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DERIVED EQUIVALENCES AND THE ABELIAN DEFECT GROUP CONJECTURE

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Abstract. Similarities between the character tables of a block with abelian defect group and of its Brauer correspondent have been observed long time ago. This led in the late 1980's to the conviction that deeper connection between the two blocks must exist. In this survey we give an introduction to the applications of categorical equivalences in modular representation theory and to Broué's abelian defect group conjecture. We present some of the methods and the recent achievements on this subject, with a focus on the techniques coming from Clifford theory.

1. Preliminaries on block theory

1.1. Let G be a finite group and p a prime number. Modular representation theory studies the relations between representations of G in characteristic zero and in characteristic p via the study of the group algebra of G over a complete discrete valuation ring $\mathcal{O}$ having residue field k of characteristic p and quotient field $\mathcal{K}$ of characteristic zero. We shall usually assume that k is algebraically closed and $\mathcal{K}$ is "big enough" that is, it contains enough roots of unity. In particular, this implies that $\mathcal{K}$ is a splitting field for the group algebra $\mathcal{K}H$, for any subgroup H of G. Many important results can be generalized to "small" fields, and indeed, rationality questions have been recentley considered in the context of this paper.

The link between representations of G over $\mathcal{K}$ and over k is then established by the functors "reduction modulo p" $k \otimes_{\mathcal{O}} -$ and "extension of scalars" $\mathcal{K} \otimes_{\mathcal{O}} -$.

All modules are assumed to be finitely generated left unitary modules, and all $\mathcal{O}$-algebras are finitely generated as $\mathcal{O}$-modules.

1.2. A *block of the finite group* G is a primitive idempotent a in the center of the group algebra $\mathcal{O}G$. The algebra $A = \mathcal{O}Ga$ is called the *block algebra* of the block a. Most of the time we shall understand by the same a the block algebra $\mathcal{O}Ga$. The group algebra $\mathcal{O}G$ decomposes into a direct product

$$\mathcal{O}G = A_0 \times A_1 \times \cdots \times A_n$$

of blocks, where $A_i = a_i\mathcal{O}G$.

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Given an indecomposable $\mathcal{O}G$-modules M, there is a unique block a_i such that $a_iM \neq 0$, and we say that M "belongs" to a_i. The trivial $\mathcal{O}G$-module $\mathcal{O}$ belongs to the *principal block* $A_0 = a_0\mathcal{O}G =: B_0(\mathcal{O}G)$.

1.3. A *defect group of the block* A is a minimal subgroup D of G such that the homomorphism $A \otimes_{\mathcal{O}D} A \to A$ induced by multiplication splits as homomorphism of (A, A)-bimodules. A *source idempotent of* A is a primitive idempotent $i \in A^D$ such that the induced map $Ai \otimes_{\mathcal{O}D} iA \to A$ is still split surjective as a homomorphism of (A, A)-bimodules. The algebra iAi, considered together with the group homomorphism $D \to \mathrm{U}(iAi)$, $u \mapsto ui$, is called a *source algebra of* A. Here we denoted

$$A^D = \{a \in A \mid uau^{-1} = a \text{ for all } u \in D\}.$$

The defect groups of A form a unique conjugation class of p-subgroups of G. The defect groups of the principal block A_0 are the Sylow p-subgroups of G. If p^d is the order of a defect group of A, then d is called the *defect* of b, and we denote $d = \mathrm{def}(A)$.

Reduction modulo p gives a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG, preserving defect groups.

The source algebra of A is Morita equivalent to A and encodes all the essential information about the local structure of the block. From Puig's point of view, the main problem of block theory is the determination of the source algebra of a block as an *interior* D*-algebra*.

The defect groups of A control the complexity of A. Defect "zero" just means that A is isomorphic to a matrix algebra $M_r(\mathcal{O})$, and A is of finite representation type if and only if D is cyclic. In the rest of the cases, A has infinite representation type, so the problem of comparing blocks of various subgroups of G appears to be more feasible.

1.4. By Maschke's theorem, $\mathcal{K}A := \mathcal{K} \otimes_{\mathcal{O}} A$ is a semisimple $\mathcal{K}$-algebra, hence a direct product of full matrix $\mathcal{K}$-algebras. Each of these matrix algebras have a unique simple module, which is said to *belong to* A.

Let V be a simple $\mathcal{K}A$-module. It is known that $\dim_{\mathcal{K}} V$ divides $[G : Z(G)]$. Dade introduced the *defect* $\mathrm{def}(V)$ of V by the equality

$$p^{\mathrm{def}(V)} = \left(\frac{|G|}{\dim_{\mathcal{K}} V}\right)_p,$$

and denote by $\mathrm{k}_\delta(B)$ the number of (isomorphism classes of) simple modules of defect δ in A.

2. Local subgroups and the Brauer construction

The Brauer construction provides a link between the "global" and the "local" structure of blocks. It has been generalized to modules by Feit and to G-algebras by Broué and Puig.

2.1. Let P a p-subgroup of G. The *Brauer homomorphism* associated with P is the surjective $\mathcal{O}$-algebra homomorphism

$$\mathrm{Br}_P^G : (\mathcal{O}G)^P \to kC_G(P), \qquad \sum_{x \in G} \alpha_x x \mapsto \sum_{x \in C_G(P)} \overline{\alpha}_x x,$$

where $\alpha_x \in \mathcal{O}$ and $\overline{\alpha}_x$ is the image of α_x in k, for any $x \in G$.

Note that D is a defect group of $A = \mathcal{O}Ga$ if and only if D is a maximal subgroup of G such that $\mathrm{Br}_D^G(a) \neq 0$.

This construction allows to define the *local structure* of a block in terms of its Brauer pairs. A *Brauer pair* of $A = \mathcal{O}Ga$ is a pair (P, e) consisting of a p-subgroup P of G and a block e of $kC_G(P)$ such that $\mathrm{Br}_P^G(a)e = e$. With a partial order relation and morphisms suitably defined, the a-Brauer pairs form a partially ordered set, and also category called the *Brauer category* of the block A.

2.2. For an arbitrary G-algebra A, the Brauer map is the canonical $N_G(P)$-algebra homomorphism

$$\mathrm{Br}_P^A : A^P \to A(P) := k \otimes_{\mathcal{O}} (A^P / \sum_{Q<P} \mathrm{Tr}_Q^P A^Q),$$

while for an $\mathcal{O}G$-module V, the Brauer map is the canonical homomorphism of $\mathcal{O}N_G(P)$-modules

$$\mathrm{Br}_P^V : V^P \to V(P) := k \otimes_{\mathcal{O}} (V^P / \sum_{Q<P} \mathrm{Tr}_Q^P V^Q),$$

where $\mathrm{Tr}_Q^P : V^Q \to V^P$ is the *trace map*.

2.3. The correspondence in the following theorem is called the *Brauer correspondence*:

Let D be a p-subgroup of G and H a subgroup of G containing $N_G(D)$. For any block a of $\mathcal{O}G$ having D as defect group there is a unique block b of $\mathcal{O}H$ having D as defect group such that $Br_D(a) = Br_D(b)$. Moreover, the correspondence $a \mapsto b$ is a bijection between the sets of blocks of $\mathcal{O}G$ and of $\mathcal{O}H$ having D as defect group.

Note that by Brauer's 3rd Main Theorem, the Brauer correspondent of the principal block A_0 is the principal block B_0 of $\mathcal{O}H$.

Subgroups of the form $N_G(P)$, where P is a p-subgroups of G are called "p-local subgroups".

2.4. More generally, the Brauer map gives a correspondence between blocks which is not a bijection. One can formulate this also in terms of bimodules. Let H be a subgroup of G, $B = b\mathcal{O}H$ a block of $\mathcal{O}H$ and $A = a\mathcal{O}G$ a block of $\mathcal{O}G$. We say that b is a *Brauer correspondent* of a and write $a = b^G$, if A is the only block of $\mathcal{O}G$ with the property that B is a direct summand of A regarded as $\mathcal{O}(H \times H)$-modules. Notice that if D is a defect group of B and H contains $C_G(D)$, then b^G is defined.

3. The conjectures of Alperin, Dade, Isaacs-Navarro. Abelian defect groups

Block theory was initiated by Brauer with the aim to provide a tool for the classification of finite simple groups. His famous list of problems is still stimulating the research in this field. Subsequent conjectures are clarification and refinements of his ideas. Brauer's problems are expressing the belief that block theoretic invariants of blocks of positive defect of $\mathcal{O}G$ should be determined from information coming from p-local subgroups. According to Alperin, the "main problem of block theory" is the determination of values of the irreducible characters of G on p-singular elements in terms of p-local information.

3.1. Alperin's weight conjecture. Let $A = a\mathcal{O}G$ be a block of $\mathcal{O}G$, $\ell(A)$ be the number of simple A-modules and $f_0(B)$ the number of projective simple A-modules. Then

$$\ell(A) = \sum_{Q,b} f_0(B),$$

where Q runs over the p-subgroups of G up to conjugacy and B over the Brauer correspondents of B in $N_G(Q)$. This formula implies that if B has positive defect, then $\ell(A)$ is indeed p-locally determined.

3.2. Dade's conjecture. There is an equivalent formulation of Alperin's conjecture in terms of p-subgroup complexes and alternating sums, due to Knörr and Robinson. Inspired by this, Dade stated the following conjecture.

Let $\mathcal{S}_p(G)$ be the set of of stricly increasing chains $C = (Q_0 < Q_1 < \cdots < Q_n)$ of nontrivial p-subgroups of G and let $|C| = n$ be the length of C. Then G acts by conjugation on $\mathcal{S}_p(G)$ and let $G_C = N_G(Q_0) \cap \cdots \cap N_G(Q_n)$ be the stabilizer of C. If the maximal normal p-subgroup $O_p(G) = 1$ and A is a block of positive defect, then for all $\delta > 0$,

$$\sum_{C,B} (-1)^{|C|}\,\mathrm{k}_\delta(B) = 0,$$

where C runs over $\mathcal{S}_p(G)$ up to conjugacy and B over the Brauer correspondents of A in $\mathcal{O}G_C$.

This implies Alperin's conjecture, and for $\delta = d$, it gives an older conjecture of Alperin and McKay.

3.3. The Isaacs-Navarro conjecture. This is another generalization of the Alperin-McKay conjecture. Let $A = a\mathcal{O}G$ be a block with defect group D, $|D| = p^d$, and let $B = b\mathcal{O}H_G(D)$ be the Brauer correspondent of A. Let k be an integer such that $p \nmid k$ and denote by $M_k(A)$ be the number of simple $\mathcal{K}A$ modules V with defect d such that $(\dim_{\mathcal{K}} V)_{p'} \equiv \pm k \pmod p$. Denote $c = |G : N_G(D)|_{p'}$. Then the conjecture predicts that $M_{ck}(A) = M_k(B)$.

There are stronger forms of this conjecture, involving the action of the Galois group of $\mathcal{K}$ on characters (see G. Navarro [52] and A. Turull [76], Tur06a). K. Uno [78] also formulated alternating sum versions of these conjectures.

3.4. Assume that A has abelian defect group D, and let $B = b\mathcal{O}N_G(D)$ be the Brauer correspondent of A. In this case, Alperin's conjecture reduces to the equalities

- $\ell(A) = \ell(B)$ (A and B have the same number of simple modules);
- $\mathrm{k}(A) = \mathrm{k}(B)$ ($\mathcal{K}A$ and $\mathcal{K}B$ have the same number of simple modules).

Dade's conjecture also reduces to the equalities

- $\mathrm{k}_\delta(A) = \mathrm{k}_\delta(B)$ ($\mathcal{K}A$ and $\mathcal{K}B$ have the same number of simple modules of defect δ), $\delta \geq 0$.

These suggest that there should be a deeper structural connection between A and B.

4. Equivalences between symmetric algebras

Let A and B be symmetric $\mathcal{O}$-algebras, free over $\mathcal{O}$ (note that blocks of $\mathcal{O}G$ satisfy these conditions). The (A, B)-bimodule is called *exact*, if ${}_AM$ and M_B are projective, finitely generated. Denote by $M^\vee$ the $\mathcal{O}$-dual of M.

In this context, Broué discusses three levels of equivalence: Morita equivalence, derived equivalence and stable equivalence of Morita type. An important progress was achieved by the fundamental work of Rickard describing the equivalences the derived categories of two algebras (regarded as triangulated categories). The characterization of derived equivalences is given either in terms or *one-sided tilting complexes* or *two-sided tilting complexes* (complexes of bimodules). Equivalences between the stable module categories are much more general; the Morita stable equivalences are those otained by tensoring with exact bimodules.

4.1. Morita equivalence. Recall that the abelian categories A-Mod and B-Mod are equivalent if and only if the following equivalent conditions hold.

- There is an exact (A, B)-bimodule M such that $M \otimes_B M^\vee \simeq A$ in $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-Mod and $M^\vee \otimes_A M \simeq B$ in $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-Mod.
- There is a progenerator ${}_AP$ such that $B \simeq \mathrm{End}_A(P)^{\mathrm{op}}$.

4.2. Rickard equivalence. The homotopy categories $\mathcal{H}^b(A\text{-mod})$ and $\mathcal{H}^b(B\text{-mod})$ are equivalent as triangulated categories if and only if the following equivalent conditions hold.

- There is a bounded complex C of exact (A, B) bimodules (called two-sided tilting complex) such that $C \otimes_B C^\vee \simeq A$ in $\mathcal{H}(A \otimes_{\mathcal{O}} A^{\mathrm{op}}\text{-mod})$ and $C^\vee \otimes_A C \simeq B$ in $\mathcal{H}(B \otimes_{\mathcal{O}} B^{\mathrm{op}}\text{-mod})$.
- There exists an object $T \in \mathcal{H}^b(A\text{-proj})$ (called one-sided tilting complex) such that $\mathrm{End}_{\mathcal{H}(A\text{-mod})}(T)^{\mathrm{op}} \simeq A$, $\mathrm{Hom}_{\mathcal{H}(A\text{-mod})}(T, T[n]) = 0$ for $n \neq 0$, and the direct summands of T generate $\mathcal{H}^b(A\text{-proj})$ as a triangulated category.

The connection between the tilting complexes C and T is much more subtle, since $\mathrm{End}_{\mathcal{H}(A\text{-mod})}(T)^{\mathrm{op}}$ acts on T only up to homotopy.

4.3. Stable Morita equivalence. By definition, the bounded complex C of exact (A,B)-bimodules induces a stable equivalence between A and B, if $C\otimes_B C^\vee \simeq A\oplus P$ in $\mathcal{H}(A\otimes_{\mathcal{O}} A^{\mathrm{op}}\text{-mod})$ and $C^\vee\otimes_A C\simeq B\oplus Q$ in $\mathcal{H}(B\otimes_{\mathcal{O}} B^{\mathrm{op}}\text{-mod})$, where P and Q are bounded complexes of projective bimodules.

Note that if $V\in B$-stmod, then the image of V in A-stmod is obtained by taking the image of $C\otimes_B V$ under the composition

$$\mathcal{H}^b(A\text{-mod})\to\mathcal{C}^b(A\text{-mod})\to\mathcal{C}^b(A\text{-mod})/\mathcal{H}^b(A\text{-proj})\simeq A\text{-stmod}$$

of exact functors. Moreover, by using an argument of Rickard (seee also subsection 8.2 below), one may replace C with an exact (A,B)-bimodule M, so that the stable equivalence is induced by the exact functor $M\otimes_B - :$ B-mod $\to A$-mod.

4.4. Here is a list of invariants preserved by a derived equivalence. Assume that A and B are derived equivalent. Then:

a) The Grothendieck groups $K_0(A)$ and $K_0(B)$ are isomorphic (Rickard).

b) The centers $Z(A)$ and $Z(B)$ are isomorphic, and more generally the Hochschild cohomology, rings $HH^*(A)$ and $HH^*(B)$ are isomorphic (Rickard).

c) The cyclic homology $HC_*(A)\simeq HC_*(B)$ (Keller).

d) $\ell(A)=\ell(B)$ (Happel).

e) $\mathrm{k}(A)=\mathrm{k}(B)$. Moreover, if A and B are blocks of group algebras, and if the simple $\mathcal{K}A$-module X corresponds to the simple $\mathcal{K}B$-module Y, then $\mathrm{def}(X)=\mathrm{def}(Y)$, that is, $\mathrm{k}_\delta(A)=\mathrm{k}_\delta(B)$ for $\delta\geq 0$ (Broué).

Moreover, if the Rickard tilting complex is defined over the ring of p-adic integers, then X and Y above have the same field of definition and p-local Schur index (Marcus [49], [49]).

f) A and B have the same representation type.

g) Cartan matrices are invariant under Morita equivalence, but not under derived equivalence.

h) Let KA be the k-vector space generated by the subset $\{xy-yx \mid xy\in A\}$. Külshammer has introduced for any integer n the spaces

$$T_n(A):=\{x\in A\mid x^{p^n}\in KA\}$$

and let $T_n(A)^\perp$ be the orthogonal space to $T_n(A)$ with respect to the symmetrizing form of A. Then the isomorphism $\phi : ZA\to ZB$ induced by an equivalence $D^b(A)\simeq D^b(B)$ of standard type has the property $\phi(T_nA^\perp)=T_nB^\perp$ for all positive integers $n\in\mathbb{N}$ (Zimmermann).

i) Let $Z_{p'}kG$ be the k-subspace of the group algebra kG spanned by all p-regular class sums in G. If A is a block with abelian defect group of kG, then $Z_{p'}A:=A\cap Z_{p'}kG$ is a subalgebra of A which is invariant under derived equivalences (Fan and Külshammer [17]).

5. Some examples

5.1. p-nilpotent groups. Assume that $G = D \rtimes H$, where D is a Sylow p-subgroup of G. Then $B_0(\mathcal{O}G) \simeq \mathcal{O}P$.

5.2. Isomorphic blocks. Let $H \trianglelefteq G$, D a Sylow p-subgroup of G, and assume that $p \nmid [G : H]$ and $HC_G(D) = G$. Then $B_0(\mathcal{O}G) \simeq B_0(\mathcal{O}H)$ via restriction.

5.3. T.I. situation. Assume that G has trivial intersection Sylow p-subgroups (i.e. $D \cap {}^gD = 1$ for $g \in G \setminus N_G(D)$, where D is a Sylow p-subgroup of G). Let $A = B_0(\mathcal{O}G)$ and $B = B_0(\mathcal{O}N_G(D)$. Then it is not very difficult to see that *the bimodule ${}_AA_B$ induces a Morita stable equivalence between A and B.*

An interesting example for the T.I. situation is the Suzuki group $G = Sz(8)$ with $p = 2$. Then A and B are stably equivalent, but not derived equivalent. In fact, $Z(kA) \simeq Z(kB)$, but $Z(A) \not\simeq Z(B)$ (Cliff).

More generally, a subgroup H of G is called *weakly embedded* if $p \nmid [G : H]$, and $N_G(Q) = N_H(Q)\,\mathrm{O}_{p'}(C_G(Q))$ for any p-subgroup $Q \neq 1$ of H. If this is the case, then again *the bimodule ${}_AA_B$ induces a Morita stable equivalence between A and B.*

A conjecture of Auslander predicts that if two self-injective algebras over a field are stably equivalent, then they have the same number of non-projective simple modules. This would imply Alperin's conjecture in the case of abelian defect groups.

By using the classification of finite simple groups, Blau and Michler proved the following theorem.

Theorem 5.4. *If G has T.I. Sylow p-subgroups, then $\ell(A) = \ell(B)$.*

6. The abelian defect group conjecture

Let $A = a\mathcal{O}G$ be a block of $\mathcal{O}G$ with defect group D, let $H = N_G(D)$, and let $B = b\mathcal{O}H$ be the block of $\mathcal{O}H$ corresponding to A.

6.1. Broué's Conjecture. *If D is abelian, then the algebras A and B are Rickard equivalent.*

Actually, it is conjectured that several refinements of 6.1 are also true, and then such refined equivalences have additional consequences.

6.2. Splendid and basic equivalences. The complex ${}_AC_B$ inducing the equivalence is *splendid*, that is, its components are relative $\delta(D)$-projective p-permutation $\mathcal{O}(G\times H)$-modules, that is, are direct summands of $\mathcal{O}(G\times H)$-modules of the form $\mathrm{Ind}_{\delta(Q)}^{G\times H}\mathcal{O}$, where $Q \leq D$ and $\delta(Q)$ is the diagonal subgroup of $Q \times Q$.

These come from equivalences between the source algebras of the blocks, and their main feature is that certain summands of $\mathrm{Br}_{\delta(Q)}^{G\times H}$ induce equivalence between certain blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ involved in $\mathrm{Br}_Q(a)$

and $\mathrm{Br}_Q(b)$, for any subgroup Q of D. This refinement relies on the fact that in the situation of 6.1, the Brauer categories of subpairs of A and B are equivalent.

Actually, one may formulate a more general version of conjecture 6.1, by letting H be a arbitrary finite group and B a block of $\mathcal{O}H$ having a defect group isomorphic to D, such that this isomorphism induces an equivalence between the Brauer categories of A and B.

By 3.4, Conjecture 6.1 implies the conjecture of Alperin and Dade in the abelian defect group case. As Broué has pointed out, the "splendid" form of 6.1 also implies the validity of the Isaacs-Navarro conjecture for blocks with abelian defect groups.

The concept of splendid equivalence introduced by Rickard in the case of principal blocks, and it was generalized (in different settings) to arbitrary blocks by Linckelmann and Harris. A splendid Morita equivalence is also called a *Puig equivalence*. Moreover, Puig [62] studied in a very general setting the local structure of equivalences between blocks, and he observes a more general class of equivalences which are compatible with the Brauer construction, the so called *basic equivalences*.

Recall that, by definition, the Morita equivalence induced by an indecomposable $(A\otimes B^{\mathrm{op}})$-module M is *basic* between A and B if its source modules have rank prime to p. Note that the existence of a basic Rickard equivalence between A and B actually implies that equivalence of the respective Brauer categories. We also refer the reader to Rouquier [73] for a detailed discussion of this topic.

A generalization of Conjecture 6.1 was formulated by Rouquier [73]. Assume for simplicity that A is the principal block (but this also generalizes to arbitrary blocks). The *hyperfocal subgroup* $\mathfrak{h}(D)$ is, by definition, the subgroup generated by the commutators $[K,Q]$, where Q runs over the subgroups of D, and K runs over the p'-subgroups of $N_G(Q)$. *If $\mathfrak{h}(D)$ is abelian, then there is a basic Rickard equivalence between A and B.*

Note that A is a nilpotent block if and only if $\mathfrak{h}(D)=1$. In this case, A is basically Morita equivalent to $\mathcal{O}D$ (where the Sylow p-subgroup D is arbitrary), but in general, there is no splendid Rickard equivalence between A and $\mathcal{O}D$.

6.3. Isotypies and p-permutation equivalences. A perfect isometry between A and B is an isometry between the character groups of A and B satisfying a certain arithmetic property with respect to p. An isotypy between A and B is a family of perfect isometries between the centralizers of subgroups $P\le D$ in G and H, which are compatible with the generalized decomposition map (see Broué [4]).

Boltje and Xu [3] defined a p-permutation equivalence between A and B as an element γ in the representation ring of p-permutation (A,B)-bimodules which satisfes $\gamma\cdot\gamma^\vee=[A]$ and $\gamma^\vee\cdot\gamma=[B]$ in the representation rings of (A,A)-bimodules and (B,B)-bimodules, respectively, where $\gamma^\vee$ denotes the dual of γ. They showed that a splendid derived equivalence between A and B induces a p-permutation equivalence (see also Boltje [2]). Moreover, if D is abelian and the Frobenius categories of A and B are equivalent, then a p-permutation equivalence between A and B induces an isotypy between A and B.

The notion of p-permutation equivalence was adapted to deal with the source algebras of A and B by Linckelmann [42], and he obtained extensions of the above results.

6.4. The equivalence in 6.1 is compatible with p'-extensions of the group G (see also 7.3 below).

6.5. The equivalence in 6.1 is compatible with p-central extensions of the group G (see also 7.4 below).

6.6. The equivalence in 6.1 takes the trivial module to the trivial module. More precisely, if A and B are the principal blocks, then $C \otimes_B \mathcal{O} \simeq \mathcal{O}$.

7. Methods

Here A and B are the same as in the preceding section.

7.1. Gluing local equivalences. With the same notations, let C be a splendid complex of (A, B)-bimodules and assume that A is the principal block. By a theorem of Rouquier, *the complex C induces a stable equivalence between A and B if and only if $Br_{\delta P}^{A\otimes B^{\mathrm{op}}} C$ induces a Rickard equivalence between $kC_H(P)a_P$ and $kC_H(P)b_P$ for every nontrivial subgroup P of D*, where $a_P = \mathrm{Br}_P(a)$ and $b_P = \mathrm{Br}_P(b)$ are the principal blocks of $kC_H(P)e_P$ and $kC_H(P)f_P$ respectively.

A particular case is the following older theorem of Broué: *the splendid (A, B)-bimodule M induces a stable Morita equivalence between A and B if and only if* $\mathrm{Br}_{\delta P}^{A\otimes B^{\mathrm{op}}} M$ *induces a Morita equivalence between $kC_H(P)a_P$ and $kC_H(P)b_P$ for every nontrivial subgroup P of D.*

Note that this holds if H is weakly embedded in G (so, in particular, if D is T.I.).

A much more general discussion of gluing splendid tilting complexes was done by Rouquier [73]. Together with the next subsection, this raises the possibility of proving Broué's conjecture by induction.

7.2. Lifting stable equivalences to Rickard equivalences. The following strategy devised by Okuyama is widely used. Assume that we have a stable Morita equivalence between A and B, and that we can construct a tilting complex for B with endomorphism ring C. Then B and C are also stably equivalent of Morita type, hence so are A and C. Assume that the stable equivalence between A and C sends simple A-modules to simple C-modules. Then, by a theorem of Linckelmann, A and C are in fact Morita equivalent.

Various useful constructions of tilting complexes have been proposed by Okuyama and Rickard. Explicit tilting complexes based on their algorithms were exhibited by Holloway [22], Robbins [69] and by M. Schaps et al. (see [1] and the references given there) in the cases $D = C_3 \times C_3$ and $C_5 \times C_5$, by making use of computational algebra software.

7.3. Clifford theory. One strategy to prove the above conjectures is to reduce them to the case of simple groups and to use the classification and the available information on their representations. To this goal, it is important to investigate the relationship between representations of a group and the representations of its normal subgroups and its factor groups. This is the object of "Clifford theory". A reduction theory for Broué's conjecture for principal blocks has been given by Marcus [44], and it is based on the properties of graded bimodules and equivalences induced by them.

a) (*Going up*) In our situation, assume that G is a normal subgroup of a group $\tilde{G}$ and let $F = \tilde{G}/G$ be a p'-group. Let also $\tilde{H} = N_{\tilde{G}}(D)$, and assume that a (and hence b) is F-invariant. The compatibility with p'-extensions 6.4 requires that the complex C inducing the equivalence between A and B extends to a complex of Δ-modules, where $\Delta = \mathcal{O}N_{\tilde{G}\times\tilde{H}}(\delta P)$. Then one may lift an equivalence between A and B to an equivalence between the F-graded algebras $a\mathcal{O}\tilde{G}$ and $b\mathcal{O}\tilde{H}$ induced by the complex $(a\mathcal{O}\tilde{G} \otimes_{\mathcal{O}} (b\mathcal{O}\tilde{H})^{\mathrm{op}}) \otimes_\Delta C$. We discuss such "group graded equivalences" in the next section. Note that graded equivalences defined over small fields preserve Turull's Clifford classes associated to characters (see [49] and [50]).

b) (*Going down*) The following situation was considered in [47], and applies to the symmetric and alternating groups. Assume that G has a normal subgroup G^+, and let $H^+ = H \cap G^+$. Assume also that $a \in \mathcal{O}G^+$ and $b \in \mathcal{O}H^+$, so A and B are F-graded algebras, where $F = G/G^+$. Let C be a complex inducing an equivalence between A and B. If C is a complex of F-graded (A, B)-bimodules, then then the 1-component (where $1 \in F$) of C induces an equivalence between $a\mathcal{O}G^+$ and $b\mathcal{O}H^+$. One useful case is when F is a cyclic p'-group. If there is an action of the group $\hat{F} = \mathrm{Hom}(F, \mathcal{K}^\times)$ on C, then C is F-graded.

7.4. Lifting equivalences to p-central extensions. Assume that $G = \hat{G}/P$ is, where P is a central p-subgroup of $\hat{G}$, and let $\hat{H}$ be the inverse image of H in $\hat{G}$. The blocks a and b lift to blocks of $\mathcal{O}\hat{G}$ and $\mathcal{O}\hat{H}$. By the compatibility with p-central extensions 6.5, one may lift an equivalence between A and B to an equivalence between $\tilde{a}\mathcal{O}\tilde{G}$ and $\tilde{b}\mathcal{O}\tilde{H}$.

8. Graded equivalences

We are concerned with the problem of constructing derived equivalences between two algebras R and S graded by the finite group G, and denote $A = R_1$ and $B = S_1$.

8.1. **G-graded tilting complexes.** Recall that equivalences between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ have been characterized by J. Rickard in terms of one-sided tilting complexes, and in terms of two-sided tilting complexes. A difficulty in the case of derived equivalences is that if T is an one-sided tilting complex of A-modules, then $\mathrm{End}_{\mathcal{H}(A)}(T)^{\mathrm{op}}$ act on T only up to homotopy.

For two-sided tilting complexes, compatibility with p'-extensions is expressed in terms of the existence of certain tilting complexes of graded bimodules. The results of [46] and [48] are motivated by constructions due to T. Okuyama [53], [54], [55], and to J. Rickard [68], aimed to lift stable equivalences between symmetric algebras to Rickard equivalences. Although they end up with two-sided tilting complexes, the methods of Okuyama and Rickard are based on constructions of one-sided tilting complexes.

8.1. Let k be a commutative ring, G a group and $R = \bigoplus_{g\in G} R_g$ a G-graded k-algebra. We denote by R-Gr the category of G-graded R-modules $M = \bigoplus_{x\in G} M_x$ and grade-preserving R-homomorphisms.

A complex $T \in \mathcal{H}(R\text{-Gr})$ is called *G-invariant* if $T(g) \simeq T$ (in the category $\mathcal{H}(R\text{-Gr})$) for all $g \in G$. T is called *weakly G-invariant* if $T(g) \in \mathrm{add}(T)$ for all $g \in G$.

Let $T \in \mathcal{H}(R\text{-Gr})$, $E = \mathrm{End}_{\mathcal{H}(R)}(T)^{\mathrm{op}}$, and assume that G is a finite group. Then

a) E is a G-graded algebra with components

$$E_g \simeq \mathrm{Hom}_{\mathcal{H}(R\text{-Gr})}(T, T(g)).$$

b) E is strongly graded (crossed product) if and only if T is weakly G-invariant (G-invariant).

8.2. A complex P of G-graded R-modules is called *perfect* if it is bounded, and its terms are finitely generated projective R-modules. We denote by R-Grperf the full subcategory of $\mathcal{D}(R\text{-Gr})$ consisting of complexes quasi-isomorphic to a perfect complex. Rickard's characterization of R-perf also holds in this situation: *a complex $P \in \mathcal{D}(R\text{-Gr})$ belongs to R-Grperf if and only if it is compact*, that is, the functor

$$\mathrm{Hom}_{\mathcal{D}(R\text{-Gr})}(P, -) \to \mathcal{A}b$$

commutes with infinite direct sums.

An object $T \in \mathcal{D}(R\text{-Gr})$ is called a *G-graded tilting complex* over R if it satisfies the following conditions:

(a) $T \in R$-Grperf (that is, T, regarded as a complex of R-modules, belongs to R-perf).
(b) $\mathrm{Hom}_{\mathcal{D}(R)}(T, T[n]) = 0$ for all $n \neq 0$.
(c) $\mathrm{add}(T)$ generates R-perf as a triangulated category.

The following theorem is the main result of [46] combined with [45, Theorem 4.7], and characterizes *graded derived equivalences.*

Theorem 8.3. *Let k be a commutative ring, G a finite group and R, S two G-graded k-algebras. The following statements are equivalent.*

(i) *There is a G-graded tilting complex $T \in \mathcal{D}(R\text{-Gr})$ and an isomorphism $S \to \mathrm{End}_{\mathcal{D}(R)}(T)^{\mathrm{op}}$ of G-graded algebras.*

(ii) *There is a complex X of G-graded (R,S)-bimodules such that the functor*

$$X\overset{\mathbf{L}}{\otimes}_S - : \mathcal{D}(S) \to \mathcal{D}(R)$$

is an equivalence.

(iii) *There are equivalences $F : \mathcal{D}(S) \to \mathcal{D}(R)$ and $F^{\mathrm{gr}} : \mathcal{D}(S\text{-Gr}) \to \mathcal{D}(R\text{-Gr})$ of triangulated categories such that F^{gr} is a G-graded functor* (that is, it commutes with g-suspensions, for all $g \in G$), *and the diagram*

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "F"] & \mathcal{D}(R) \\ \mathcal{D}(S\text{-Gr}) \arrow[u, "\mathcal{U}"] \arrow[r, "F^{\mathrm{gr}}"'] & \mathcal{D}(R\text{-Gr}) \arrow[u, "\mathcal{U}"'] \end{tikzcd}$$

is commutative, where $\mathcal{U}$ is the ungrading functor.

(iv) *There are equivalences*

$$F_{\mathrm{perf}} : \mathcal{D}(S\text{-perf}) \to \mathcal{D}(R\text{-perf})$$

and

$$F^{\mathrm{gr}}_{\mathrm{perf}} : \mathcal{D}(S\text{-Grperf}) \to \mathcal{D}(S\text{-Grperf})$$

of triangulated categories such that $F^{\mathrm{gr}}_{\mathrm{perf}}$ is a G-graded functor and $\mathcal{U} \circ F^{\mathrm{gr}}_{\mathrm{perf}} = F_{\mathrm{perf}} \circ \mathcal{U}$.

(v) (provided that R and S are strongly graded) *There are (bounded) complexes X_1 of $\Delta(R \otimes_k S^{\mathrm{op}})$ modules and Y_1 of $\Delta(S \otimes_k R^{\mathrm{op}})$ modules, and isomorphisms*

$$X_1\overset{\mathbf{L}}{\otimes}_{S_1} Y_1 \simeq R_1 \quad \text{in} \quad \mathcal{D}^b(\Delta(R \otimes_k R^{\mathrm{op}}))$$

and

$$Y_1\overset{\mathbf{L}}{\otimes}_{R_1} X_1 \simeq S_1 \quad \text{in} \quad \mathcal{D}^b(\Delta(S \otimes_k S^{\mathrm{op}})).$$

The next result is concerned with the induction one-sided tilting complexes, and the first statement is an analogue of the equivalence between (ii) and (v) in the above theorem.

Proposition 8.4. *Assume that G is finite and R is strongly graded. Let T be a G-invariant object of $\mathcal{H}^b(A)$, and denote $\tilde{T} = R \otimes_A T$ and $S = \mathrm{End}_{\mathcal{H}(R)}(\tilde{T})^{\mathrm{op}}$.*

1) *T is a tilting complex for A if and only if $\tilde{T}$ is a G-graded tilting complex for R.*

2) *If T is a tilting complex for A and R is a finite dimensional symmetric crossed product, then S is a symmetric crossed product of $B := S_1 \simeq \mathrm{End}_{\mathcal{H}(A)}(T)^{\mathrm{op}}$ and G.*

8.2. **Stable equivalences and Rickard equivalences between symmetric algebras.** We shall consider the situation when R and S are G-graded symmetric crossed products over the algebraically closed field k, where G is a finite group of order not divisible by p.

8.5. Let $T^\bullet$ be an one-sided tilting complex of G-graded R-module with endomorphism ring $\mathrm{End}_{\mathcal{H}(R)}(T^\bullet)^{\mathrm{op}} \simeq S$. Denote $A = R_1$, $B = S_1$, and by Δ the diagonal subalgebra

$$\Delta = \Delta(R \otimes_k S^{\mathrm{op}}) = \bigoplus_{g\in G} R_g \otimes_k S_{g^{-1}}$$

of $R\otimes_k S^{\mathrm{op}}$. There exists a two-sided tilting complex $X^\bullet$ of G-graded (R, S)-bimodules. Then $X^\bullet_1$ is a complex of Δ-modules, and also a two-sided tilting complex of (R_1, S_1)-bimodules. Let $Y^\bullet_1$ be a projective resolution of $X^\bullet_1$ as Δ-modules. It is possible to truncate $Y^\bullet_1$ in order to obtain a bounded complex

$$Z^\bullet_1 := (\cdots \to 0 \to \mathrm{Ker} d^n \to Y^n_1 \to Y^{n+1}_1 \to \cdots),$$

of Δ-modules quasi-isomorphic to $X^\bullet_1$, such that all the terms of $Z^\bullet_1$ but $\mathrm{Ker}\, d^n$ are projective Δ-modules, and $\mathrm{Ker}\, d^n$ is projective as an R_1-module and as a right S_1-module. Let

$$M_1 := \Omega^n(\mathrm{Ker} d^n), \quad N_1 := \Omega^{-n}(\mathrm{Hom}_{R_1}(\mathrm{Ker} d^n, R_1)),$$

and

$$Z^\bullet := (R \otimes_k S^{\mathrm{op}}) \otimes_\Delta Z^\bullet_1.$$

Then we have:

(a) The functor
$$Z^\bullet \otimes_S - : \mathcal{H}^b(S) \to \mathcal{H}^b(R)$$
is an equivalence, and it is also a graded functor. The complex $Z^\bullet$ is called a *Rickard tilting complex* or a split endomorphism tilting complex. The inverse equivalence is induced by the k-dual of $Z^\bullet$.

(b) M_1 is a Δ-module, $N_1 \simeq M_1^\vee$ as $\Delta(S \otimes_k R^{\mathrm{op}})$-modules, and M_1 and N_1 induce a stable Morita equivalence between R_1 and S_1.

(c) It follows that $M := (R \otimes_k S^{\mathrm{op}}) \otimes_\Delta M_1$ and its k-dual induce a graded stable Morita equivalence between R and S.

This can be extended to Rouquier's notion of stable equivalences induced by complexes.

Proposition 8.6. *Assume that G is a p'-group, and let C and D be bounded complexes of G-graded (R, S)-bimodules such that C induces a stable equivalence between R and S, D induces a derived equivalence between R and S, and the stable equivalence between A and B induced by D_1 agrees on each simple module, up to isomorphism, with that induced by C_1.*

Then there is a bounded complex X of finitely generated G-graded (R, S)-bimodules such that

1) *$X = C \oplus P$, where P is a complex of G-graded projective bimodules;*

2) *X induces a G-graded homotopy equivalence between R and S;*

3) *In the derived category of G-graded (R, S)-bimodules, X is isomorphic to the composition between D and a G-graded Morita autoequivalence of R.*

8.7. In this context, T. Okuyama has devised several methods of constructing tilting complexes, starting with a stable equivalence and a choice of a subset of the set of simple modules. Let $\{S_i \mid i \in I\}$ be a set of representatives for the isomorphism classes of simple A-modules, and let P_i be a projective cover of S_i. The index set I becomes a G-set via the action of G on simple A-modules. Let I_0 be a subset of I, and assume that the (A, B)-bimodule M_1 induces a stable Morita equivalence between A and B. In [46] and [48] we showed that Okuyama's complexes are compatible with p'-extensions provided that the following two conditions hold.

(1) I_0 is a G-subset of I.

(2) M_1 is a Δ-module.

For instance, in the T.I. situation, the bimodule $M_1 = {}_AA_B$ induces a stable Morita equivalence between A and B, and M_1 is clearly a Δ-module.

8.8. We now discuss Rickard's construction [66] and its extension to group graded algebras [48]. Under a derived equivalence between the k-algebras A and B, the objects $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, corresponding to simple B modules must satisfy the following conditions.

(a) $\mathrm{Hom}(X_i, X_j[m]) = 0$ for $m < 0$.

(b) $\mathrm{Hom}(X_i, X_j) = k$ if $i = j$ and 0 otherwise.

(c) X_i, $i \in I$, generate $\mathcal{D}^b(A\text{-mod})$ as a triangulated category.

In order to obtain a graded derived equivalence, we also need to consider the action of G. The next results from [48] should be compared with Robbins [69], and apply to the examples considered by J. Chuang [10], J. Rickard [68] and M. Holloway [22].

Theorem 8.9. *Let R be a symmetric crossed product of A and G, let I be a finite G-set, and let $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, be objects satisfying* (8.8. a, b, c)*. Assume that the objects X_i satisfy the additional condition*

(8.9.a) *$R_g \otimes_A X_i \simeq X_{^gi}$ in $\mathcal{D}^b(A\text{-mod})$, for all $i \in I$ and $g \in G$.*

Then there is another symmetric crossed product R' of A' and G, and a G-graded derived equivalence between R and R', whose restriction to A sends X_i, $i \in I$, to the simple A'-modules.

In order to lift a stable equivalence to a graded derived equivalence by using Okuyama's strategy, in our general setting we need to assume that p does not divide the order of G.

Corollary 8.10. *Let R and S be two G-graded symmetric crossed products, and denote $A = R_1$ and $B = S_1$. Assume that G is a p'-group and I is a G-set. Let M be a G-graded (R, S)-bimodule inducing a Morita stable equivalence between R and S, and let $\{S_i \mid i \in I\}$ be a set of representatives for the simple B-modules.*

If there are objects $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, satisfying the conditions (8.8. a,b,c) *and* (8.9. a)*, and such that X_i is stably isomorphic to $M_1 \otimes_B S_i$, for all $i \in I$, then there is a G-graded derived equivalence between R and S.*

Remark 8.11. a) The setting of Broué' s conjecture and splendid equivalences actually leads to graded equivalences. Let D be an abelian Sylow p-subgroup of G, $H = N_G(D)$, fix a subgroup p of D, and denote here $X = C_G(P)$, $\tilde{X} = N_G(P)$, $Y = C_H(P)$, $\tilde{Y} = N_H(P)$, so $\tilde{X}/X \simeq \tilde{Y}/Y$. Let $C^\bullet$ be a splendid tilting complex of (kGe, kHf)-bimodules, where e and f are the principal blocks, and let e_P and f_P be the principal blocks of kX and kY respectively. Observe that $(e_P \otimes f_P)k(N_{G\times H}(\delta P))$ is the diagonal subalgebra of $k\tilde{X}e_P \otimes_k (k\tilde{Y}f_P)^{\mathrm{op}}$. By the functoriality of the Brauer construction it follows that the complex $\mathrm{Ind}_{N_{G\times H}(\delta P)}^{N_G(P)\times N_H(P)} \mathrm{Br}_{\delta P}^{G\times H}(C^\bullet)$ induces an $\tilde{X}/X$-graded Rickard equivalence between $k\tilde{X}e_P$ and $k\tilde{Y}f_P$.

b) A particular case of application of Theorem 8.3 is Okuyama-Gollan [56, Lemma 1.5 and Proposition 1.6]. Assume that for the fixed subgroup P of D, we have $C_G(P) = P \times G_1$ for some subgroup G_1 of G. It follows that $C_H(P) = P \times H_1$, where $H_1 = H \cap G_1$. Let e_1 be the principal block of kG_1 and f_1 the principal block of kH_1, and regard $R = kC_G(P)e_1$ and $S = kC_H(P)f_1$ as P-graded algebras with $R_1 = kG_1e_1$ and $S = kH_1f_1$. Observe that

$$\Delta(R \otimes_k S^{\mathrm{op}}) = (e_1 \otimes f_1)k(\delta(P)(G_1 \times H_1)),$$

and since $\delta(P) \cap (G_1 \times H_1) = 1$, scalar restriction is an isomorphism of categories from $\Delta(R\otimes_k S^{\mathrm{op}})$-mod to $R_1 \otimes_k S_1^{\mathrm{op}}$-mod, the inverse being defined by the trivial action of $\delta(P)$ of an (R_1, S_1)-bimodule. Consequently, if $C^\bullet$ is a (splendid) tilting complex of (R_1, S_1)-bimodules, then $\mathrm{Ind}_{\Delta(R\otimes_k S^{\mathrm{op}})}^{R\otimes_k S^{\mathrm{op}}} C^\bullet$ is a (splendid) tilting complex of P-graded (R, S)-bimodules.

9. The status of Broué's conjecture

9.1. Broué's conjecture appears to be very hard to check even on particular examples. The following list contains cases where Broué's conjecture has been verified. Here again A is a block of $\mathcal{O}G$ with defect group D, and B is the block of $\mathcal{O}N_G(D)$ corresponding to A.

(1) D cyclic (Rickard, Linckelmann, Rouquier). Moreover, Rickard tilting complexes defined over the p-adic number field $\mathbb{Q}_p$ exist in this case [50], and this implies Turull's strengthening the McKay conjecture [77, Theorem 2.2] for cyclic blocks.
(2) $D \simeq C_2 \times C_2$ (Erdmann, Linckelmann, Rickard, Rouquier).
(3) A principal block, $p = 2$ (Rouquier, Okuyama, Okuyama-Golan, and the reduction theorem by Marcus).
(4) A principal block with defect group $D \simeq C_3 \times C_3$ (Okuyama, Koshitani, Kunugi, Waki, and the reduction theorem by Marcus).

(5) G a symmetric group (Rickard, Chuang-Kessar, Chuang-Rouquier) or an alternating group (Marcus).
(6) A faithful 3-block of defect group $C_3 \times C_3$ in the central extension of the Mathieu group M_{22} by C_4 (J. Müller and M. Schaps).
(7) G p-solvable (there is a Morita equivalence) (Dade, Puig, Harris-Linckelmann).
(8) non-principal blocks of $G = HS$, $G = O'N$, $G = He$, $G = Suz$ and J_4, $p = 3$, $D \simeq C_3 \times C_3$ (Koshitani, Kunugi, Waki, Holm).
(9) non-principal blocks of $G = 2.J_2$ and $G = \mathrm{SL}_2(p^2)$, $p = 5$, $D \simeq C_5 \times C_5$ (Holloway).
(10) A any p-blocks with abelian defect group of the Tits group ${}^2F_4(2)'$, an then the Rickard equivalences can be lifted to prove the conjecture for the group ${}^2F_4(2)$ (the case $p = 5$ for principal blocks is done by Robbins [69]).
(11) A any p-blocks with abelian defect group of the Janko simple group J_4 [33].
(12) principal blocks of $G = J_2$ and $G = \mathrm{Sp}_4(4)$, $p = 5$, $D \simeq C_5 \times C_5$ (Holloway).
(13) principal blocks of $G = \mathrm{G}_2(2^n)$, $D \simeq C_5 \times C_5$, $5 \mid 2^n + 1$, $25 \nmid 2^n + 1$ (Usami-Yoshida).
(14) principal p-blocks of $G = \mathrm{G}_2(q)$, $D \simeq C_{p^a} \times C_{p^a}$, $p^a \| q+1$ (Okuyama).
(15) G a connected reductive algebraic group over $\mathbb{F}_q$, where p divides $q - 1$ but p does not divide the order of the Weil group (Puig).
(16) Unipotent blocks of weight 2 of $G = \mathrm{GL}_n(q)$, $p \nmid q$, $D \simeq C_{p^\alpha} \times C_{p^\alpha}$ (Turner, Hida-Miyachi).
(17) principal block of $G = \mathrm{SL}_2(p^n)$ (Okuyama); this lifts to the principal block of $\mathrm{GL}_2(p^n)$ (Marcus).
(18) principal block of $G = \mathrm{SU}_3(q^2)$, $p > 3$ and $p \mid q + 1$, $D \simeq C_{p^\alpha} \times C_{p^\alpha}$, where $p^\alpha = (q + 1)_p$ (Kunugi-Waki).
(19) principal block of $G = \mathrm{Sp}_4(q)$, $p > 2$ and $p \mid q + 1$, $D \simeq C_{p^\alpha} \times C_{p^\alpha}$, where $p^\alpha = (q + 1)_p$ (Kunugi-Okuyama-Waki).
(20) $G = \mathrm{GL}_2(q)$, $p \neq 2$ prime dividing $q - 1$ or $q + 1$ (Puig, Rouquier, Gonard).

9.2. Let (D, e) be a maximal (G, A)-Brauer pair (so e is a block of $\mathcal{O}C_G(D)$ corresponding to A), and set $N := N_G(P, e)$. Let $E := N/DC_G(D)$ be the *inertial quotient* of A. In some cases weaker types of equivalences exist.

(1) Assume that D is abelian. By the work of Usami and Puig, A and B are isotypic in the following cases:
   - $|E| = 2$;
   - $|E| = 3$;
   - $|E| = 4$;
   - E is a dihedral group of order 6;
   - $E \simeq C_4 \times C_2$ and $p \geq 7$;
   - $E \simeq C_3 \times C_3$ and $p \neq 2, 7$;

- E is an elementary abelian 2-group and $p \neq 2, 3$;
- E is a dihedral group of order 8 and $p \neq 3$.

(2) If D is abelian and the hyperfocal subgroup $[N, D]$ is cyclic, then A and B are isotypic. Actually, this is true under the following weaker conditions: E is cyclic, $C_E(x) = 1$ for $1 \neq x \in [N, P]$, and $|E| = \ell(A)$ (Watanabe [86]).

(3) Assume that D has order at most p^2 and and the number $\ell(B)$ of simple B-modules is 1. Then $\ell(A) = 1$, E is abelian, the decomposition matrices of A and B are equal, and there is a p-permutation equivalence between A and B inducing an isotypy between A and B all of whose signs are positive (Kessar and Linckelmann [27].

9.3. Interesting equivalences still exist in some cases of non-abelian defect groups.

(1) Consider the Chevalley group $G = G_2(q)$, where q is a power of 2 and $q \equiv 2, 5 \pmod 9$. Here $p = 3$ and the Sylow 3-subgroup D of $G_2(q)$ is the extraspecial group $M(3)$ of order 27 and exponent 3. Then there is a Morita stable equivalence between the principal block of $\mathcal{O}G_2(q)$ and the principal block of $\mathcal{O}N_G(Z(D))$. Moreover there is a Puig equivalence between the principal block of $\mathcal{O}N_G(D)$ and the principal block of $\mathcal{O}N_G(Z(D))$. This can be used to show that the principal blocks of $\mathcal{O}G_2(q)$ and $\mathcal{O}G_2(2)$ are Puig equivalent (Usami-Nakabayashi).

(2) Let $G = \mathrm{PGL}_3(4)$, $p = 3$, $D \simeq M(3)$, $H = N_G(P) \simeq \mathrm{PGU}_3(4)$, where $P = D \cap \mathrm{PSL}_3(4) \simeq C_3 \times C_3$ is a Sylow 3-subgroup of $\mathrm{PSL}_3(4)$. Then the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ are splendidly Rickard equivalent (Kunugi-Usami).

(3) Let $G = \mathrm{PSL}_3(q^2)$, $H = \mathrm{PSU}_3(q^2)$, $p = 3 \mid q + 1$, where q is a power of a prime. Then the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ are splendidly Rickard equivalent. This equivalence lifts to a splendid Rickard equivalence between the principal blocks of $\mathrm{SL}_3(q^2)$ and $\mathrm{SU}_3(q^2)$ (Kunugi-Okuyama).

(4) Let $G = \mathrm{GL}_2(q)$, $q \equiv \pm 3 \pmod 8$, $p = 2$, and let P be a Sylow 2-subgroup of $\mathrm{SL}_2(q)$ (so P is abelian). Then there is a splendid Rickard equivalence between the principal blocks of $\mathcal{O}G$ and $\mathcal{O}N_G(P)$ (Gonard).

(5) Let b be a block of kG with quaternion defect group Q_8. Let Z be the unique subgroup of order 2 of Q_8, let $H = C_G(Z)$, and let $c = \mathrm{Br}_Z(b)$ be the Brauer correspondent of c. Then the blocks kGb and kHc are Morita equivalent. This is due to Linckelmann and Kessar [26], and relies on earlier work of Cabanes and Picaronny [7] showing the existence of perfect isometries, and on Erdmann's classification of tame blocks [16].

(6) Assume that A has a non-abelian TI defect group D with $|D| = p^3$. Then there is a "generalized" isotypy between A and B (Eaton

[15]), and it is conjectured that the result is true for any TI defect group. This version of isotypy is related to the Isaacs-Navarro [23] refinement of the Alperin-McKay conjecture.

References

[1] A. Bar-Ilan, T. Berrebi, G. Chereshnya, R. Leabovich, M. Cohen and M. Schaps, *Explicit tilting complexes for the Broué's conjecture on* 3*-blocks*, Groups St. Andrews 2005. Vol. 1, 207–223, London Math. Soc. Lecture Note Ser., 339, Cambridge Univ. Press, Cambridge, 2007.

[2] R. Boltje, *Chain complexes for Alperin's weight conjecture and Dade's ordinary conjecture in the abelian defect group case*, to appear in Journal of Group Theory.

[3] R. Boltje and B. Xu, *On p-permutation equivalences: between Rickard equivalences and isotypies*, preprint 2006.

[4] M. Broué, *Isométries parfaites, Types de blocs, Catégories dérivées*, Astérisque **181-182** (1990), 61–92.

[5] M. Broué, *Equivalences of blocks of group algebras*, V. Dlab and L. Scott eds. Proc. of the NATO Advanced Research Workshop on Representation of Algebras and Related Topics, Kluwer Academic Publishers, Dordrecht (1991), 1–26.

[6] M. Broué, *Rickard equivalences and block theory*, C.M. Cambbell et al eds. Groups 1993 Galway/St. Andrews Proc. of the NATO Advanced Research Workshop on Representation of Algebras and Related Topics , Cambridge University Press London Math. Soc. Lecture Notes 211 **2** (1995), 58–79.

[7] M. Cabanes and C. Picaronny, *Types of blocks with dihedral or quaternion defect groups*, J. Fac. Sci. Univ. Tokyo 39 (1992) 141–161. Corrected version (1999): http://www.math. jussieu.fr/ cabanes/.

[8] J.F. Carlson and R. Rouquier, *Self-equivalences of stable module categories*, Math. Z. **233** (2000), 165–178.

[9] J. Chuang, *The derived categories of some blocks of symmetric groups and a conjecture of Broué*, J. Algebra **217** (1999), 114–155.

[10] J. Chuang, *Derived equivalence in* $\mathrm{SL}_2(p^2)$, Trans. Amer. Math. Soc. **353** (2001), 2897–2913.

[11] J. Chuang and R. Kessar, *Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture*, Bull. London Math. Soc. **34** (2002), 174–184.

[12] J. Chuang and J. Rickard, *Representations of finite groups and tilting*, in *Handbook of Tilting Theory* (L. Angeleri-Hügel, D. Happel and H. Krause eds.), Cambridge Univ. Press 2007, 359–391.

[13] J. Chuang and R. Rouquier, *Derived equivalences for the symmetric groups and* $\mathfrak{sl}_2$*-categorification*, Annals of Math., **167** (2008), 245–298.

[14] E.C. Dade, *Counting characters in blocks II.9*, in R. Solomon editor, Representation Theory of finite groups (Columbus, OH, 1995), Ohio State Univ. Math. Res. Inst. Publ. **6**, de Gruyter, Berlin 1997, 45–59.

[15] C.W. Eaton, *Perfect generalized characters inducing the AlperinMcKay conjecture*, Journal of Algebra **320** (2008), 2301–2327.

[16] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Math. 1428, Springer, 1990.

[17] Y. Fan and B. Külshammer *A note on blocks with abelian defect groups*, Journal of Algebra **317** (2007), 250–259.

[18] H. Gollan and T. Okuyama, *Derived equivalences for the smallest Janko group*, preprint (2000).

[19] B. Gonard, *Catégories dérivées des blocs à défaut non abélien de* $\mathrm{GL}(2,q)$, Thèse Univ. Paris 7, 2002.

[20] M.E. Harris, *Splendid equivalences for arbitrary blocs of finite groups*, J. London Math. Soc. (2) **60** (1999), 71–82.

[21] M.E. Harris and M. Linckelmann, *Splendid Rickard equivalences for blocks of p-solvable groups*, J. London Math. Soc. **62** (2000), 85–96.

[22] M. Holloway, *Broué's conjecture for the Hall-Janko group and its double cover*, Proc. London Math. Soc. (3) **86** (2003), 109–130.

[23] I.M. Isaacs and G. Navarro, *New refinements of the McKay conjecture for arbitrary finite groups*, Ann. of Math. (2) **156** (2002), 333–344.

[24] Keller, B., *On the construction of triangle equivalences*, in S. König and A. Zimmermann, Derived Equivalences for Group Rings, Springer-Verlag, Berlin 1998, 155–176.

[25] Keller, B., *Bimodule complexes via strong homotopy actions*, Algebras and Representation theory, **3** (2000), 357–376.

[26] R. Kessar and M. Linckelmann, *On perfect isometries for tame blocks*, Bull. London Math. Soc. **34** (2002) 46–54.

[27] R. Kessar and M. Linckelmann, *On stable equivalences and blocks with one simple modules*, preprint 2008.

[28] S. Koshitani and N. Kunugi, *The principal 3-blocks of the 3-dimensional projective special linear groups in non-defining characteristic*, J. Reine Angew. Math. **539** (2001), 1–27.

[29] S. Koshitani and N. Kunugi, *Broué's conjecture holds for principal 3-blocks with elementary abelian defect groups of order 9*, J. Algebra **248** (2002), 575–604.

[30] S. Koshitani and N. Kunugi, *Blocks of p-central group extensions*, Proc. Amer. Math. Soc. **133** (2004), 21–26.

[31] S. Koshitani, N. Kunugi and K. Waki, *Broué's conjecture for non-principal 3-blocks of finite groups*, J. Pure Appl. Algebra **173** (2002), 177–211.

[32] S. Koshitani, N. Kunugi and K. Waki, *Broué's abelian defect group conjecture for the Held group and the sporadic Suzuki group*, J. Algebra **279** (2004), 638–666.

[33] S. Koshitani, N. Kunugi and K. Waki, *Broué's Abelian defect group conjecture holds for the Janko simple group J_4*, J. Pure Appl. Algebra **212** (2008), 1438–1456.

[34] S. Koshitani and H. Miyachi, *The principal 3-blocks of four- and five-dimensional projective special linear groups in non-defining characteristic*, J. Algebra **226** (2000), 788–806.

[35] S. König and A. Zimmermann, *Derived Equivalences for Group Rings*, Springer-Verlag, Berlin 1998.

[36] N. Kunugi, *Morita equivalent 3-blocks of the 3-dimensional projective special linear groups*, Proc. London Math. Soc. **80** (2000), 575–589.

[37] N. Kunugi and K. Waki, *Derived equivalences for the 3-dimensional special unitary groups in non-defining characteristic*, J. Algebra **240** (2001), 251–267.

[38] M. Linckelmann, *Derived equivalence for cyclic blocks over a p-adic ring*, Math. Z. **207** (1991), 293–304.

[39] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223** (1996), 87–100.

[40] M. Linckelmann, *On derived equivalences and local structure of blocks of finite groups*, Turkish J. Math. **22** (1998), 93–107.

[41] M. Linckelmann, *On splendid derived and stable equivalences between blocks of finite groups*, J. Algebra **242** (2001), 819–843.

[42] M. Linckelmann, *Trivial source bimodule rings for blocks and p-permutation equivalences*, rans. Amer. Math. Soc. **361** (2009) 1279–1316.

[43] M. Linckelmann and R. Kessar, *On perfect isometries for blocks*, Bull. London Math. Soc. **34** (2002), 46–54.

[44] A. Marcus, *On equivalences between blocks of group algebras: reduction to the simple components*, J. Algebra **184** (1996), 372–396.

[45] A. Marcus, *Equivalences induced by graded bimodules*, Comm. Algebra **26** (1998), 713–731.
[46] A. Marcus, *Tilting complexes for group graded algebras*, J. Group Theory **6** (2003), 175–193.
[47] A. Marcus, *Broué's abelian defect group conjecture for alternating groups*, Proc. Amer. Math. Soc. **132** (2004), 7–14.
[48] A. Marcus, *Tilting complexes for group graded algebras. II*, Osaka J. Math. **42** (2005), 453–462.
[49] A. Marcus, *Characters and equivalence classes of central simple group graded algebras*, Communications in Algebra **36** (2008), 1394–1412.
[50] A. Marcus, *Derived invariance of Clifford classes*, Journal of Group Theory, to appear.
[51] J. Müller and M. Schaps, *The Broue Conjecture for the faithful* 3*-blocks of* $4.M_{22}$, J. Algebra **319** (2008), 3588–3602.
[52] G. Navarro, *The McKay Conjecture and Galois automorphisms*, Ann. of Math. **160** (2004), 1–12.
[53] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint (1998).
[54] T. Okuyama, *Remarks on splendid tilting complexes*, S. Koshitani ed. Representation Theory of Finite Groups and Related Topics , RIMS Kokyuroku, Kyoto Univ. **1149** (2000), 53–59.
[55] T. Okuyama, *Derived equivalence in* $\mathrm{SL}(2,q)$, preprint (2000).
[56] T. Okuyama and H.W. Gollan, *Derived equivalences for the smallest Janko group*, preprint (2000).
[57] T. Okuyama and K. Waki, *Decomposition numbers of* $Sp(4,q)$, J. Algebra **199** (1998), 544–555.
[58] L. Puig, *Une correspondance de modules pour les blocs à groupes de défaut abéliens*, Geom. Dedicata **37** (1991), 9–43.
[59] L. Puig, *Algébres de source de certains blocs des groupes de Chevalley*, Astérisque **181-182** (1990), 221–236.
[60] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and Klein four inertial quotients*, J. Algebra **160** (1993), 192–225.
[61] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.
[62] L. Puig, *On the local structure of Morita and Rickard equivalences between Brauer blocks*, Birkhäuser Verlag, Basel 1999.
[63] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. **61** (1989), 303–317.
[64] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), 303–317.
[65] J. Rickard, *Derived equivalences as derived functors* , J. London Math. Soc. **43** (1991), 37–48.
[66] J. Rickard, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math. Soc. **43** (1996), 37–48.
[67] J. Rickard, *The abelian defect group conjecture*, Proceedings of the I.C.M. Berlin 1998, vol. II, 121–128.
[68] J. Rickard, *Equivalences of derived categories for symmetric algebras*, J. Algebra **257** (2002), 460–481.
[69] D. Robbins, *Broués abelian defect group conjecture for the Tits group*, preprint 2008.
[70] R. Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, C.M. Campbell et al eds. Groups 1993 Galway/St. Andrews Proc. of the NATO Advanced Research Workshop on Representation of Algebras and Related Topics, Cambridge University Press **2** (1995), 512–523.

[71] R. Rouquier, *Some examples of Rickard complexes*, Representation theory of groups, algebras, and orders (Constanţa, 1995). An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **4** (1996), 169–173.

[72] R. Rouquier, *The derived category of blocks with cyclic defect groups*, in S. König and A. Zimmermann, Derived Equivalences for Group Rings, Springer-Verlag Berlin 1998, 199–220.

[73] R. Rouquier, *Block theory via stable and Rickard equivalences*, Modular representation theory of finite groups (Charlottesville, VA, 1998), de Gruyter Berlin 2001, 101–146.

[74] R. Rouquier, *Derived equivalences and finite dimensional algebras*, Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol II, pp. 191-221, EMS Publishing House, 2006.

[75] W. Turner, *Equivalent blocks of finite general linear groups in non-describing characteristic*, J. Algebra **247** (2002), 244–267.

[76] A. Turull, *Reduction theorems for Clifford classes*, J. Group Theory **9** (2006), 27-47.

[77] A. Turull, *Strengthening the McKay Conjecture to include local fields and local Schur indices*, J. Algebra **319** (2008) 4853–4868.

[78] K. Uno, *Variations of conjectures on counting irreducible characters of finite groups*, Algebraic Combinatorics, RIMS Kokyoroku, Kyoto Univ. **1327** (2003), 181–190.

[79] Y. Usami, *On p-blocks with abelian defect groups and inertial index* 2 *or* 3*, I, II*, J. Algebra **119** (1988), 123–146; **122** (1989), 98–105.

[80] Y. Usami, *Perfect isometries for blocks with abelian defect groups and dihedral inertial quotients of order* 6, J. Algebra **172** (1995), 113–125.

[81] Y. Usami, *Perfect isometries and isotypies for blocks with abelian defect groups and the inertial quotients isomorphic to* $\mathbb{Z}_4 \times \mathbb{Z}_2$, J. Algebra **181** (1996), 727–759.

[82] J. Usami, *Perfect isometries and isotypies for blocks with abelian defect groups and the inertial quotients isomorphic to* $\mathbb{Z}_3 \times \mathbb{Z}3$, J. Algebra **182** (1996), 140–164.

[83] Y. Usami, *Perfect isometries for principal blocks with abelian defect groups and elementary abelian* 2*-inertial quotients*, Journal of Algebra **196** (1997), 646–681.

[84] Y. Usami and M. Nakabayashi, *Morita equivalent principal 3-blocks of the Chevalley groups* $G_2(q)$, Proc. London Math. Soc. (3) **86** (2003), 397–434.

[85] Usami, Y. and Yoshida, N., *Morita equivalent principal* 5*-blocks of the Chevalley groups* $G_2(2^n)$, Preprint 2003.

[86] A. Watanabe, *On perfect isometries for blocks with abelian defect groups and cyclic hyperfocal subgroups*, Kumamoto J. Math. **18** (2005), 85–92.

[87] A. Zimmermann, A., *Two-sided tilting complexes for Gorenstein orders*, J. Algebra **209** (1998), 585–621.

[88] A. Zimmermann, A., *Tilted symmetric orders are symmetric orders*, Arch. Math. **73** (1999), 15–17.

[89] A. Zimmermann, *Invariance of generalized Reynolds ideals under derived equivalences*, Mathematical Proceedings of the Royal Irish Academy **107A** (2007), 1–9.

[90] A. Zimmermann, *Fine Hochschild invariants of derived categories for symmetric algebras*, J. Algebra **308** (2007), 350–367.

[91] A. Zimmermann, *Hochschild homology invariants of Külshammer type of derived categories*, preprint 2007.

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