MATHEMATICA, 65 (88), N° 1, 2023, pp. 66–74

A NEW APPROACH TO MULTIPLICATION MODULES VIA (δ-)SMALL SUBMODULES

MOTAHAREH IRANI, YAHYA TALEBI, and ALI REZA MINIRI HAMZEKOLAEE

Abstract. Let R be a commutative ring and M an R-module. In this work we introduce two new generalizations of multiplication modules via δ-small submodules and small submodules of a fixed module. A module M is said to be (δ-)small multiplication provided for every (δ-)small submodule N of M, there is an ideal I of R such that $N = IM$. We study some general properties of both δ-small multiplication modules and also small multiplication modules. A counterexample is presented to state the fact that the class of all δ-small multiplication modules lies exactly between the class of multiplication modules and small multiplication modules. We show that any direct summand of a (δ-)small multiplication module inherits the property.

MSC 2020. 16D10, 16D40, 16D80.

Key words. Small submodule, multiplication module, small multiplication module, δ-small submodule, δ-small multiplication module.

1. INTRODUCTION

Throughout this paper R denotes an arbitrary commutative ring with identity and all modules are unitary R-modules. Let M be an R-module and N a submodule of M. Then N is said to be *small* in M denoted by $N \ll M$ in case $M = N + K$ implies $M = K$. A module M is called hollow provided every proper submodule of M is small in M. Examples of hollow modules contains linearly ordered modules such as the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_{p^n}$.

Let M be a module over a ring R. Then M is called *multiplication* in case for every submodule N of M, there exists an ideal I of R such that $N = MI$. General properties and some of nice characterizations of multiplication modules were studied and investigated in [2]. It is proved that a module M is multiplication if and only if for each cyclic submodule mR of M, there is an I ideal of R with $mR = MI$. It is also shown that a finitely generated module M is multiplication if and only if M/JM is multiplication. An important fact in [2] states that a direct sum of modules M_i for $i \in I$ is multiplication if and

The authors thank the referee for his helpful comments and suggestions.
Corresponding author: Yahya Talebi.

DOI: 10.24193/mathcluj.2023.1.07

only if each M_i is multiplication and for each $i \in I$ there exists an ideal λ_i of R such that $M_i = M_i\lambda_i$ and $(\oplus_{j\neq i}M_j)\lambda_i = 0$.

Multiplication modules ant their generalizations also studied by several authors. For more information about these modules we may refer the readers to [3, 5].

Motivated by mentioned works on multiplication modules, in this work we are interested in studying these modules via small submodules and also δ-small submodules. As a generalization of small submodules, a submodule K of M is called (δ-)*small* in M in case $M = K + L$ with M/L singular implies that $M = L$. In fact, instead of any submodule of a module, we may choose just (δ-)small submodules of that module. By the way, we call a module M, (δ-)small multiplication in case for every (δ-)small submodule N of a module M, there is an ideal of R such that $N = MI$. We show that the class of all small multiplication modules contains properly the class of all multiplication modules. Note that a δ-small multiplication module is a small multiplication module. So, by presenting a counterexample we show that the converse may not true. In section 2, we study some general properties of small multiplication modules. We prove that under an assumption, a homomorphic image of a small multiplication module inherits the property. We also show that a direct summand of a small multiplication module is small multiplication. It is also proved that a direct summand of a small multiplication module inherits the property.

In section 3, we introduce a new class of modules via the concept of singular modules which lies between the class of multiplication modules and small multiplication modules. We say that a module M is δ-small multiplication in case for every δ-small submodule K of M, there is an ideal I of R such that $K = MI$. Some examples of such modules are provided. We prove that a direct summand of a δ-small multiplication module is δ-small multiplication. We also prove that if M is δ-small multiplication and $N \ll_\delta M$, then M/N is δ-small multiplication.

2. MULTIPLICATION MODULES VIA SMALL SUBMODULES

In last decades multiplication modules and their various generalizations have been studied extensively and their properties have been fully investigated. As the concept of small submodules plays a key role in module theory and also multiplication modules are valuable for studying, in this work we are interested in introducing a new generalization of multiplication modules via small submodules.

DEFINITION 2.1. Let M be a module. Then we call M, small multiplication in case for every small submodule N of M, there is an ideal I of R such that $N = MI$.

We denote $\mathrm{Rad}(M)$ as radical of module M.

EXAMPLE 2.2. (1) Every module M with $\mathrm{Rad}(M) = 0$, is obviously small multiplication. Therefore, any semisimple module is small multiplication. In particular, every module over a V-ring is small multiplication.

(2) Let $M = \mathbb{Q}$ as an $\mathbb{Z}$-module and N a finitely generated submodule of M. It is known that N is a small submodule of M. If there is an ideal $n\mathbb{Z}$ of $\mathbb{Z}$ such that $N = n\mathbb{Q}$, then $N = M$ which is a contradiction. It follows that M can not be small multiplication. Generally, over a principal ideal domain R an injective R-module M with $\mathrm{Rad}(M) \neq 0$ can not be small multiplication since for every $x \in \mathbb{R}$, $xM = M$.

(3) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_n$ where n is not a square-free natural number. Then for every submodule N of M, there is an integer $x \in \mathbb{N}$ such that $N = x\mathbb{Z}M$. It follows that M is small multiplication.

(4) Recall that a module M is hollow in case, every proper submodule of M is small in M. By the way, for a hollow module M the two concepts small multiplication and multiplication, coincide. In particular, any injective $\mathbb{Z}$-module neither multiplication nor small multiplication by (2).

The following example introduces a large class of small multiplication modules while they are not multiplication.

EXAMPLE 2.3. Any semisimple module M over a local ring R is small multiplication while M is not a multiplication module. As M is semisimple, then $M \cong \oplus_{i\in I} R/J(R)$ for an indexed set I. Suppose that N is a nonzero submodule of M. If M is multiplication, then there is an ideal I of R such that $K = I(\oplus_{i\in I} R/J(R)) \cong N$. Then $K = \oplus_{i\in I} I(R/J(R)) = \oplus_{i\in I}(IR + J(R))/J(R) = \oplus_{i\in I} J(R)/J(R) = 0$. which is a contradiction (note that R is local and I must be contained in $J(R)$). This should be noticed that M is small multiplication by Example 2.2.

LEMMA 2.4. *Let M be a Noetherian module such that for every cyclic submodule xR there is an ideal I of R such that $xR = IM$. Then M is small multiplication. In particular, every finitely generated $\mathbb{Z}$-module is small multiplication.*

Proof. Let M be Noetherian and N a small submodule of M. Then $N = x_1R + \ldots + x_tR$. By assumption, for each x_i there is an ideal J_i of R such that $x_iR = J_iM$. Now, $N = J_1M + \ldots + J_tM = (J_1 + \ldots + J_t)M$. This completes the proof. □

For a subset X of a module M over a ring R, the ideal $\{r \in R | X.r = 0\}$ is called the annihilator of X in R; it is denoted by $\mathrm{Ann}_R(X)$.

For two subsets X and Y of a module M over a ring R, the subset $\{r \in R | X.r \subseteq Y\}$ of R is denoted by $(Y : X)$. If Y is a submodule of M, then it is directly verified that for any subset X of M, the set $(Y : X)$ is a ideal of R.

PROPOSITION 2.5. *For a module M, the following conditions are equivalent:*
(1) *M is small multiplication;*

(2) *For every small submodule N of M, we have $N \subseteq M(N : M)$;*

(3) *For each small submodule N of M, we have $N = M(N : M) = M\mathrm{Ann}_R(M/N)$.*

Proof. (1) $\Rightarrow$ (2) Since M is small multiplication then for every small submodule N of M there exist an ideal I of R such that $N = IM$. It follows that $I \subseteq (N : M)$. Therefore, $N = IM \subseteq (N : M)M$.

(2) $\Rightarrow$ (3) Let N be a small submodule of M and $x = \sum_{i=1}^{n} x_i r_i \in M(N : M)$ such that $x_i \in M$ and $r_i \in (N : M)$. As $r_i \in (N : M)$, we conclude that $x_i r_i \in N$. It follows that x belongs to N. Hence $N = M(N : M)$. Note also that $(N : M) = \mathrm{Ann}_R(M/N)$.

(3) $\Rightarrow$ (1) Take $(N : M)$ as the ideal related to N. □

Proposition 2.6. *For a right module M over a ring R, the following conditions are equivalent:*

(1) *M is small multiplication;*

(2) *For every ideal I of R such that $I \subseteq \mathrm{Ann}_R(M)$, the R/I-module M is small multiplication;*

(3) *There exist an ideal I of R such that $I \subseteq \mathrm{Ann}_R(M)$ and M is a small multiplication R/I-module.*

Let R be a ring and M be an R-module. Recall that a submodule L of M is called *fully invariant* provided $\varphi(L) \subseteq L$ for every endomorphism φ of M. Clearly 0 and M are fully invariant submodules of M. Note that since R is a commutative ring, each submodule of the R-module R is fully invariant.

Proposition 2.7. *Let M be a small multiplication module and N a small submodule of M. Then M/N is small multiplication.*

Proof. Let M be a small multiplication module and N a small submodule of M. Suppose that K/N be a small submodule of M/N where K is a submodule of M containing N. As $N \ll M$, it follows that $K \ll M$. Being M a small multiplication module implies that there is an ideal I of R such that $K = MI$. It is not hard to check that $K/N = (M/N)I$. Hence M/N is small multiplication. □

Proposition 2.8. *Let M be a small multiplication module. Then:*

(1) *Every small submodule of M is a fully invariant submodule of M.*

(2) *Let N be a submodule of M such that $N \cap MI = NI$ for every ideal I of R. Then N is a small multiplication module.*

Proof. (1) Suppose that N is a small submodule of M and φ is an endomorphism of M. Being M small multiplication implies existing an ideal I of R such that $N = MI$. It follows that $\varphi(N) = \varphi(IM) = \varphi(M)I \subseteq MI = N$. This shows that N is fully invariant.

(2) Let N be a submodule of M with stated property and $K \ll N$. Note that K will be a small submodule of M, as well. As M is small multiplication,

there exists an ideal I of R such that $K = MI$. Therefore, $K = MI = K \bigcap MI \subseteq N \bigcap MI = NI \subseteq MI = K$. In fact, $K = NI$, as required. □

Let M be a small multiplication module that does not contain a nontrivial fully invariant submodule. Then the only small submodule of M is zero submodule by Proposition 2.8. In fact, $\mathrm{Rad}(M) = 0$.

Corollary 2.9. *Every direct summand of a small multiplication module, inherits the property.*

Proof. Suppose that M is a small multiplication module and N a direct summand of M. Set $M = N \oplus N'$. Then for each ideal I of R, $MI = NI \oplus N'I$. It follows that $N \cap MI = N \cap (NI \oplus N'I)$, which implies $N \cap MI = NI$. The rest follows from Proposition 2.8.

□

Proposition 2.10. *Let M be a small multiplication R-module and P is an ideal of R such that $M \neq MP$. If for each cyclic submodule Y of M, we have $\mathrm{Rad}(Y) \neq 0$, then there exists a cyclic submodule X of M such that $P \nsubseteq \mathrm{Ann}_R(M/X)$.*

Proof. Suppose that M be an R-module with stated properties. As $M \neq MP$, there exist a cyclic submodule X of M that is not contained in the module MP. Since M is small multiplication, there exists an ideal I of R such that $X = MI$. As MP does not include X, we conclude that $I \nsubseteq P$. Note also that $X = IM$ implies $I \subseteq \mathrm{Ann}_R(M/X)$. If in the contrary, $P \supseteq \mathrm{Ann}_R(M/X)$, then $MP \supseteq X$ causing a contradiction. □

Proposition 2.11. *For an R-module M the following conditions are equivalent;*

(1) *M is small multiplication;*

(2) *For every cyclic small submodule Y of M, there exists an ideal B of R such that $X = MB$;*

(3) *For every small submodule X of M, there exists a set $\{X_i\}_{i\in I}$ of submodules of X and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = MB_i$ for each $i \in I$.*

Proof. The implication (1) ⇒ (2) is obvious.

(2) ⇒ (3) Let X be a small submodule of M. Then each cyclic submodule of X is a small submodule of M. Note that $X = \sum_{y\in X} yR$. By (2), for each $y \in X$, there exists an ideal B_y of R such that $yR = MB_y$. This completes the proof.

(3) ⇒ (1) Consider an arbitrary small submodule X of M and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = MB_i$ for each $i \in I$. We denote by B the sum of all ideals B_i of R. Then $X = \sum_{i\in I} X_i = \sum_{i\in I} MB_i = M(\sum_{i\in I} B_i) = MB$. Hence M is small multiplication. □

The following presents an easy characterization of small multiplication modules.

Lemma 2.12. *Let M be an R-module. Then M is small multiplication if and only if for every small submodule N of M and each $m \in M$, there is an ideal I of R such that $N \cap mR = MI$.*

Proof. Suppose that M is small multiplication, $N \ll M$ and $m \in M$. Then $N \cap mR$ is a small submodule of M, as well as N. So that will be an ideal I of R such that $N \cap mR = MI$. For the converse, let N be a small submodule of M. Then for each $x \in N$, there is an ideal I_x of R such that $xR = MI_x$. This can be easily verified that $N = \sum_{x \in N} xR = MI$ where I is an ideal of R which is the sum of all ideas I_x. $\square$

Remark 2.13. Let R be a ring with just one non-trivial ideal I. Let M be a small multiplication R-module and let N be a small submodule of M. Consider an arbitrary $0 \neq x \in N$. Then xR is small in M as well as N. Since R has just one nontrivial ideal I combining with M is small multiplication, we conclude that $xR = MI = N$. This shows that N is a simple submodule of M. In particular, for any small multiplication module over the ring $R = \mathbb{Z}_{p^2}$, every small submodule is simple.

Proposition 2.14. *Let M be an Artinian small multiplication module and J is jacobson radical of R. Then the following assertions hold.*

(1) *If $JM \ll M$, then the module M/JM is a semisimple small multiplication module.*

(2) *If JM is small submodule of M and M/JM is cyclic, then M is a cyclic module.*

(3) *If M is a finitely generated module, then M is a cyclic module.*

Proof. (1) The module M/JM is a small multiplication module by Proposition 2.7. Since M/JM is a semiprimitive Artinian module, M/JM is a finitely generated semisimple module.

(2) Since M/JM is a cyclic R-module, there exists a cyclic submodule X of M such that $M = X + JM$. By assumption JM is a small submodule of M. Therefor, $M = X$.

(3) Being M a finitely generated module implies $JM \ll M$. Now by (2), M is a cyclic module. $\square$

3. MULTIPLICATION MODULES AND δ-SMALL SUBMODULES

In last section we try to make a connection between the definition of a multiplication module and the concept of small submodules. It can be of interest that we consider the δ-small submodules instead of small submodules. By the way, it should be noted that any small submodule of a module is δ-small in that module. Zhou in [7] introduced the concept of δ-small submodules. A submodule N of M is said to be δ-small in M (denoted by $N \ll_\delta M$) provided $M \neq N + K$ for any proper submodule K of M with M/K singular. General properties and some useful characterizations of δ-small submodules of a module have been provided in [7]. The sum of all δ-small submodules of M

is denoted by $\delta(M)$. Also $\delta(M)$ is the reject of the class of all simple singular modules in M.

By the way, we are interested in introducing a new generalization of multiplication modules via δ-small submodule. We say that a module M is δ-small multiplication provided for every δ-small submodule K of M, there is an ideal I of R such that $K = MI$.

It is obvious that every δ-small multiplication module is small multiplication. So we have the following hierarchies:

multiplication module $\Rightarrow$ δ-small multiplication module $\Rightarrow$ small multiplication module

Example 3.1. (1) Let M be a module with $\delta(M) = 0$. Then the only δ-small submodule of M is zero submodule and we have $0 = M0$. Hence M is δ-small multiplication.

(2) Let M be a δ-hollow module, i.e. every proper submodule of M is δ-small in M. Then M is δ-small multiplication if and only if M is multiplication. In particular, for a hollow module the three concepts δ-small multiplication, small multiplication and multiplication coincide.

(3) For a singular module M, we have M is δ-small multiplication if and only if M is small multiplication, since for a singular module M, a submodule N is δ-small in M if and only if N is small in M.

(4) Generally, over a principal ideal domain R, an injective (divisible) R-module M with $\delta(M) \neq 0$ can not be δ-small multiplication since for every $x \in \mathbb{R}$, $xM = M$.

We present an example of a small multiplication module which is not δ-small multiplication showing that the class of all small multiplication modules contains properly the class of all δ-small multiplication modules.

Example 3.2. Let F be a field. Set $M = \oplus_{i=1}^{\infty} F$. Then M is a semisimple projective F-module and $\mathrm{Rad}(M) = 0$. Hence M is a small multiplication F-module. Consider the submodule $N = F \oplus F$ of M. It is not hard to check that N is a δ-small submodule of M. If there is an ideal I of F such that $N = MI$, then $N = 0$ or $N = M$ that both of them contradict choosing N. It follows that M is not δ-small multiplication.

Lemma 3.3. *Let M be an R-module. Then M is δ-small multiplication if and only if for every δ-small submodule N of M and each $m \in M$, there is an ideal I of R such that $N \cap mR = MI$.*

Proof. The proof is the same as the proof of Lemma 2.12. □

An easy characterization of δ-small multiplication modules via the annihilator of modules is presented below.

Proposition 3.4. *For a module M, the following conditions are equivalent:*
(1) *M is δ-small multiplication;*

(2) *For every δ-small submodule N of M, we have $N \subseteq M(N : M)$;*

(3) *For each δ-small submodule N of M, we have $N = M(N : M) = M\mathrm{Ann}_R(M/N)$.*

Proof. (1) $\Rightarrow$ (2) Since M is δ-small multiplication then for every δ-small submodule N of M there exist an ideal I of R such that $N = IM$. It follows that $I \subseteq (N : M)$. Therefore, $N = IM \subseteq (N : M)M$.

(2) $\Rightarrow$ (3) Let N be a δ-small submodule of M and $x = \sum_{i=1}^{n} x_i r_i \in M(N : M)$ such that $x_i \in M$ and $r_i \in (N : M)$. As $r_i \in (N : M)$, we conclude that $x_i r_i \in N$. It follows that x belongs to N. Hence $N = M(N : M)$. Note also that $(N : M) = \mathrm{Ann}_R(M/N)$.

(3) $\Rightarrow$ (1) Take $(N : M)$ as the ideal related to N. □

Let M be a module $N \leq K \leq M$. If K/N is a δ-small submodule of M/N and $N \leq_\delta M$, then it can be easily shown that $K \leq_\delta M$.

Proposition 3.5. *Let M be a δ-small multiplication module and a N a δ-small submodule of M. Then M/N is δ-small multiplication.*

Proof. Let M be a δ-small multiplication module and N a δ-small submodule of M. Suppose that K/N is a δ-small submodule of M/N where K is a submodule of M containing N. As $N \ll_\delta M$, it follows that $K \ll_\delta M$. Being M a δ-small multiplication module implies that there is an ideal I of R such that $K = MI$. It is not hard to check that $K/N = (M/N)I$. Hence M/N is δ-small multiplication. □

Proposition 3.6. *Suppose that M is a δ-small multiplication module. Then the following hold.*

(1) *Every δ-small submodule of M is a fully invariant submodule of M.*

(2) *Let N be a submodule of M such that $N \cap MI = NI$ for every ideal I of R. Then N is a small multiplication module. For instance, every direct summand of a δ-small multiplication module, inherits the property.*

Proof. (1) Let $N \leq_\delta M$ and φ be an endomorphism of M. Being M δ-small multiplication implies existing an ideal I of R such that $N = MI$. It follows that $\varphi(N) = \varphi(IM) = \varphi(M)I \subseteq MI = N$. This shows that N is fully invariant.

(2) Let N be a submodule of M such that $N \cap MI = NI$ for every ideal I of R. Suppose that K is a δ-small submodule of N. Note that K will be a δ-small submodule of M, as well. As M is δ-small multiplication, there exists an ideal I of R such that $K = MI$. Therefore, $K = MI = K \bigcap MI \subseteq N \bigcap MI = NI \subseteq MI = K$. In fact, $K = NI$, as required. For the latter, set $M = N \oplus N'$ where $N' \leq M$. Then for each ideal I of R, $MI = NI \oplus N'I$. It follows that $N \cap MI = N \cap (NI \oplus N'I)$, which implies $N \cap MI = NI$. Hence N is δ-small multiplication. □

Let M be a module with no nontrivial fully invariant submodule. If M is δ-small multiplication, then either $\delta(M) = 0$ or M is a semisimple projective module. For, if N is a δ-small submodule of M, then $N = 0$ or $N = M$. Note that $M \ll_\delta M$ implies M is semisimple projective. Especially, the $\mathbb{Z}$-module $\mathbb{Q}$ can not be δ-small multiplication (note that $\mathbb{Q}$ is neither semisimple projective nor $\delta(\mathbb{Q}) = 0$.

PROPOSITION 3.7. *For an R-module M the following conditions are equivalent;*

(1) *M is δ-small multiplication;*

(2) *For every cyclic δ-small submodule Y of M, there exists an ideal B of R such that $X = MB$;*

(3) *For every δ-small submodule X of M, there exists a set $\{X_i\}_{i\in I}$ of submodules of X and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = MB_i$ for each $i \in I$.*

Proof. Similar to the proof of Proposition 2.11. □

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.

[2] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, **16** (1988), 755–779.

[3] S. Ebrahimi Atani, S. Khojasteh and G. Ghaleh, *On multiplication modules*, Int. Math. Forum, **1** (2006), 1175–1180.

[4] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Mathematical Society Lecture Note Series, Vol. 147, Cambridge, University Press, 1990.

[5] A. A. Tuganbaev, *Multiplication modules*, J. Math. Sci. (N.Y.), **123** (2004), 3839–3905.

[6] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[7] Y. Zhou, *Generalizations of perfect, semiperfect, and semiregular rings*, Algebra Colloq., **7** (2000), 305–318.

Received September 29, 2021
Accepted March 26, 2022

University of Mazandaran
Department of Mathematics
Babolsar, Iran
E-mail: motahareh.irani64@yahoo.com
E-mail: talebi@umz.ac.ir
E-mail: a.monirih@umz.ac.ir