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$\theta(\star)$-PRECONTINUITY

CHAWALIT BOONPOK

Abstract. This paper is concerned with the notion of $\theta(\star)$-precontinuous functions. Some characterizations of $\theta(\star)$-precontinuous functions are investigated. Moreover, the relationships between $\theta(\star)$-precontinuous functions and weakly $\star$-precontinuous functions are discussed.

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1. INTRODUCTION

The concept of continuity is one of the most important tools for the study in topological spaces and different forms of generalizations of continuity have been introduced and investigated. As weak forms of continuity in topological spaces, weak continuity [12], quasi continuity [13], semi-continuity [11] and almost continuity in the sense of Husain [7] are well-known. In [14], almost continuity is called precontinuity by Mashhour et al. Nasef and Noiri [15] introduced a new class of functions called almost precontinuous functions. The class of almost precontinuity is a generalization of precontinuity [14]. Jafari and Noiri [8] investigated the further properties of almost precontinuous functions. Janković [9] introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri [18] introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. Paul and Bhattacharyya [17] called weakly precontinuous functions quasi-precontinuous and obtained the further properties of quasi-precontinuity. Noiri [16] introduced a new class of functions called θ-precontinuous functions which is contained in the class of weakly precontinuous functions and contains the class of almost precontinuous functions and showed that the θ-precontinuous image of a p-closed space is quasi H-closed. The concept of ideals in topological spaces, which is one of the important areas of research in the branch of mathematics, has been introduced and studied by Kuratowski [10] and Vaidyanathaswamy [19].

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Corresponding author: Chawalit Boonpok.

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Janković and Hamlett [9] introduced the notion of $\mathscr{I}$-open sets in ideal topologial spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [3] introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. Yüksel et al. [21] introduced a new type of functions called strongly θ-pre-continuous functions and investigated some characterizations of such functions. In [20], the present authors introduced a strong form in ideal topological spaces of weak θ-pre-continuity called weak θ-pre-$\mathscr{I}$-continuity and shown that weak θ-pre-$\mathscr{I}$-continuity is strictly weaker than strong θ-pre-$\mathscr{I}$-continuity. In this paper, we introduce the notion of $\theta(\star)$-precontinuous functions and investigate some characterizations of $\theta(\star)$-precontinuous functions. Furthermore, the relationships between $\theta(\star)$-precontinuous functions and weakly $\star$-precontinuous functions are discussed.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ foreveryopenneighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [10], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by the base $\mathscr{B}(\mathscr{I}, \tau) = \{U - I' \mid U \in \tau \text{ and } I' \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I}, \tau)$ is not always a topology [19]. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-closed [9] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

Definition 2.1. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:

(i) pre-$\mathscr{I}$-open [4] if $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(A))$;
(ii) semi-$\mathscr{I}$-open [6] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$;
(iii) R-$\mathscr{I}$-open [5] if $A = \mathrm{Int}(\mathrm{Cl}^\star(A))$.

By $p\mathscr{I}O(X)$ (resp. $s\mathscr{I}O(X)$, $r\mathscr{I}O(X)$), we denote the family of all pre-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open, R-$\mathscr{I}$-open) sets of an ideal topological space $(X, \tau, \mathscr{I})$.

The complement of a pre-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open, R-$\mathscr{I}$-open) set is called pre-$\mathscr{I}$-closed (resp. semi-$\mathscr{I}$-closed, R-$\mathscr{I}$-closed).

Definition 2.2 ([23]). Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$.

(i) The intersection of all pre-$\mathscr{I}$-closed sets of X containing A is called the *pre-$\mathscr{I}$-closure* of A and is denoted by $p\imath\mathrm{Cl}(A)$.

(ii) The union of all pre-$\mathscr{I}$-open sets of X contained in A is called the *pre-$\mathscr{I}$-interior* of A and is denoted by $p\imath\mathrm{Int}(A)$.

Lemma 2.3 ([23]). *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $x \in p\imath\mathrm{Cl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every pre-$\mathscr{I}$-open set U of X containing x;*
(2) *A is pre-$\mathscr{I}$-closed if and only if* $A = p\imath\mathrm{Cl}(A)$*;*
(3) $p\imath\mathrm{Cl}(X - A) = X - p\imath\mathrm{Int}(A)$*;*
(4) $p\imath\mathrm{Int}(X - A) = X - p\imath\mathrm{Cl}(A)$.

Lemma 2.4. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $p\imath\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}^\star(A))$ [21]*;*
(2) $p\imath\mathrm{Int}(A) = A \cap \mathrm{Int}(\mathrm{Cl}^\star(A))$.

Definition 2.5 ([21]). Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. A point x of X is called a *pre-θ-$\mathscr{I}$-cluster point* of A if $p\imath\mathrm{Cl}(U) \cap A \neq \emptyset$ for every pre-$\mathscr{I}$-open set U of X containing x. The set of all pre-θ-$\mathscr{I}$-cluster points of A is called the *pre-θ-$\mathscr{I}$-closure* of A and is denoted by $p\imath\mathrm{Cl}_\theta(A)$.

Definition 2.6 ([24]). Let $(X, \tau, \mathscr{I})$ be an ideal topological space, S a subset of X and x a point of X.

(i) x is called a *δ-$\mathscr{I}$-cluster point* of S if $\mathrm{Int}(\mathrm{Cl}^\star(U)) \cap S \neq \emptyset$ for each open neighbourhood of x.

(ii) The family of all δ-$\mathscr{I}$-cluster point of S is called the *δ-$\mathscr{I}$-closure* of S and is denoted by $\delta\mathrm{Cl}_{\mathscr{I}}(S)$.

(iii) A subset S is said to be *δ-$\mathscr{I}$-closed* if $\delta\mathrm{Cl}_{\mathscr{I}}(S) = S$. The complement of a δ-$\mathscr{I}$- closed set of X is said to be *δ-$\mathscr{I}$-open.*

Lemma 2.7 ([24]). *For subsets A and B of an ideal topological space $(X, \tau\mathscr{I})$, the following properties hold:*

(1) $\mathrm{Int}(\mathrm{Cl}^\star(A))$ *is R-$\mathscr{I}$-open.*
(2) *If A and B are R-$\mathscr{I}$-open, then $A \cap B$ is R-$\mathscr{I}$-open.*
(3) *If A is regular open, then it is R-$\mathscr{I}$-open.*
(4) *If A is R-$\mathscr{I}$-open, then it is δ-$\mathscr{I}$-open.*

(5) *Every δ-$\mathscr{I}$-open set is the union of a family of R-$\mathscr{I}$-open sets.*

Lemma 2.8 ([24]). *For subsets A and B of an ideal topological space $(X, \tau\mathscr{I})$, the following properties hold:*

(1) $A \subseteq \delta\mathrm{Cl}_{\mathscr{I}}(A)$.
(2) *If $A \subseteq B$, then $\delta Cl_{\mathscr{I}}(A) \subseteq \delta\mathrm{Cl}_{\mathscr{I}}(B)$.*
(3) $\delta\mathrm{Cl}_{\mathscr{I}}(A) = \cap\{F \subseteq X \mid A \subseteq F$ and F is $\delta-\mathscr{I}-$closed$\}$.
(4) *If F_α is δ-$\mathscr{I}$- closed set of X for each $\alpha \in \nabla$, then $\cap\{F_\alpha \mid \alpha \in \nabla\}$ is δ-$\mathscr{I}$-closed.*
(5) *$\delta\mathrm{Cl}_{\mathscr{I}}(A)$ is δ-$\mathscr{I}$-closed.*

Definition 2.9 ([21]). Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point $x \in X$ is called a *θ-$\mathscr{I}$-cluster point* of A if $\mathrm{Cl}^\star(U) \cap A \neq \emptyset$ for every $U \in \tau$ containing x.The set of all θ-$\mathscr{I}$-cluster points of A is called the *θ-$\mathscr{I}$-closure* of A and is denoted by $\mathrm{Cl}_{\theta\imath}(A)$.

Definition 2.10 ([2]). Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point $x \in X$ is called a *pre-$\mathscr{I}$-θ-cluster point* of A if $p\imath\mathrm{Cl}(U) \cap A \neq \emptyset$ for every pre-$\mathscr{I}$-open set U of X containing x. The set of all pre-$\mathscr{I}$-θ-cluster points of A is called the *pre-$\mathscr{I}$-θ-closure* of A and is denoted by $p\imath\mathrm{Cl}_\theta(A)$.

Definition 2.11 ([2]). Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point $x \in X$ is called a *pre-$\mathscr{I}$-θ-interior point* of A if there exists a pre-$\mathscr{I}$-open set U of X containing x such that $x \in p\imath\mathrm{Cl}(U) \subseteq A$. The set of all pre-$\mathscr{I}$-$\theta$-interior points of A is called the *pre-$\mathscr{I}$-θ-interior* of A and is denoted by $p\imath\mathrm{Int}_\theta(A)$.

Lemma 2.12 ([20]). *For subsets A and B of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $p\imath\mathrm{Cl}_\theta(p\imath\mathrm{Cl}_\theta(A)) = p\imath\mathrm{Cl}_\theta(A)$.
(2) *If $A \subseteq B$, then $p\imath\mathrm{Cl}_\theta(A) \subseteq p\imath\mathrm{Cl}_\theta(B)$.*
(3) $p\imath\mathrm{Int}_\theta(X - A) = X - p\imath\mathrm{Cl}_\theta(A)$.
(4) $p\imath\mathrm{Cl}_\theta(X - A) = X - p\imath\mathrm{Int}_\theta(A)$.

3. CHARACTERIZATIONS OF $\theta(\star)$-PRECONTINUOUS FUNCTIONS

In this section, we introduce the notion of $\theta(\star)$-precontinuous functions. Moreover, we discuss some characterizations of $\theta(\star)$-precontinuous functions.

Definition 3.1. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *$\theta(\star)$-precontinuous* if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$.

Theorem 3.2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $\theta(\star)$-precontinuous;*
(2) *$p\imath\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$ for every subset B of Y;*

(3) $f(p\imath\mathrm{Cl}(A)) \subseteq \mathrm{Cl}_{\theta\imath}(f(A))$ *for every subset* A *of* X.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$. Then $f(x) \notin \mathrm{Cl}_{\theta\imath}(B)$ and there exists an open set V containing $f(x)$ such that $\mathrm{Cl}^\star(V) \cap B = \emptyset$. Since f is $\theta(\star)$-precontinuous, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. Therefore, we have $f(p\imath\mathrm{Cl}(U)) \cap B = \emptyset$ and $p\imath\mathrm{Cl}(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin p\imath\mathrm{Cl}_\theta(f^{-1}(B))$. Thus, $p\imath\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$.

(2) ⇒ (1): Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since $\mathrm{Cl}^\star(V) \cap (Y - \mathrm{Cl}^\star(V)) = \emptyset$, $f(x) \notin \mathrm{Cl}_{\theta\imath}(Y - \mathrm{Cl}^\star(V))$ and

$$x \notin f^{-1}(\mathrm{Cl}_{\theta\imath}(Y - \mathrm{Cl}^\star(V))).$$

By (2), $x \notin p\imath\mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}^\star(V)))$ and there exists a pre-$\mathscr{I}$-open set U of X containing x such that $p\imath\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}^\star(V)) = \emptyset$; hence $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. Consequently, we obtain f is $\theta(\star)$-precontinuous.

(2) ⇒ (3): Let A be any subset of X. By (2),

$$p\imath\mathrm{Cl}_\theta(A) \subseteq p\imath\mathrm{Cl}_\theta(f^{-1}(f(A))) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(f(A)))$$

and hence $f(p\imath\mathrm{Cl}(A)) \subseteq \mathrm{Cl}_{\theta\imath}(f(A))$.

(3) ⇒ (2): Let B be any subset of Y. By (3), we have

$$f(p\imath\mathrm{Cl}_\theta(f^{-1}(B))) \subseteq \mathrm{Cl}_{\theta\imath}(f(f^{-1}(B))) \subseteq \mathrm{Cl}_{\theta\imath}(B)$$

and hence $p\imath\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$. □

THEOREM 3.3. *A function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is* $\theta(\star)$*-precontinuous if and only if* $f^{-1}(V) \subseteq p\imath\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}^\star(V)))$ *for every open set* V *of* Y.

Proof. Let V be an open set of Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists a pre-$\mathscr{I}$-open set U of X containing x such that

$$f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V).$$

Therefore, $x \in U \subseteq p\imath\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}^\star(V))$. Thus, $x \in p\imath\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}^\star(V)))$. It follows that $f^{-1}(V) \subseteq p\imath\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}^\star(V)))$.

Conversely, let $x \in X$ and let V be an open set of Y containing $f(x)$. By the hypothesis, $f^{-1}(V) \subseteq p\imath\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}^\star(V)))$ and $x \in p\imath\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}^\star(V)))$. There exists a pre-$\mathscr{I}$-open set U of X containing x such that $p\imath\mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}^\star(V))$ and hence $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. This shows that f is $\theta(\star)$-precontinuous. □

LEMMA 3.4 ([22]). *Let* $(X, \tau, \mathscr{I})$ *be an ideal topological space and* $A, B \subseteq X$. *Then* $\mathrm{Cl}^\star(A) \times \mathrm{Cl}^\star(B) \subseteq \mathrm{Cl}^\star(A \times B)$.

THEOREM 3.5. *Let* $(X, \tau, \mathscr{I})$ *and* $(Y, \sigma, \mathscr{J})$ *be ideal topological spaces. Let* $f : X \to Y$ *be a function and* $h : X \to X \times Y$ *the graph function of* f *defined by* $h(x) = (x, f(x))$ *for each* $x \in X$. *Then* h *is* $\theta(\star)$*-precontinuous if and only if* f *is* $\theta(\star)$*-precontinuous.*

Proof. Suppose that h is $\theta(\star)$-precontinuous. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $h(x)$. Since h is $\theta(\star)$-precontinuous, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $h(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(X \times V)$. It follows Lemma 3.4 that $\mathrm{Cl}^\star(X \times V) \subseteq X \times \mathrm{Cl}^\star(V)$ and hence $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. This show that f is $\theta(\star)$-precontinuous.

Conversely, suppose that f is $\theta(\star)$-precontinuous. Let $x \in X$ and let W be an open set of $X \times Y$ containing $h(x)$, there exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $h(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is $\theta(\star)$-precontinuous, there exists a pre-$\mathscr{I}$-open set U_2 of X containing x such that $f(p\imath\mathrm{Cl}(U_2)) \subseteq \mathrm{Cl}^\star(V)$. Let $U = U_1 \cap U_2$, then U is pre-$\mathscr{I}$-open set of X containing x. Therefore, we obtain $h(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(U_1) \times f(p\imath\mathrm{Cl}(U_2)) \subseteq \mathrm{Cl}^\star(U_1) \times \mathrm{Cl}^\star(V) \subseteq \mathrm{Cl}^\star(W)$. This shows that h is $\theta(\star)$-precontinuous. □

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, we denote by $\tau_{|A}$ the relative topology on A and $\mathscr{I}_{|A} = \{A \cap I' \mid I' \in \mathscr{I}\}$ is an ideal on A.

Lemma 3.6 ([23]). *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and let A and X_0 be subsets of X.*

(1) *If $A \in p\mathscr{I}O(X)$ and $X_0 \in s\mathscr{I}O(X)$, then $A \cap X_0 \in p\mathscr{I}O(X_0)$.*
(2) *If $A \in p\mathscr{I}O(X_0)$ and $X_0 \in p\mathscr{I}O(X)$, then $A \in p\mathscr{I}O(X)$.*

Lemma 3.7 ([23]). *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X_0 \subseteq X$ and $p\imath\mathrm{Cl}_{X_0}(A)$ denote the pre-$\mathscr{I}$-closure of A in X_0.*

(1) *If $X_0 \in s\mathscr{I}O(X)$, then $p\imath\mathrm{Cl}_{X_0}(A) \subseteq p\imath\mathrm{Cl}(A)$.*
(2) *If $A \in p\mathscr{I}O(X_0)$ and $X_0 \in p\mathscr{I}O(X)$, then $p\imath\mathrm{Cl}(A) \subseteq p\imath\mathrm{Cl}_{X_0}(A)$.*

Theorem 3.8. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous and $G \in s\mathscr{I}O(X)$, then the restriction $f_{|G} : (G, \tau_{|G}, \mathscr{I}_{|G}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$, since f is $\theta(\star)$-precontinuous. Put $U_0 = U \cap G$, then by Lemma 3.6 and 3.7, $U_0 \in p\mathscr{I}O(G)$ containing x and $p\imath\mathrm{Cl}_G(U_0) \subseteq p\imath\mathrm{Cl}(U_0)$. Therefore, we obtain $f_{|G}(p\imath\mathrm{Cl}_G(U_0)) = f(p\imath\mathrm{Cl}_G(U_0)) \subseteq f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. This shows that $f_{|G}$ is $\theta(\star)$-precontinuous. □

Theorem 3.9. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous if for each $x \in X$, there exists a pre-$\mathscr{I}$-open set G of X containing x such that the restriction $f_{|G} : (G, \tau_{|G}, \mathscr{I}_{|G}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. There exists a pre-$\mathscr{I}$-open set G of X containing x such that

$$f_{|G} : (G, \tau_{|G}, \mathscr{I}_{|G}) \to (Y, \sigma, \mathscr{J})$$

is $\theta(\star)$-precontinuous. Thus, there exists a pre-$\mathscr{I}$-open set U of G containing x such that $f_{|G}(p\imath\mathrm{Cl}_G(U)) \subseteq \mathrm{Cl}^\star(V)$. By Lemma 3.6 and 3.7, U is a pre-$\mathscr{I}$-open set of X containing x such that $p\imath\mathrm{Cl}(U) \subseteq p\imath\mathrm{Cl}_G(U)$. Hence, we have $f(p\imath\mathrm{Cl}(U)) = f_{|G}(p\imath\mathrm{Cl}(U)) \subseteq f_{|G}(p\imath\mathrm{Cl}_G(U)) \subseteq \mathrm{Cl}^\star(V)$. This shows that $f_{|G}$ is $\theta(\star)$-precontinuous. □

Definition 3.10. A subset K of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$p(\star)$-closed* (resp. *$\mathcal{H}(\star)$-closed*) *relative to* X if for each cover

$$\{V_\alpha \mid \alpha \in \nabla\}$$

of K by pre-$\mathscr{I}$-open sets of X, there exists finite subset ∇_0 of ∇ such that $K \subseteq \cup\{p\imath\mathrm{Cl}(V_\alpha) \mid \alpha \in \nabla\}$ (resp. $K \subseteq \cup\{\mathrm{Cl}^\star(V_\alpha) \mid \alpha \in \nabla\}$).

Theorem 3.11. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $\theta(\star)$-precontinuos function and K is $p(\star)$-closed relative to X, then $f(K)$ is quasi $\mathcal{H}(\star)$-closed relative to Y.*

Proof. Suppose that $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuos and K is $p(\star)$-closed relative to X. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $f(K)$ by open sets of Y. For each $x \in K$, there exists $\alpha(x) \in \nabla$ such that $f(x) \in V_{\alpha(x)}$. Since f is $\theta(\star)$-precontinuos, there exists a pre-$\mathscr{I}$-open set U_x of X containing x such that $f(p\imath\mathrm{Cl}(U_x)) \subseteq \mathrm{Cl}^\star(V_{\alpha(x)})$. The family $\{U_x \mid x \in K\}$ is a cover of K by pre-$\mathscr{I}$-open sets of X and hence there exists a finite subset K_0 of K such that $K \subseteq \cup_{x\in K_0} p\imath\mathrm{Cl}(U_x)$. Therefore, we obtain $f(K) \subseteq \cup_{x\in K_0}\mathrm{Cl}^\star(V_{\alpha(x)})$. This shows that $f(K)$ is quasi $\mathcal{H}(\star)$-closed relative to Y. □

Corollary 3.12. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $\theta(\star)$-precontinuos surjection and X is $p(\star)$-closed relative to X, then Y is quasi $\mathcal{H}(\star)$-closed relative to Y.*

Next, we introduce the notion of almost $\star$-precontinuous functions. Moreover, several characterizations of almost $\star$-precontinuous functions are discussed.

Definition 3.13. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called *almost $\star$-precontinuous* if for each $x \in X$ and each R-$\mathscr{J}$-open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq V$.

Theorem 3.14. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\star$-precontinuous;*
(2) *for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(V))$;*
(3) *$f^{-1}(F)$ is pre-$\mathscr{I}$-closed in X for every R-$\mathscr{J}$-closed set F of Y;*
(4) *$f^{-1}(V)$ is pre-$\mathscr{I}$-open in X for every R-$\mathscr{J}$-open set V of Y.*

Proof. The proof is obvious and is thus omitted. □

THEOREM 3.15. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\star$-precontinuous;*
(2) *$f(p\imath\mathrm{Cl}(A)) \subseteq \delta\mathrm{Cl}_{\mathscr{J}}(f(A))$ for every subset A of X;*
(3) *$p\imath\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\delta\mathrm{Cl}_{\mathscr{J}}(B))$ for every subset B of Y;*
(4) *$f^{-1}(F)$ is pre-$\mathscr{I}$-closed in X for every δ-$\mathscr{J}$-closed set F of Y;*
(5) *$f^{-1}(V)$ is pre-$\mathscr{I}$-open in X for every δ-$\mathscr{J}$-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X. By Lemma 2.8(5), we have $\delta\mathrm{Cl}_{\mathscr{J}}(f(A))$ is δ-$\mathscr{J}$-closed in Y and by Lemma 2.7(5),

$$\delta\mathrm{Cl}_{\mathscr{J}}(f(A)) = \cap\{F_\alpha \mid F_\alpha \text{ is } R\text{−}\mathscr{J}\text{−closed}, \alpha \in \nabla\},$$

where ∇ is an index set. By Theorem 3.14, we have

$$A \subseteq f^{-1}(\delta\mathrm{Cl}_{\mathscr{J}}(f(A))) = \cap\{f^{-1}(F_\alpha) \mid \alpha \in \nabla\}$$

and $\cap\{f^{-1}(F_\alpha) \mid \alpha \in \nabla\}$ is pre-$\mathscr{I}$-closed in X. Thus,

$$p\imath\mathrm{Cl}(A) \subseteq f^{-1}(\delta\mathrm{Cl}_{\mathscr{J}}(f(A)))$$

and hence $f(p\imath\mathrm{Cl}(A)) \subseteq \delta\mathrm{Cl}_{\mathscr{J}}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), we have

$$f(p\imath\mathrm{Cl}(f^{-1}(B))) \subseteq \delta\mathrm{Cl}_{\mathscr{J}}(f(f^{-1}(B))) \subseteq \delta\mathrm{Cl}_{\mathscr{J}}(B)$$

and hence $p\imath\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\delta\mathrm{Cl}_{\mathscr{J}}(B))$.

(3) $\Rightarrow$ (4): Let F be a δ-$\mathscr{J}$-closed set of Y. By (3), $p\imath\mathrm{Cl}(f^{-1}(F)) \subseteq f^{-1}(\delta\mathrm{Cl}_{\mathscr{J}}(F)) = f^{-1}(F)$. Thus, $f^{-1}(F)$ is pre-$\mathscr{I}$-closed in X.

(4) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): Let V be a R-$\mathscr{J}$-open set of Y. By Lemma 2.7(4), V is δ-$\mathscr{J}$-open in Y. By (5), we have $f^{-1}(V)$ is pre-$\mathscr{I}$-open in X and hence by Theorem 3.14, f is almost $\star$-precontinuous. □

DEFINITION 3.16. For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the graph

$$G(f) = \{(x, f(x)) \mid x \in X\}$$

is called *strongly almost $\star$-preclosed* if for each $(x, y) \in X \times Y - G(f)$, there exist a pre-$\mathscr{I}$-open set U of X containing x and a R-$\mathscr{J}$-open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

LEMMA 3.17. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ has the strongly almost $\star$-perclosed graph if and only if for each $x \in X$ and $y \in Y$ such that $f(x) \neq y$, there exist a pre-$\mathscr{I}$-open set U of X containing x and a R-$\mathscr{J}$-open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of the above definition. □

THEOREM 3.18. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is almost $\star$-precontinuous and $(Y, \sigma, \mathscr{J})$ is Hausdorff, then $G(f)$ is strongly almost $\star$-preclosed.*

Proof. Suppose that $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since $(Y, \sigma, \mathscr{J})$ is Hausdorff, there exist open sets V_1 and V_2 in Y such that $y \in V_1$, $f(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since V_1 and V_2 are disjoint, $\mathrm{Int}(\mathrm{Cl}^\star(V_1)) \cap \mathrm{Cl}^\star(V_2) = \emptyset$. Since f is almost $\star$-precontinuous, by Theorem 3.14, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}^\star(V_2)$. Thus, $f(U) \cap \mathrm{Int}(\mathrm{Cl}^\star(V_1)) = \emptyset$. It follows from Lemma 3.17 that $G(f)$ is strongly almost $\star$-preclosed. □

As a generalization of $\theta(\star)$-precontinuous functions, we introduce the notion of weakly $\star$-precontinuous functions and investigate some characterizations of weakly $\star$-precontinuous.

Definition 3.19. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called *weakly $\star$-precontinuous* if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}^\star(V)$.

Theorem 3.20. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-precontinuous;*
(2) *$f^{-1}(V) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(f^{-1}(\mathrm{Cl}^\star(V))))$ for every open set V of Y;*
(3) *$f^{-1}(V) \subseteq p\imath\mathrm{Int}(f^{-1}(\mathrm{Cl}^\star(V)))$ for every open set V of Y.*

Proof. (1) ⇒ (2): Let V be an open set of Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and by (1), there exists a pre-$\mathscr{I}$-open set of X containing x such that $f(U) \subseteq \mathrm{Cl}^\star(V)$. Thus, we obtain $U \subseteq f^{-1}(\mathrm{Cl}^\star(V))$ and hence

$$x \in \mathrm{Int}(\mathrm{Cl}^\star(f^{-1}(\mathrm{Cl}^\star(V)))).$$

This shows that $f^{-1}(V) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(f^{-1}(\mathrm{Cl}^\star(V))))$.

(2) ⇒ (3): Let V be an open set of Y. By (2) and Lemma 2.4(2),

$$f^{-1}(V) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(f^{-1}(\mathrm{Cl}^\star(V)))) \cap f^{-1}(\mathrm{Cl}^\star(V)) = p\imath\mathrm{Int}(f^{-1}(\mathrm{Cl}^\star(V))).$$

(3) ⇒ (1): Let $x \in X$ and let V be an open set of Y containing $f(x)$. By (3), we have $x \in f^{-1}(V) \subseteq p\imath\mathrm{Int}(f^{-1}(\mathrm{Cl}^\star(V)))$. Put $U = p\imath\mathrm{Int}(f^{-1}(\mathrm{Cl}^\star(V)))$, then U is a pre-$\mathscr{I}$-open set U of X containing x and $f(U) \subseteq \mathrm{Cl}^\star(V)$. This shows that f is weakly $\star$-precontinuous. □

Theorem 3.21. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-precontinuous;*
(2) *$f(p\imath\mathrm{Cl}(A)) \subseteq \mathrm{Cl}_{\theta\imath}(f(A))$ for every subset A of X;*
(3) *$p\imath\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$ for every subset B of Y.*

Proof. (1) ⇒ (2): Let A be any subset of X. Suppose that $x \in p\imath\mathrm{Cl}(A)$ and G is an open set of Y containing $f(x)$. Since f is weakly $\star$-precontinuous, there there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}^\star(G)$. Since $x \in p\imath\mathrm{Cl}(A)$, $U \cap A \neq \emptyset$. It follows that $\emptyset \neq f(U) \cap f(A) \subseteq \mathrm{Cl}^\star(G) \cap f(A)$. Thus, $\mathrm{Cl}^\star(G) \cap f(A) \neq \emptyset$ and $f(x) \in \mathrm{Cl}_{\theta\imath}(f(A))$. This shows that $f(p\imath\mathrm{Cl}(A)) \subseteq \mathrm{Cl}_{\theta\imath}(f(A))$.

(2) ⇒ (3): Let B be any subset of Y. By (2), we have

$$f(p\imath\mathrm{Cl}(f^{-1}(B))) \subseteq \mathrm{Cl}_{\theta\imath}(f(f^{-1}(B))) \subseteq \mathrm{Cl}_{\theta\imath}(B)$$

and hence $p\imath\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}_{\theta\imath}(B))$.

(3) ⇒ (1): Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since

$$\mathrm{Cl}^\star(V) \cap (Y - \mathrm{Cl}^\star(V)) = \emptyset,$$

we have $f(x) \notin \mathrm{Cl}_{\theta\imath}(Y - \mathrm{Cl}^\star(V))$ and hence $x \notin f^{-1}(\mathrm{Cl}_{\theta\imath}(Y - \mathrm{Cl}^\star(V)))$. By (3), $x \notin p\imath\mathrm{Cl}(f^{-1}(Y - \mathrm{Cl}^\star(V)))$ and thus there exists a pre-$\mathscr{I}$-open set U of X containing x such that $U \cap f^{-1}(Y - \mathrm{Cl}^\star(V)) = \emptyset$; hence $f(U) \cap (Y - \mathrm{Cl}^\star(V)) = \emptyset$. This implies that $f(U) \subseteq \mathrm{Cl}^\star(V)$. Therefore, f is weakly $\star$-precontinuous. □

THEOREM 3.22. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous, then f is weakly $\star$-precontinuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since f is $\theta(\star)$-precontinuous, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(p\imath\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V)$. Thus, $f(U) \subseteq \mathrm{Cl}^\star(V)$. This shows that f is weakly $\star$-precontinuous. □

The converse of Theorem 3.22 is not true as shown by the following example.

EXAMPLE 3.23. Let $\mathbf{R}$ be the set of real numbers, $\tau = \{\emptyset\} \cup \{U \subseteq \mathbf{R} \mid 0 \in U\}$ and $\mathscr{I} = \{\emptyset\}$. Let $Y = \{p, q, r\}$, $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$ and $\mathscr{J} = \{\emptyset, \{r\}\}$. Define a function $f : (\mathbf{R}, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ as follows:

$$f(x) = \begin{cases} p, & \text{if } x < 0, \\ q, & \text{if } x = 0, \\ r, & \text{if } x > 0. \end{cases}$$

Then f is a weakly $\star$-precontinuous function which is not $\theta(\star)$-precontinuous.

LEMMA 3.24 ([2]). *An ideal topological space $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-regular if and only if each pre-$\mathscr{I}$-open neighbourhood U of x, there exists a pre-$\mathscr{I}$-open set V containing x such that $x \in V \subseteq p\imath\mathrm{Cl}(V) \subseteq U$.*

THEOREM 3.25. *Let $(X, \tau, \mathscr{I})$ be a pre-$\mathscr{I}$-regular space. For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties hold:*

(1) *If f is weakly $\star$-precontinuous, then f is $\theta(\star)$-precontinuous.*
(2) *If f is almost $\star$-precontinuous, then f is $\theta(\star)$-precontinuous.*

Proof. We prove only the first, the second being proved analogously. Suppose that f is weakly $\star$-precontinuous. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Then, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}^\star(V)$. Since $(X, \tau, \mathscr{I})$ is pre-$\mathscr{I}$-regular, there exists a pre-$\mathscr{I}$-open set W of X containing x such that $x \in W \subseteq p\imath\mathrm{Cl}(W) \subseteq U$. Therefore, we obtain $f(p\imath\mathrm{Cl}(W)) \subseteq \mathrm{Cl}^\star(V)$. This shows that f is $\theta(\star)$-precontinuous. □

LEMMA 3.26 ([24]). *An ideal topological space $(X, \tau, \mathscr{I})$ is an $A\mathscr{I}$-R space if and only if for each $x \in X$ and each R-$\mathscr{I}$-open neighbourhood of x, there exists a R-$\mathscr{I}$-open neighbourhood U of x such that $x \in U \subseteq \mathrm{Cl}^\star(U) \subseteq \mathrm{Cl}(U) \subseteq V$.*

THEOREM 3.27. *Let $(Y, \sigma, \mathscr{J})$ be an $A\mathscr{J}$-R space. If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-precontinuous, then f is almost $\star$-precontinuous.*

Proof. Suppose that f is $\theta(\star)$-precontinuous. Let $x \in X$ and let V be a R-$\mathscr{J}$-open set of Y containing $f(x)$. Since $(Y, \sigma, \mathscr{J})$ is an $A\mathscr{J}$-R space, there exists a R-$\mathscr{J}$-open set W containing $f(x)$ such that $f(x) \in W \subseteq \mathrm{Cl}^\star(W) \subseteq \mathrm{Cl}(W) \subseteq V$. Since f is $\theta(\star)$-precontinuous, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(p\imath\mathrm{Cl}(U) \subseteq \mathrm{Cl}^\star(W)$; hence $f(U) \subseteq V$. This shows that f is almost $\star$-precontinuous. □

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Mahasarakham University
Mathematics and Applied Mathematics
Research Unit
Department of Mathematics, Faculty of Science
Maha Sarakham, 44150, Thailand
E-mail: chawalit.b@msu.ac.th
https://orcid.org/0000-0002-4094-727X