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A NOTE ON APPROXIMATION PROPERTIES OF DERIVATIVES OF SCHOENBERG SPLINES

HEINER GONSKA, MICHAEL WOZNICZKA, and FRANK ZEILFELDER

Abstract. We analyze approximation properties of derivatives of variation-diminishing Schoenberg splines with emphasis on the case of purely equidistant knots. New direct inequalities regarding simultaneous approximation up to the second derivative are obtained in terms of the classical second order modulus of smoothness. For adequate polynomial degree and sufficiently smooth functions these *quantitative* estimates imply a *simultaneous approximation order* which is *quadratic* with respect to mesh size. These results remain valid if we drop the general requirement of data given outside the basic interval. Numerical tests verify our theoretical error bounds.

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Key words. Schoenberg operator, splines, derivatives, degree of approximation, simultaneous approximation, modulus of smoothness, Hestenes extension.

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University of Duisburg-Essen
Faculty of Mathematics
47048 Duisburg, Germany
E-mail: `heiner.gonska@uni-due.de`
E-mail: `michael.wozniczka@uni-due.de`

University of Mannheim
Institute of Mathematics
68131 Mannheim, Germany
E-mail: `frank.zeilfelder@gmx.de`