ such that for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,*

$$|u_\varepsilon|_p^p \le \begin{cases} C\varepsilon^{sp} & \text{if } N > sp^2 \\ C\varepsilon^{sp}\log\left(\frac{1}{\varepsilon}\right) & \text{if } N = sp^2 \\ C\varepsilon^{\frac{N-sp}{p-1}} - C\varepsilon^{sp} & \text{if } N < sp^2. \end{cases}$$

LEMMA 2.4 ([8]). *There exists $C = C(N, s, r) > 0$ such that for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,*

$$|u_\varepsilon|_r^r \geq \begin{cases} C\varepsilon^{N-\frac{r(N-sp)}{p}} & \text{if} \quad r > \frac{N(p-1)}{N-sp} \\ C\varepsilon^{N-\frac{r(N-sp)}{p}}|\log(\varepsilon)| & \text{if} \quad r = \frac{N(p-1)}{N-sp} \\ C\varepsilon^{\frac{r(N-sp)}{p(p-1)}} & \text{if} \quad r < \frac{N(p-1)}{N-sp}. \end{cases}$$

LEMMA 2.5. *We have*

$$\max_{t\geq 0} I(tu_\varepsilon) < \frac{s}{N} S_*^{\frac{N}{sp}}.$$

In particular $c_* < \frac{s}{N} S_*^{\frac{N}{sp}}$.

Proof. With the help of Lemmas 2.2, 2.3, 2.4 and (f_3), the proof is quite similar to that presented in [1, Lemma 3.2], and so will be omitted. □

LEMMA 2.6. *Let $\{u_n\} \subset W^{s,p}(\mathbb{R}^N)$ be the $(Ce)_c$ sequence of I with $c \in \mathbb{R}$, that is,*

$$I(u_n) \to c \text{ and } (1 + ||u_n||_{s,p})||I'(u_n)||_* \to 0. \tag{4}$$

Then $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$.

Proof. From (4) and (f_4), we have

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*}\right)\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{1}{p}f(u_n)u_n - F(u_n)\right)\mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{p_s^*}\right)\int_{\mathbb{R}^N} |u_n|^{p_s^*}\mathrm{d}x, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$. For $\varepsilon > 0$, by $(f_1) - (f_2)$, there exists $C_\varepsilon > 0$ such that

$$|f(t)t| \leq \varepsilon|t|^p + C_\varepsilon|t|^{p_s^*} \text{ for all } t \in \mathbb{R}.$$

By (4) and the Sobolev embeddings, for some $C_p > 0$,

$$\begin{aligned} o_n(1) &= \langle I'(u_n), u_n\rangle \\ &\geq ||u_n||_{s,p}^p - |u_n|_{p_s^*}^{p_s^*} - \varepsilon|u_n|_p^p - C_\varepsilon|u_n|_{p_s^*}^{p_s^*} \\ &\geq (C_p - \varepsilon)||u_n||_{s,p}^p - (1 + C_\varepsilon)|u_n|_{p_s^*}^{p_s^*}, \end{aligned}$$

here we choose $\varepsilon < C_p$. Hence $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$, since it is bounded in $L^{p_s^*}(\mathbb{R}^N)$. □

We will use the following variant Lions lemma, whose proof is similar to [11, Lemma 1.21].

Lemma 2.7. *Let $\{u_n\}$ be a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ such that*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n|^p\mathrm{d}x=0,$$

for some $R>0$. Then $u_n\to 0$ in $L^\nu(\mathbb{R}^N)$ for any $\nu\in(p,p_s^*)$.

Lemma 2.8. *Let $\{u_n\}\subset W^{s,p}(\mathbb{R}^N)$ be a $(Ce)_c$ sequence of I, with $c\in\left(0,\frac{s}{N}S_*^{\frac{N}{sp}}\right)$. Then*

$$\eta:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^p\mathrm{d}x>0.$$

Proof. If $\eta=0$, by virtue of Lemma 2.7, $u_n\to 0$ in $L^\nu(\mathbb{R}^N)$ for any $\nu\in(p,p_s^*)$. Using $(f_1)-(f_2)$, we deduce that $\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x\to 0$. Up to a subsequence, we have $|u_n|_{p_s^*}^{p_s^*}\to\mathfrak{A}$ and $||u_n||_{s,p}^p\to\mathfrak{B}$. By definition of S_*, we infer that $S_*\mathfrak{A}^{\frac{p}{p_s^*}}\le\mathfrak{B}$. We know that $\langle I'(u_n),u_n\rangle\to 0$, thus $\mathfrak{A}=\mathfrak{B}$. Since $c>0$, we deduce that $\mathfrak{B}>0$ and so $S_*^{\frac{N}{sp}}\le\mathfrak{B}$. It follows from (f_4) that

$$\begin{aligned}c&=\lim_{n\to\infty}I(u_n)\\&\ge\lim_{n\to\infty}\left[\frac{1}{p}||u_n||_{s,p}^p-\frac{1}{p_s^*}|u_n|_{p_s^*}^{p_s^*}-\frac{1}{p}\int_{\mathbb{R}^N}f(u_n)u_n\mathrm{d}x\right]\\&=\mathfrak{B}\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\ge\frac{s}{p}S_*^{\frac{N}{sp}},\end{aligned}$$

which contradicts our hypothesis. □

Proposition 2.9. *Assume that $(f_1)-(f_4)$ hold. Then equation (3) has at least one nontrivial solution.*

Proof. Let $\{u_n\}\subset W^{s,p}(\mathbb{R}^N)$ a $(Ce)_{c_*}$ sequence of I. Then, according to Lemmas 2.5 and 2.8, going if necessary to a subsequence, we may assume the existence of $\{y_n\}\subset\mathbb{R}^N$ such that

$$\int_{B_1(0)}|v_n|^p\mathrm{d}x=\int_{B_1(y_n)}|u_n|^p\mathrm{d}x>\frac{\eta}{2},$$

where $v_n=u_n(\cdot+y_n)$. Since $\{v_n\}$ is also bounded in $W^{s,p}(\mathbb{R}^N)$, $v_n\rightharpoonup v$ in $W^{s,p}(\mathbb{R}^N)$ and $v_n\to v$ in $L^p(B_1(0))$. Therefore $v\neq 0$ and by a standard way, we know that v is a solution of (3). □

Theorem 2.10. *Suppose that f satisfies $(f_1)-(f_4)$. Then*

1) *If $N<2sp$, for all $a,b>0$ and any nontrivial solution v of equation (3), there exists $\lambda_v>a^{\frac{1}{s}}$ such that (v,λ_v) is a solution of system (2);*

2) *If* $N = 2sp$, *there exists* $b_0 \in (0, +\infty]$ *such that for any* $a > 0$ *and* $b \in (0, b_0)$, *there exist a nontrivial solution* v *of equation (3) and* $\lambda_v > 0$ *such that* (v, λ_v) *is a solution of system (2) and has no solution if* $b \geq b_0$ *with* $b_0 < \infty$;
3) *If* $N > 2sp$, *for any nontrivial solution* v *of equation (3) there exists* $\sigma_0 > 0$ *such that for some* $\lambda_v^1, \lambda_v^2 > a^{\frac{1}{s}}$, (v, λ_v^1) *and* (v, λ_v^2) *are solutions of system (2) if* $ab^{\frac{sp}{N-2sp}} < \sigma_0$, *has at least one solution* $(v, \lambda_{\min}(v))$ *if* $ab^{\frac{sp}{N-2sp}} = \sigma_0$ *and has no solution if* $ab^{\frac{sp}{N-2sp}} > \gamma_0$, *where* $\gamma_0 := \frac{N-2sp}{N-sp}\left(\frac{spb_0}{N-sp}\right)^{\frac{sp}{N-2sp}}$ *with* $b_0 < \infty$.

Proof. For each $u \in W^{s,p}(\mathbb{R}^N)\backslash\{0\}$, we define the function G_u on $[0, +\infty)$ by

$$G_u(\lambda) = a + b\lambda^{\frac{N-sp}{p}} T(u) - \lambda^s.$$

Case 1. $N < 2sp$. Let v be a nontrivial solution of equation of (3). Then $G_v(\lambda) \to -\infty$, as $\lambda \to +\infty$. Since $G_v(\lambda) > 0$ for all $\lambda \in (0, a^{\frac{1}{s}}]$, there exists $\lambda_v > a^{\frac{1}{s}}$ such that $G_v(\lambda_v) = 0$. Thus (v, λ_v) is a solution of system (2).

Case 2. $N = 2sp$. Then $G_u(\lambda) = a - \lambda^s(1 - bT(u))$. Therefore for any $a > 0$ and $0 < b < b_0$, where

$$b_0 := \sup\left\{\frac{1}{T(u)} : u \text{ is a nontrivial solution of (3)}\right\},$$

there exists a solution v of equation of (3) such that $G_v(\lambda_v) = 0$, where $\lambda_v = \left(\frac{a}{1-bT(v)}\right)^{\frac{1}{s}}$. Hence (v, λ_v) is a solution of system (2).

If $b \geq b_0$ with $b_0 < \infty$, then for any nontrivial solution u of equation (3) and $\lambda > 0$, we have

$$G_u(\lambda) \geq a - (1 - bT(u))\lambda^s \geq a - \left(1 - \frac{b}{b_0}\right)\lambda^s \geq a > 0.$$

Thus system (2) has no solution.

Case 3. $N > 2ps$. Let v be a nontrivial solution of equation of (3). Note that for $G_v(\lambda) > 0$ for all $\lambda \in (0, a^{\frac{1}{s}}]$ and $G_v(\lambda) \to +\infty$ as $\lambda \to +\infty$. By simple computation, the minimum value of G_v is achieved only at the point

$$\lambda_{\min}(v) := \left(\frac{sp}{(N-sp)bT(v)}\right)^{\frac{p}{N-2sp}}$$

and

$$\min_{\lambda \geq 0} G_v(\lambda) = G_v(\lambda_{\min}(v)) = a - \frac{N-2sp}{N-sp}\left(\frac{sp}{(N-sp)bT(v)}\right)^{\frac{sp}{N-2sp}}.$$

Set

$$\sigma_0 = \sigma_v := \frac{N-2sp}{N-sp}\left(\frac{sp}{(N-sp)T(v)}\right)^{\frac{sp}{N-2sp}}.$$

If $ab^{\frac{sp}{N-2sp}} < \sigma_0$, then $G_v(\lambda_{\min}(v)) < 0$, thus there exist $\lambda_v^1 \in (a^{\frac{1}{s}}, \lambda_{\min}(v))$ and $\lambda_v^2 \in (\lambda_{\min}(v), +\infty)$ such that $G_v(\lambda_v^1) = G_v(\lambda_v^2) = 0$. Consequently (v, λ_v^1) and (v, λ_v^2) are solutions of system (2).

If $ab^{\frac{sp}{N-2sp}} = \sigma_0$, then $G_v(\lambda_{\min}(v)) = 0$, hence $(v, \lambda_{\min}(v))$ is a solution of system (2).

If $ab^{\frac{sp}{N-2sp}} > \gamma_0 := \frac{N-2sp}{N-sp}\left(\frac{spb_0}{N-sp}\right)^{\frac{sp}{N-2sp}}$ with $b_0 < \infty$, then for any nontrivial solution u of equation (3) and $\lambda > 0$,

$$G_u(\lambda) \geq G_u((\lambda_{\min}(u)) \geq a - \frac{N-2sp}{N-sp}\left(\frac{spb_0}{(N-sp)b}\right)^{\frac{sp}{N-2sp}} > 0,$$

which yields that system (2) has no solution. $\square$

Proof of Theorem 1.1. Theorem 1.1 is a direct consequence of Proposition 2.1 and Theorem 2.10. $\square$

REFERENCES

[1] V. Ambrosio and T. Isernia, *On the multiplicity and concentration for p-fractional Schrödinger equations*, Appl. Math. Lett., **95** (2019), 13–22.

[2] H. Jin and W. Liu, *Fractional Kirchhoff equation with a general critical nonlinearity*, Appl. Math. Lett., **74** (2017), 140–146.

[3] G. Kirchhoff, *Vorlesungen uber Mechanik*, 3rd edition, Teubner, Leipzig, 1883.

[4] Q. Li, K. Teng, X. Wu and W. Wang, *Existence of nontrivial solutions for fractional Schrödinger equations with critical or supercritical growth*, Math. Methods Appl. Sci., **42** (2019), 1480–1487.

[5] Z. Liu, M. Squassina and J. Zhang, *Ground states for fractional Kirchhoff equations with critical nonlinearity in low dimension*, NoDEA Nonlinear Differential Equations Appl., **24** (2017), 1–32.

[6] M. Massar and M. Talbi, *Radial solutions for a fractional Kirchhoff type equation in* $\mathbb{R}^N$, Indian J. Pure Appl. Math., **52** (2021), 897–902.

[7] L. Medeiros, J. Limaco and S. Menezes, *Vibrations of elastic strings: Mathematical aspects. Part one*, Journal of Computational Analysis and Applications, **4** (2002), 91–127.

[8] S. Mosconi, K. Perera, M. Squassina and Y. Yang, *The Brezis-Nirenberg problem for the fractional p-Laplacian*, Calc. Var. Partial Differential Equations, **55** (2016), 55–105.

[9] P. Pucci, M. Xiang and B. Zhang, *Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian*, Adv. Calc. Var., **12** (2019), 253–275.

[10] E. Silva and G. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with critical growth*, Calc. Var. Partial Differential Equations, **39** (2010), 1–33.

[11] M. Willem, *Minimax theorems, Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, 1996.

[12] M. Xiang, B. Zhang and M. Yang, *A fractional Kirchhoff-type problem in* $\mathbb{R}^N$ *without the (AR) condition*, Complex Var. Elliptic Equ., **61** (2016), 1481–1493.

[13] L. Zhang, X. Tang and S. Chen, *Multiple solutions for fractional Kirchhoff equation with critical or supercitical nonlinearity*, Appl. Math. Lett., **119** (2021), 1–7.
[14] F. Zhou and M. Yang, *Solutions for a Kirchhoff type problem with critical exponent in* $\mathbb{R}^N$, J. Math. Anal. Appl., **494** (2021), 1–7.
[15] J. Zuo, T. An and M. Li, *Superlinear Kirchhoff-type problems of the fractional p-Laplacian without the (AR) condition*, Bound. Value Probl., **180** (2018), 1–13.

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