

OLD AND RECENT RESULTS
ON FINITE BOLYAI-LOBACHEVSKY PLANES

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Abstract. The revolutionary ideas of János Bolyai opened the way for a far more general and abstract approach to geometry than had previously been pursued. In the spirit of Bolyai's ideas, axioms with their mutual relationships and impacts on geometry had intensively been studied and discussed for a long time. The historical development is treated in the expository papers appeared in the volume [75] which commemorated the 200th anniversary of the birth of János Bolyai, written by leading scientists of non-Euclidean geometry, its history, and its applications. A recent survey on Bolyai's work is also found in the survey paper [44].

Axiom systems proposed for general Bolyai-Lobachevsky planes appeared in the literature for the first time in the 1940's. These attempts were strongly influenced by the classical point of view in geometry in that time, and the proposed definitions included sufficiently many postulates to exclude finite planes, that is, geometries on a finite set of points; see Topel [88], De Baggis [33] and Baer [5].

It was only in 1962 Graves' paper [43] that an axiom system for Bolyai-Lobachevsky planes was proposed that admitted finite geometry. Since then, a number of models for finite Bolyai-Lobachevsky planes have been constructed; some of them present interesting properties from different points of view.

The present paper is an account of the known results on finite Bolyai-Lobachevsky planes. We focus on the finite analogs of the well known models of the classical Bolyai-Lobachevsky plane, and show that the finite Beltrami-Cayley and Poincaré models are related to current research in Finite geometry. We also discuss some more, typically finite, models arising from unitary polarities and maximal (k, n) -arcs of finite projective planes. In this context, we investigate those models which have a large symmetry group. An extensive list of bibliographic references on finite Bolyai-Lobachevsky planes is also provided.

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