MATHEMATICA, 65 (88), N° 1, 2023, pp. 60–65

ON THE RECURRENCES OF THE JACOBSTHAL SEQUENCE

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Abstract. In the present work, two new recurrences of the Jacobsthal sequence are defined. Some identities of these sequences which we call the Jacobsthal array is examined. Also, the generating and series functions of the Jacobsthal array are obtained.

MSC 2020. 11B39, 05A15, 11B83.

Key words. Fibonacci numbers, Jacobsthal Numbers, Binet like formula, generating function, series, sums.

1. INTRODUCTION

The Fibonacci sequence, one of the most famous number sequences in mathematics, has many applications. It has also undergone many generalizations until today. One of these generalizations is the Jacobsthal sequence. The Jacobsthal numbers are an integer sequence named after the German mathematician Ernst Jacobsthal. The Jacobsthal sequence is an additive sequence similar to the Fibonacci sequence, have many interesting properties and applications to almost every fields of science, nature and art. The Jacobsthal sequence has charming applications to combinatorics, graph theory, and number theory. There are many studies on the Jacobsthal sequence and its generations (for details see [1, 5–12, 14, 15, 17–20]). The Jacobsthal sequence $\{J_n\}_{n\geq 0}$ is defined by the initial values $J_0 = 0$ and $J_1 = 1$ and the recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n, \qquad n \geq 0. \tag{1}$$

The first few terms of this sequence are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341$.

The relation (1) involves the characteristic equation

$$x^2 - x - 2 = 0 \tag{2}$$

with roots

$$\alpha = 2 \quad \text{and} \quad \beta = -1 \tag{3}$$

so that,

$$\alpha + \beta = 1, \quad \alpha - \beta = 3 \quad \text{and} \quad \alpha\beta = -2. \tag{4}$$

The authors thank the referee for his helpful comments and suggestions.

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DOI: 10.24193/mathcluj.2023.1.06

The Binet formula of the Jacobsthal sequence is

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{5}$$

The Cassini-like formula of the Jacobsthal numbers is given [10] by

$$J_{n+1}J_{n-1} - J_n^2 = -(-2)^{n-1}. \tag{6}$$

Some identities of the Jacobsthal numbers are as follows (see [3, 4, 13, 16]):

$$J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n \tag{7}$$

$$J_{2n} = J_n J_{n+1} + 2J_{n-1}J_n \tag{8}$$

$$J_{2n+1} = J_{n+1}^2 + 2J_n^2. \tag{9}$$

2. RECURRENCES OF THE JACOBSTHAL SEQUENCE

A Fibonacci array was defined in [2] by Carlitz and some of their identities were examined. In this section, we defined the recurrences of the Jacobsthal sequence and examined their some properties.

A Jacobsthal array $\{j_{m,n}\}_{m\geq 0, n\geq 0}$ is defined by the two recurrences

$$j_{m,n} = j_{m,n-1} + 2j_{m,n-2}, \quad n \geq 2 \tag{10}$$

$$j_{m,n} = j_{m-1,n} + 2j_{m-2,n}, \quad m \geq 2, \tag{11}$$

where

$$j_{0,n} = J_n, \quad j_{1,n} = J_{n+2} \tag{12}$$

are the 0-th and 1-th rows of the Jacobsthal array, respectively.

The following table is readily computed:

$m \diagdown n$	0	1	2	3	4	5	6	7
0	0	1	1	3	5	11	21	43
1	1	3	5	11	21	43	85	171
2	1	5	7	17	31	65	127	257
3	3	11	17	39	73	151	297	599
4	5	21	31	73	135	281	451	1013
5	11	43	65	151	281	583	1145	2311
6	21	85	127	297	451	1145	2047	4337
7	43	171	257	599	1013	2311	4337	8959

Table 1 – The first few members of the Jacobsthal array

As it can be seen from the table, the symmetry property

$$j_{m,n} = j_{n,m}$$

is readily proved by making use of (10) and (11).

Proposition 2.1. *The following identity is valid:*

$$j_{m,n} = 2J_{m-1}J_n + J_mJ_{n+2}. \tag{13}$$

Proof. The relationship of the Jacobsthal arrays with the Jacobsthal number is obtained by using (11) and (12). □

Theorem 2.2. *The Binet-like formula for the Jacobsthal array is*

$$j_{m,n} = \frac{5\alpha^{m+n} - 2\alpha^m\beta^n - 2\alpha^n\beta^m - \beta^{m+n}}{9}.$$

Proof. Considering (3), (4), (5) and (13) we write

$$\begin{aligned}
j_{m,n} &= 2J_{m-1}J_n + J_mJ_{n+2}\\
&= 2\left(\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta}\right)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right)\left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right)\\
&= 2\left(\frac{\alpha^{m+n-1} - \alpha^{m-1}\beta^n - \alpha^n\beta^{m-1} + \beta^{m+n-1}}{9}\right)\\
&+ \left(\frac{\alpha^{m+n+2} - \alpha^m\beta^{n+2} - \alpha^{n+2}\beta^m + \beta^{m+n+2}}{9}\right)\\
&= \left(\frac{\alpha^{m+n} - \alpha^m\beta^n + 2\alpha^n\beta^m - 2\beta^{m+n}}{9}\right)\\
&+ \left(\frac{4\alpha^{m+n} - \alpha^m\beta^n - 4\alpha^n\beta^m + \beta^{m+n}}{9}\right)\\
&= \frac{5\alpha^{m+n} - 2\alpha^m\beta^n - 2\alpha^n\beta^m - \beta^{m+n}}{9}.
\end{aligned}$$

□

Proposition 2.3. *The following identities are valid:*

1. $j_{m+1,m-1} - j_{m,m} = (-2)^m$,
2. $j_{m,n} = J_{m+n} + 2J_mJ_n$,
3. $j_{n,n} = J_{2n} + 2J_n^2$,
4. $j_{n+1,n} = J_{2n+1} + 2J_{n+1}J_n$.

Proof. 1. Considering (13) and (6) we have

$$\begin{aligned}
j_{m+1,m-1} - j_{m,m} &= 2J_mJ_{m-1} + J_{m+1}J_{m+1} - 2J_{m-1}J_m - J_mJ_{m+2}\\
&= J_{m+1}J_{m+1} - J_mJ_{m+2}\\
&= (-2)^m.
\end{aligned}$$

2. Considering (13) and (7) we get

$$\begin{aligned}
j_{m,n} &= 2J_{m-1}J_n + J_mJ_{n+2}\\
&= 2J_{m-1}J_n + J_m(J_{n+1} + 2J_n)\\
&= J_{m+n} + 2J_mJ_n.
\end{aligned}$$

3. Considering (13) and (8), for $m = n$, the identity is proved.
4. Considering (13) and (9), for $m = n + 1$, the identity is obtained.

□

THEOREM 2.4. *The generating function of the Jacobsthal array is*

$$G_j(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} j_{m,n} x^n y^m = \frac{xy + y + x}{(1 - x - 2x^2)(1 - y - 2y^2)}.$$

Proof. Let

$$j_m(x) = \sum_{n=0}^{\infty} j_{m,n} x^n.$$

In particular, it follows from (12) that

$$j_0(x) = \sum_{n=0}^{\infty} j_{0,n} x^n = \sum_{n=0}^{\infty} J_n x^n = \frac{x}{1 - x - 2x^2}, \tag{14}$$

$$j_1(x) = \sum_{n=0}^{\infty} j_{1,n} x^n = \sum_{n=0}^{\infty} J_{n+2} x^n = \frac{2x + 1}{1 - x - 2x^2},$$

and by (11) we have also

$$j_m(x) = j_{m-1}(x) + 2j_{m-2}(x). \tag{15}$$

Using (14) and (15), we prove easily that

$$j_m(x) = \sum_{n=0}^{\infty} j_{m,n} x^n = \frac{(x + 1)J_m + xJ_{m+1}}{1 - x - 2x^2}.$$

So,

$$\begin{aligned} G_j(x) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} j_{m,n} x^n y^m \\ &= \frac{x + 1}{1 - x - 2x^2} \sum_{m=0}^{\infty} J_m y^m + \frac{x}{1 - x - 2x^2} \sum_{m=0}^{\infty} J_{m+1} y^m \\ &= \frac{xy + y + x}{(1 - x - 2x^2)(1 - y - 2y^2)}. \end{aligned}$$

□

THEOREM 2.5. *The Jacobsthal array series is*

$$S_j(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{j_{m,n}}{x^n y^m} = \frac{x^2 y + xy + xy^2}{(x^2 - x - 2)(y^2 - y - 2)}.$$

Proof. Let

$$j_m(x) = \sum_{n=0}^{\infty} \frac{j_{m,n}}{x^n}.$$

In particular, it follows from (12) that

$$\begin{aligned} j_0(x) &= \sum_{n=0}^{\infty} \frac{j_{0,n}}{x^n} = \sum_{n=0}^{\infty} \frac{J_n}{x^n} = \frac{x}{x^2 - x - 2}, \\ j_1(x) &= \sum_{n=0}^{\infty} \frac{j_{1,n}}{x^n} = \sum_{n=0}^{\infty} \frac{J_{n+2}}{x^n} = \frac{x^2 + 2x}{x^2 - x - 2}, \end{aligned} \tag{16}$$

and by (11) we have also

$$j_m(x) = j_{m-1}(x) + 2j_{m-2}(x). \tag{17}$$

Using (16) and (17), we prove easily that

$$j_m(x) = \sum_{n=0}^{\infty} \frac{j_{m,n}}{x^n} = \frac{(x^2 + x)J_m + xJ_{m+1}}{x^2 - x - 2}.$$

So,

$$\begin{aligned} S_j(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{j_{m,n}}{x^n y^m} &= \frac{x^2 + x}{x^2 - x - 2} \sum_{m=0}^{\infty} \frac{J_m}{y^m} + \frac{x}{x^2 - x - 2} \sum_{m=0}^{\infty} \frac{J_{m+1}}{y^m} \\ &= \frac{x^2 y + xy + xy^2}{(x^2 - x - 2)(y^2 - y - 2)}. \end{aligned}$$

□

THEOREM 2.6. *The following equation is valid.*

$$T_j = \sum_{t=0}^{m} \sum_{k=0}^{n} j_{t,k} = \frac{2J_{n+2}J_{m+1} - 2J_{m+1} + J_{n+4}J_{m+2} - J_{n+4} - 3J_{m+2}}{4}.$$

Proof. Considering (13)

$$\begin{aligned} T_j &= 2\sum_{t=0}^{m} J_{t-1} \sum_{k=0}^{n} J_k + \sum_{t=0}^{m} J_t \sum_{k=0}^{n} J_{k+2} \\ &= (J_{n+2} - 1) \sum_{t=0}^{m} J_{t-1} + \frac{J_{n+4} - 3}{2} \sum_{t=0}^{m} J_t \\ &= (J_{n+2} - 1) \left(\frac{J_{m+1}}{2} \right) + \left(\frac{J_{n+4} - 3}{2} \right) \left(\frac{J_{m+2} - 1}{2} \right) \\ &= \frac{2J_{n+2}J_{m+1} - 2J_{m+1} + J_{n+4}J_{m+2} - J_{n+4} - 3J_{m+2}}{4} \end{aligned}$$

takes place. □

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Received September 7, 2021
Accepted February 28, 2022

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