MATHEMATICA, 65 (88), N$^{\circ}$ 1, 2023, pp. 85–93

ON THE COMBINATORIAL NATURE OF TREE REPRESENTATIONS OF EUCLIDEAN QUIVERS

ÁBEL LŐRINCZI

Abstract. We verify computationally a conjecture on the field independence of tree representations of Euclidean quivers, with dimension vector bounded by the minimal radical vector of the quiver. This includes a large class of exceptional representations, in particular all the regular non-homogeneous exceptionals. In addition we also present some thought-provoking findings, which further confirms the combinatorial nature of the category of representations of tame quivers.

MSC 2020. 16G20, 16G70.

Key words. Tree representations, exceptional modules, indecomposable modules

1. INTRODUCTION

Let k be an arbitrary field and Q a quiver. Recall that an indecomposable representation $M = (M_i, M_\alpha)$ of Q is a tree representation if its matrices M_α consist only of elements 0 and 1, such that the total number of non-zero elements is $d-1$, where d is the length of M. In [13] Ringel proves that every exceptional module has a tree representation, hence they are also called tree modules. One of the steps in the proof involves a choice of basis, which seems to depend on the underlying field. Ringel posed the question (see Problems 1. and 2. from Section 9. of [13]) whether there exist tree representations that are independent of this choice of basis, hence being "field independent". This problem remains open in general, but in some particular cases it has been settled: tree representations for the canonically oriented Euclidean quivers $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{D}}_m$ were given in [4] and [5], respectively, along with their appendix [6], where all the given representations were proven to be field independent, thus giving an affirmative answer to Ringel's question in these cases.

We note that the representations in the articles mentioned above were obtained by experimentation in $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and were not specifically constructed to be field independent. This is probably not a lucky coincidence, and makes us believe that every tree representation must be field independent.

The author was supported by the Collegium Talentum 2020 Programme of Hungary and would like to thank I. Szöllősi for his help in preparing this manuscript.

Corresponding author: Ábel Lőrinczi.

DOI: 10.24193/mathcluj.2023.1.09

The proof in [13] is based on a result by Schofield (see [14]), stating that if a non-simple module M is exceptional, then there are exceptional modules X and Y with the properties

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}(Y,X) = \mathrm{Ext}^1(X,Y) = 0$$

and an exact sequence of the following form:

$$0 \longrightarrow X^u \longrightarrow M \longrightarrow Y^v \longrightarrow 0,$$

where u and v are positive integers. There are precisely $s(M) - 1$ such sequences, where $s(M)$ is the number of nonzero components in $\underline{\dim} M$. We call these short exact sequences *Schofield sequence* and the pair (Y, X) a *Schofield pair* associated to M. The original proof of Schofield assumes an algebraically closed field, but Ringel later gave a proof in [12] which works for arbitrary fields.

Szántó and Szöllősi gave a full list of Schofield pairs for all exceptional modules in the Euclidean case in [16]. Moreover, they proved that if X, Y and M are exceptional indecomposables such that $u\underline{\dim}X + v\underline{\dim}Y = \underline{\dim}M$, then we have a Schofield sequence

$$0 \longrightarrow X^u \longrightarrow M \longrightarrow Y^v \longrightarrow 0$$

if and only if

$$\langle \underline{\dim}X, \underline{\dim}Y \rangle = 0.$$

Thus proving that Schofield sequences and pairs depend only on the dimension vectors of indecomposables, hence their existence is field independent, which further confirms Ringel's question.

All of the cases mentioned above reassured our initial feeling, that every tree representation is field independent. In order to tackle this question, we used computational methods to check whether every exceptional tree representation with dimension vector smaller than the minimal radical of the quiver is field independent or not.

An affirmative answer in general would reveal a deep combinatorial nature of the category of representations of tame quivers.

2. TREE REPRESENTATIONS

Let k be an arbitrary field, Q a quiver and consider its path algebra kQ. The category $\mathrm{mod}\, kQ$ of finite dimensional right modules over kQ can be identified with the category $\mathrm{rep}\, kQ$ of the finite dimensional k-representations of the quiver Q.

Recall that a k-representation of Q is the set of finite dimensional k-vector spaces $\{M_i \mid i \in Q_0\}$ associated to the vertices, together with k-linear maps $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ associated to the arrows (here Q_0 denotes the set of vertices and Q_1 is the set of arrows). Given two representations $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$, a morphism $f : M \to N$ is a family of k-linear maps $f_i : M_i \to N_i$, such that $N_\alpha f_{s(\alpha)} = f_{t(\alpha)} M_\alpha$ for all $\alpha \in Q_1$.

The dimension vector of a representation $M = (M_i, M_\alpha)$ is the vector

$$\underline{\dim} M = (d_i)_{i \in Q_0} \in \mathbb{Z}^n, \text{ where } d_i = \dim_k M_i.$$

In this case the length of M is $d = \sum_{i \in Q_0} d_i$.

The Euler form of Q is the bilinear form defined on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$ as

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

Its quadratic Tits form q_Q is independent of the orientation of Q and in the Euclidean case it is positive semidefinite with radical $\mathbb{Z}\delta$, where δ is called the minimal radical vector. The defect of $x \in \mathbb{Z}Q_0$ is then $\partial x = \langle \delta, x \rangle$. An indecomposable module with dimension vector x is preprojective, preinjective or regular if $\partial x < 0$, $\partial x > 0$, $\partial x = 0$ respectively.

If an indecomposable module M has no self-extensions, that is, if $\mathrm{Ext}^1 (M, M) = 0$, then it is called exceptional. In the Euclidean case the exceptional modules are exactly the indecomposable preprojective and preinjective ones and the indecomposable regulars, which lie on the non-homogeneous tubes with their dimension vector bounded by δ. Also for these modules we have that $\dim_k \mathrm{End}(M) = 1$. The dimension vector of these modules (which is a positive real root of the Tits form) will be called an exceptional root. For more details we refer to [1] and [15].

Below we recall the definition of a coefficient quiver.

Let $M = (M_i, M_\alpha)$ be a representation of a quiver Q and $B = \cup_{i \in Q_0} B(i)$ a collection of bases $B(i)$ of M_i. This set B is a basis of the vector space $\oplus_{i \in Q_0} M_i$ and we will call it a basis of M. The coefficient quiver $\Gamma(M, B)$ of M with respect to the basis B is a quiver, whose set of vertices is the set B and the arrows are defined in the following way. For every arrow $\alpha : i \to j$ of Q and every $b \in B(i)$ we expand

$$M_\alpha(b) = \sum_{b' \in B_j} c_{b'} b'$$

in the basis $B(j)$ of M_j and we put an arrow, denoted by α from b to $b' \in B(j)$ in $\Gamma(B, M)$ if the coefficient $c_{b'}$ is non-zero.

We will call an indecomposable representation M of Q over k a tree representation, provided there exists a basis B of M such that the coefficient quiver $\Gamma(M, B)$ is a tree. This definition is equivalent to M having exactly $d - 1$ non-zero equal to 1 in its matrices, while the remaining entries are 0, where d is the length of M.

Even though the proof presented in [13] by Ringel doesn't give an explicit method for constructing tree representations in general, in some particular cases they are known.

We mention here the influential paper [3] by Gabriel, where he gave a full list of indecomposable representations for the Dynkin quivers using $0-1$ matrices.

Excluding 4 of them, all the representations were tree representations. This list was later completed by Crawley-Boevey in [2].

Regarding the Euclidean case, Mróz gave a full list of the indecomposable tree representations for the quiver $\widetilde{\mathbb{D}}_4$ in [11]. In [10] Kussin and Meltzer described a method to explicitly determine the indecomposable preprojective and preinjective representations of $\widetilde{\mathbb{D}}_m$ and $\widetilde{\mathbb{E}}_6$ over an arbitrary field, but these representations are not tree representations in general. Later, in [9] Kędzierski and Meltzer generalized these results and gave a method for calculating indecomposable preprojective and preinjective representations of $\widetilde{\mathbb{E}}_8$ over any field and all indecomposable representations for algebraically closed fields. However these methods don't result in tree representations in general.

Using a computer generated proof, together with Sz. Lénárt and I. Szöllősi we managed to describe explicitly, in a field independent manner, all the exceptional tree representations in the case of the canonically oriented $\widetilde{\mathbb{E}}_6$ quiver in [4]. We also conjectured in that article that every tree representation of a Euclidean quiver is field independent.

We later gave a complete and general list corresponding to the exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\widetilde{\mathbb{D}}_6$ and a method to obtain tree representations for exceptionals in the canonically oriented general case $\widetilde{\mathbb{D}}_m$ from that list, see [5].

3. COMPUTATIONAL FINDINGS AND CONJECTURES

Let k be an arbitrary field, Q a Euclidean quiver, and x an exceptional root over Q. We introduce the following notation for the set of all tree representations having dimension vector x over k:

$$T_k(x) = \{\ M \in \operatorname{rep} kQ \mid \underline{\dim} M = x \text{ and } M \text{ is a tree representation } \}.$$

Remark 3.1. Recall that the notion of tree representation includes indecomposability.

Proposition 3.2. *Let Q denote a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$. Let x be an exceptional root over Q, smaller than the minimal radical δ. If we regard the matrices of the representations as formal 2-dimensional arrays of the symbols 0 and 1, then the set $T_k(x)$ has the same elements over any field, that is $T_k(x) = T_{k'}(x)$ for any two fields k and k'.*

Proof. We have checked this proposition using a specially developed software, implemented using the GAP computer algebra system (see [17]). The algorithm takes as input the quiver Q (represented through its set of arrows) and an exceptional root $x \in \mathbb{Z}^n$ where n is the number of vertices of Q. Then it performs the following steps:

(1) Using the backtracking method, it generates all the possible set of matrices $\{M_\alpha \mid \alpha \in Q_1\}$ where each matrix M_α consists only of the elements 1 and 0 and has maximal rank (if regarded over any field), moreover the total number of ones present in the matrices is $(\sum_{i\in Q_0} x_i) - 1$.

(2) For every such set of matrices it builds the corresponding representation $M = (M_i, M_\alpha)$ and checks for indecomposability using the fact that if x is a root of Q such that $x < \delta$ and $\underline{\dim} M = x$, then M is indecomposable if and only if $\dim \mathrm{End}(M) = 1$. This may be done by writing the matrix A of the homogeneous system of linear equations defining $\mathrm{End}(M)$ and showing that the co-rank of A is one (i.e. the solution space is one dimensional) – again, if regarded over any field.

(3) The "field independent" tree representations found are added to the set $T(x)$ along the way and this set is returned as the result at the end.

Note that both in steps (1) and (2) one has to check the ranks of a matrices in a "field independent way". In order to compute the rank of a matrix, it must be echelonized (brought to row or column echelon form) using elementary operations on rows and/or columns. This means that every single elementary operation used in the process of echelonizing the matrix must be such that the elements in the resulting matrix are either 0, 1 or -1 and the result is exactly the same if performed in any field k. For example (taken from [13]) if in the case of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ we perform the following elementary row operations, then we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

if performed in $\mathbb{R}$, or

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

if performed in $\mathbb{Z}_2$. Hence it has different ranks if considered over different fields. A crucial element of the algorithm is to ensure something like this never happens, but the result of every single elementary operation performed is formally the same matrix, independently of the field it is considered in.

Also note that in step (3) we do not filter out tree representations in any way: it just simply happens that all representations found are field independent. An error would be signalled upon finding the first representation containing a matrix with field dependent rank or with a field dependent co-rank of the linear system defining $\mathrm{End}(M)$. $\square$

As a result of the previous proposition we formulate the following conjecture:

Conjecture 3.3. Let x be an exceptional root over an arbitrary Euclidean quiver Q smaller than the minimal radical vector δ. If we regard the matrices of the representations as formal 2-dimensional arrays of the symbols 0 and 1, then the set $T_k(x)$ has the same elements over any field, that is $T_k(x) = T_{k'}(x)$ for any two fields k and k'.

In the case of the (computationally verified) quivers we could omit the index k and denote the set only as $T(x)$.

We provide the result of our computations (the contents of the sets $T(x)$) as an appendix to this article (see [7]), making it possible to verify the results of our computations using an independent implementation.

Remark 3.4. Note that this set is non-empty, since the existence of tree representations for every exceptional root is guaranteed by the theorem of Ringel. In addition, because every tree representation contains only $0-1$ matrices, this set is finite.

Let z be an exceptional root of the quiver Q and $Z \in T(z)$ a tree representation. We define the set $S(z)$, which will contain the pairs of dimension vectors of every (non-special) Schofield pair belonging to Z. More precisely:

$$\begin{aligned} S(z) = \{\ (x, y) \mid x, y \text{ are exceptional roots of } Q \text{ and } (Y, X) \text{ is a Schofield} \\ \text{pair belonging to } Z, \text{ where } Z \in T(z),\ Y \in T(y) \text{ and} \\ X \in T(x) \text{ with } \underline{\dim} X = x,\ \underline{\dim} Y = y,\ \underline{\dim} Z = z\ \} \end{aligned}$$

Note that while the representations $X, Y, Z \in \operatorname{mod} kQ$ exist within the context of a base field k, the conditions stated in Proposition 7 from [16] depend only on the value of the roots (dimension vectors), hence the set $S(z)$ may be used in a field independent context.

If the root z is smaller than the minimal radical vector δ, then we have only so-called non-special Schofield sequences of the form $0 \to X \to Z \to Y \to 0$ (see Propositions 7 and 9 from [16]) and the set $S(z)$ may be given in the following way:

$$S(z) = \{\ (x, y) \mid x, y \text{ are exceptional roots of } Q,\ x + y = z,\ \langle x, y \rangle = 0\ \}$$

In what follows we define a set of representations constructed using Schofield pairs. Let x and y be exceptional roots of the quiver Q and consider arbitrary tree representations $X \in T(x)$ and $Y \in T(y)$. We construct a new representation $R_{XY}^{\alpha ij}$, as follows ($\alpha \in Q_1$ and i, j being row respectively column indices in the upper right block of the matrix M_α):

$$R_{XY}^{\alpha ij} = (M_v, M_a)_{\substack{v \in Q_0 \\ a \in Q_1}} = \left((X_v \oplus Y_v)_{v \in Q_0}, \left(\begin{bmatrix} X_a & E_a^{ij} \\ 0 & Y_a \end{bmatrix} \right)_{a \in Q_1} \right)$$

where for the upper right block E_a^{ij} is true that $E_a^{ij} = 0$ for $a \neq \alpha$ and E_α^{ij} contains exactly one non-zero entry 1 in the ith row and jth column and it is zero elsewhere. Using this notaion we introduce the following the set $E_k(x,y) \subseteq \operatorname{mod} kQ$:

$$E_k(x,y) = \{ \, R^{\alpha ij}_{XY} \mid \alpha \in Q_1, \; i,j \text{ are row resp. column indices}, \\ X \in T_k(x), \; Y \in T_k(y), \; R^{\alpha ij}_{XY} \in T_k(x+y) \, \}$$

For given tree representations X and Y, the representation $R^{\alpha ij}_{XY}$ is the construction given by Ringel in Section 6 of [13]. As mentioned there, the position of the single nonzero entry specified by α, i and j involves a choice of basis and could very well depend on the base field k. To our surprise, however, this seems not to be the case:

PROPOSITION 3.5. *Let Q denote a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$. Let x and y be exceptional roots over Q, smaller than the minimal radical δ. If we regard the matrices of the representations as formal 2-dimensional arrays of the symbols 0 and 1, then the set $E_k(x,y)$ has the same elements over any field, that is $E_k(x,y) = E_{k'}(x,y)$ for any two fields k and k'.*

Proof. This statement was checked computationally using the following "brute-force" method: for any given representations $X \in T(x)$, $Y \in T(y)$ construct all the possible representations $R^{\alpha ij}_{XY}$ and check whether $R^{\alpha ij}_{XY} \in T(x+y)$ with the field independent rank computation described after Proposition 3.2. $\square$

Based on our findings we conjecture that Proposition 3.5 holds for arbitrary tame quivers and exceptional roots. In the case of the (computationally verified) quivers we could omit the index k and denote the set only as $E(x,y)$.

Further advancing with our "computational inquiry" into the problem of field independence we may ask for a method to construct the set of tree representations, other than the "exhaustive search" we have performed. Ringel in his proof used Schofield induction to construct tree representations (see Section 6. of [13]), and we may ask the question whether there are other methods for obtaining them, or does his construction result in every possible tree representation. Permuting the basis vectors is a field independent operation, so we introduce the following:

DEFINITION 3.6. Let $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$ be representations of a quiver Q. Then we call them *permutation-similar*, provided there exists a family of permutation matrices $\{A_i \mid i \in Q_0\}$ such that the following diagram is commutative for every arrow $\alpha \in Q_1$:

$$\begin{tikzcd} M_i \arrow[r, "M_\alpha"] \arrow[d, "A_i"] & M_j \arrow[d, "A_j"] \\ N_i \arrow[r, "N_\alpha"] & N_j \end{tikzcd}$$

REMARK 3.7. Note that M and N are isomorphic representations over any field, and permutation-similarity defines an equivalence relation. Moreover, if we have a tree representation M, then every representation permutation-similar to M is also tree.

Let $Z \in T(z)$ be a tree representation, we denote by $\pi(Z)$ the set of all representations that are permutation-similar to Z. From Remark 3.7 it follows that every element of the set $\pi(Z)$ is a tree representation.

Using the notations introduced above, we state the following proposition, giving a method to inductively construct the sets of tree representations:

PROPOSITION 3.8. *Let z be an exceptional root of a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_4$, $\widetilde{\mathbb{D}}_5$ or $\widetilde{\mathbb{E}}_6$, such that $z < \delta$. Then we have*

$$T(z) = \bigcup_{\substack{(x,y)\in S(z)\\ Z\in E(x,y)}} \pi(Z).$$

Proof. Using the sets $T(x)$ already computed, this can be done by direct verification. One can also use the list given in the Appendix of [16] for obtaining the sets $S(z)$. Also note that we have $\underline{\dim} Z = z$ in the statement of the proposition above. □

We conjecture that Proposition 3.8 also holds true for every exceptional root of any Euclidean quiver.

REMARK 3.9. There is nothing “special” about the minimal radical vector δ of the quiver Q. We have chosen δ as an upper limit for our computations for practical reasons: running time and because in this way we have tackled all the non-homogeneous regulars. Given more time and computer resources we could run our algorithms for bigger dimension vectors and quivers of type $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. However, the cases verified gave us enough confidence in the conjectures stated in this section, and we believe further efforts should be directed towards a “theoretical” proof of the propositions (in general).

REFERENCES

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Mathematical Society Student Texts, Vol. 65, Cambridge University Press, Cambridge, 2006.

[2] W. W. Crawley-Boevey, *Matrix Problems and Drozd's Theorem*, Banach Center Publ., **26** (1990),199–222.

[3] P. Gabriel, *Unzerlegbare Darstellungen I.*, Manuscripta Math., **6** (1972), 71–103.

[4] Sz. Lénárt, Á. Lőrinczi and I. Szöllősi, *Tree representations of the quiver* $\widetilde{\mathbb{E}}_6$, Colloq. Math., **164** (2021), 221–250.

[5] Sz. Lénárt, Á. Lőrinczi, Cs. Szántó and I. Szöllősi, *Tree representations of the quiver* $\widetilde{\mathbb{D}}_m$, Colloq. Math., **167** (2022), 261–302.

[6] Sz. Lénárt, Á. Lőrinczi, Cs. Szántó and I. Szöllősi, *Proof of the tree module property for exceptional representations of tame quivers*, arXiv:2001.00016.

[7] Á. Lőrinczi, www.math.ubbcluj.ro/∼lorinczi/trees.zip.
[8] Á. Lőrinczi and Cs. Szántó, *The indecomposable preprojective and preinjective representations of the quiver* $\widetilde{\mathbb{D}}_n$, Mathematica, **57 (80)** (2015), 54–66.
[9] D. Kędzierski and H. Meltzer, *Indecomposable representations for extended Dynkin quivers of type* $\widetilde{\mathbb{E}}_8$, Colloq. Math., **124** (2011), 95–116.
[10] D. Kussin and H. Meltzer, *Indecomposable representations for extended Dynkin quivers*, arXiv:math/0612453.
[11] A. Mróz, *The dimensions of the homomorphism spaces to indecomposable modules over the four subspace algebra*, arXiv:1207.2081.
[12] C. M. Ringel, *Exceptional objects in hereditary categories*, in *Proceedings: 82 References Representation Theory of Groups, Algebras, and Orders, September 25 - October 6, 1995, Constanta*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **2** (1996), 150–158.
[13] C. M. Ringel, *Exceptional modules are tree modules*, Linear Algebra Appl., **275–276** (1998), 471–493.
[14] A. Schofield, *Semi-Invariants of Quivers*, J. Lond. Math. Soc., **s2-43** (1991), 385–395.
[15] D. Simson and A. Skowroński, *Elements of Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type*, London Mathematical Society Student Texts, Vol. 70, Cambridge University Press, Cambridge, 2007.
[16] Cs. Szántó and I. Szöllősi, *Schofield sequences in the Euclidean case*, J. Pure Appl. Algebra, **225** (2021), 1–123.
[17] The GAP Group, *GAP – Groups, Algorithms and Programming*, Version 4.11.1, 2021, http://gap-system.org.

Received March 1, 2022
Accepted July 3, 2022

Babeş–Bolyai University
Faculty of Mathematics and Computer Science
Department of Mathematics
Cluj-Napoca, Romania
E-mail: `abel.lorinczi@ubbcluj.ro`