Morita equivalences induced by bimodules over Hopf-Galois extensions

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Let H be a Hopf algebra, and A, B right H-comodule algebras. Assume that A and B are faithfully flat H-Galois extensions.

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1. If A and B are Morita equivalent, does it follow that $A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}$ are also Morita equivalent?
2. Conversely, if $A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}$ are Morita equivalent, when does it follow that A and B are Morita equivalent?

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Motivation

Problems raised in the modular representation theory of finite groups lead to the consideration of these questions in the context of strongly group graded algebras.
The results of the present paper generalize the results of Marcus (1998).

Overview

Given a right H-comodule algebra A, and a left H-comodule algebra B, we consider $(A \otimes B, H)$-Hopf modules. These are left $A \otimes B$-modules and right H-comodules, with a suitable compatibility condition.
They are also Doi-Hopf modules over a certain Doi-Hopf datum.

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Main results

- Section 2: we prove a structure Theorem for $(A \otimes B, H)$-Hopf modules, stating that the category of $(A \otimes B, H)$-Hopf modules is equivalent to the category of left modules over the cotensor product $A \square_H B$, under the condition that A is a faithfully flat H-Galois extension. This is the main tool used during the rest of the paper.

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- Section 3: we show that $(A \otimes B, H)$-Hopf modules can also be viewed as comodules over a coring.
- Section 4: we apply the results of Section 2 to relative Hopf bimodules. Let A and B be right H-comodule algebras, and consider (A, B)-bimodules with a right H-coaction, satisfying a certain compatibility condition. These are $(A \otimes B^{\mathrm{op}}, H)$-Hopf modules. We state the Structure Theorem for relative Hopf bimodules. We investigate the compatibility of the category equivalence with the Hom and tensor functors.

Overview

Main results

- Section 5: we apply our results to discuss the two problems stated above. We introduce the notion of *H*-Morita contexts, and we show that if two faithfully flat *H*-Galois extensions are connected by a (strict) *H*-Morita context, then the algebras of coinvariants are also connected by a (strict) Morita context.
  Our main result is the following converse result: if the algebras of coinvariants are Morita equivalent, in such a way that the bimodule structure on one of the connecting modules can be extended to a left-action by the cotensor product $A\square_H B^{\mathrm{op}}$, then A and B are *H*-Morita equivalent.

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- Section 6: we show that the Morita equivalence coming from a strict H-Morita context between two faithfully flat H-Galois extensions respects the Miyashita-Ulbrich action.

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- Section 6: we show that the Morita equivalence coming from a strict H-Morita context between two faithfully flat H-Galois extensions respects the Miyashita-Ulbrich action.
- Section 7: we investigate the behavior of H-Morita equivalences with respect to Hopf subalgebras.

Notation

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- For a right H-coaction ρ (respectively a left H-coaction λ) on a k-module M, we denote

$$\rho(m) = m_{[0]} \otimes m_{[1]} \quad \text{and} \quad \lambda(m) = m_{[-1]} \otimes m_{[0]}.$$

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- Let A be a right H-comodule algebra. Then ${}_A\mathcal{M}^H$ and $\mathcal{M}_A^H$ are the categories of left and right relative Hopf modules.
- We have two pairs of adjoint functors:
  $(F_1 = A \otimes_{A^{\mathrm{co}H}} -,\ G_1 = (-)^{\mathrm{co}H})$ between ${}_{A^{\mathrm{co}H}}\mathcal{M}$ and ${}_A\mathcal{M}^H$;
  $(F_2 = - \otimes_{A^{\mathrm{co}H}} A,\ G_2 = (-)^{\mathrm{co}H})$ between $\mathcal{M}_{A^{\mathrm{co}H}}$ and $\mathcal{M}_A^H$.

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- The unit and counit of the adjunction (F_1, G_1) are given by

$$\eta_{1,N} : \ N \to (A \otimes_{A^{\mathrm{co}H}} N)^{\mathrm{co}H}, \quad \eta_{1,N}(n) = 1 \otimes n;$$

$$\varepsilon_{1,M} : \ A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \to M, \quad \varepsilon_{1,M}(a \otimes m) = am.$$

Hopf Galois extensions

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Theorem (Schneider)

Let A be a right H-comodule algebra. Consider the map

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The following statements are equivalent:

1. (F_1, G_1) *is a pair of inverse equivalences;*
2. (F_2, G_2) *is a pair of inverse equivalences;*
3. can *is an isomorphism and A is faithfully flat as a left $A^{\mathrm{co}H}$-module;*
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Definition

If the equivalent conditions of the above theorem hold, then A is called a faithfully flat H-Galois extension.

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Definition

The cotensor product $M\Box_H N$ is the k-module

$$M\Box_H N = \{\sum_i m_i \otimes n_i \in M \otimes N \mid \sum_i \rho(m_i) \otimes n_i = \sum_i m_i \otimes \lambda(n_i)\}.$$

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If H is cocommutative, then $M\square_H N$ is also a right (or left) H-comodule.

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Proposition

Let R be a k-algebra. Assume that $P \in \mathcal{M}_R$ is flat.
Let $M \in {}_R\mathcal{M}^H$ and $N \in {}^H\mathcal{M}$.
Assume that we have a right H-coaction on M that is left R-linear.

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Then the map

$$P \otimes_R (M\Box_H N) \to (P \otimes_R M)\Box_H N, \quad p \otimes (\sum_i m_i \otimes n_i) \mapsto \sum_i (p \otimes m_i) \otimes n_i$$

is bijective.

H as a left $H \otimes H^{\mathrm{cop}}$-module coalgebra.

- $H \otimes H^{\mathrm{cop}}$ is also a Hopf algebra, and H is a left $H \otimes H^{\mathrm{cop}}$-module coalgebra; the left $H \otimes H^{\mathrm{cop}}$-action is given by

$$(k \otimes l) \cdot h = khS(l).$$

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- $H \otimes H^{\mathrm{cop}} \in {}_{H\otimes H^{\mathrm{cop}}}\mathcal{M}_H$, with right H-action induced by the comultiplication on H, and $k \in {}_H\mathcal{M}$ via ε.
  So we have the left $H \otimes H^{\mathrm{cop}}$-module $(H \otimes H^{\mathrm{cop}}) \otimes_H k$.

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- $(H \otimes H^{\mathrm{cop}}) \otimes_H k$ is a coalgebra with comultiplication and counit

  $$\Delta((h \otimes h') \otimes_H 1) = (h_{(1)} \otimes h'_{(2)}) \otimes_H 1 \otimes (h_{(2)} \otimes h'_{(1)}) \otimes_H 1;$$

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- Then $(H \otimes H^{\mathrm{cop}}) \otimes_H k$ is an $H \otimes H^{\mathrm{cop}}$-module coalgebra.

Proposition

$(H \otimes H^{\mathrm{cop}}) \otimes_H k$ and H are isomorphic as $H \otimes H^{\mathrm{cop}}$-module coalgebras. The isomorphisms are defined by

$$f : \ (H \otimes H^{\mathrm{cop}}) \otimes_H k \to H, \quad f((h \otimes h') \otimes_H 1) = hS(h');$$

$$g : \ H \to (H \otimes H^{\mathrm{cop}}) \otimes_H k, \quad g(h) = (h \otimes 1) \otimes_H 1.$$

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- $A \otimes B$ is a right $H \otimes H^{\mathrm{cop}}$-comodule algebra, with coaction

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- $(H \otimes H^{\mathrm{cop}}, A \otimes B, H)$ is a left-right Doi-Hopf datum.
- We consider the category ${}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$ of Doi-Hopf modules. The objects are k-modules M with a left $A \otimes B$-action and a right H-coaction such that

$$\rho((a \otimes b)m) = (a_{[0]} \otimes b_{[0]})m_{[0]} \otimes a_{[1]}m_{[1]}S(b_{[-1]}).$$

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Definition

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$(A \otimes B, H)$-Hopf modules

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Lemma

With notation as above, we have that $(A \otimes B)^{\mathrm{co}H} = A \Box_H B$.

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$$f: \; A \otimes_{A^{\mathrm{co}H}} (A\Box_H B) \to A \otimes B,$$

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$$f(a \otimes (\sum_i a_i \otimes b_i)) = \sum_i a a_i \otimes b_i.$$

If A is a faithfully flat H-Galois extension, then f is an isomorphism.

$(A \otimes B, H)$-Hopf modules

- We have a pair of adjoint functors (F, G):

$$F: \ {}_{A\Box_H B}\mathcal{M} \to {}_{A\otimes B}\mathcal{M}(H\otimes H^{\mathrm{cop}})^H, \ \ F(N) = (A\otimes B)\otimes_{A\Box_H B} N;$$

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Assume that H is flat as a k-algebra.
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If A is a faithfully flat H-Galois extension, then (F, G) is a pair of inverse equivalences between the categories ${}_{A\Box_H B}\mathcal{M}$ and ${}_{A\otimes B}\mathcal{M}(H\otimes H^{\mathrm{cop}})^H$.

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- Let A be a ring. An A-coring $\mathcal{C}$ is a comonoid in the monoidal category ${}_A\mathcal{M}_A$.

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Furthermore, $A \otimes B$ is faithfully flat as a right $A \square_H B$-module.

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Lemma

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Let H be a Hopf algebra over the commutative ring k, with bijective antipode, and consider two right H-comodule algebras A and B.

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2. *If A is a faithfully flat H-Galois extension, then (F, G) is a pair of inverse equivalences.*

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Remark

Assume that both A and B are faithfully flat H-Galois extensions. Via appropriate transport of structure, the functors

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is an isomorphism, so $M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}H}$ is an $A \square_H C^{\mathrm{op}}$-module.

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Corollary

Let A, B, C be right H-comodule algebras, and assume that A and B are faithfully flat H-Galois extensions.

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$$M = (A \otimes B^{\mathrm{op}}) \otimes_{A\Box_H B^{\mathrm{op}}} M_1 \in {}_A\mathcal{M}_B^H;$$

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$$M_1 \otimes_{B^{\mathrm{co}H}} N_1 \in {}_{A\Box_H C^{\mathrm{op}}}\mathcal{M},$$

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Let $M_1 \in {}_{A\square_H B^{\mathrm{op}}}\mathcal{M}$ and $N_1 \in {}_{B\square_H C^{\mathrm{op}}}\mathcal{M}$, and denote

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$$N = (B \otimes C^{\mathrm{op}}) \otimes_{B\square_H C^{\mathrm{op}}} N_1 \in {}_B\mathcal{M}_C^H.$$

Then

$$M_1 \otimes_{B^{\mathrm{co}H}} N_1 \in {}_{A\square_H C^{\mathrm{op}}}\mathcal{M},$$

$$M \otimes_B N \cong (A \otimes C^{\mathrm{op}}) \otimes_{A\square_H C^{\mathrm{op}}} (M_1 \otimes_{B^{\mathrm{co}H}} N_1).$$

Corollary

Let A, B, C be right H-comodule algebras, and assume that A and B are faithfully flat H-Galois extensions. Let $r: M_1 \to M_1'$ be a map of left $A\square_H B^{\mathrm{op}}$-modules, and $s: N_1 \to N_1'$ a map of left $B\square_H C^{\mathrm{op}}$-modules. Then $r \otimes_{B^{\mathrm{co}H}} s: M_1 \otimes_{B^{\mathrm{co}H}} N_1 \to M_1' \otimes_{B^{\mathrm{co}H}} N_1'$ is left $A\square_H C^{\mathrm{op}}$-linear.

Hopf bimodules

From now on, let H be a projective Hopf algebra.
Let A be a right H-comodule algebra, and $M, N \in {}_A\mathcal{M}^H$.

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$$(h^* \cdot f)(m) = \langle h^*, S^{-1}(m_{[1]}) f(m_{[0]})_{[1]} \rangle f(m_{[0]})_{[0]}.$$

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- It is the subspace of ${}_A\mathrm{Hom}(M, N)$ consisting of left A-linear $f : M \to N$ for which there exists a (unique) $f_{[0]} \otimes f_{[1]} \in {}_A\mathrm{Hom}(M, N) \otimes H$ such that

$$f_{[0]}(m) \otimes f_{[1]} = f(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]}) f(m_{[0]})_{[1]},$$

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$$f_{[0]}(m) \otimes f_{[1]} = f(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]})f(m_{[0]})_{[1]},$$

  for all $m \in M$.
- ${}_A\operatorname{HOM}(M, N)$ is a right H-comodule.
- If H is finitely generated projective, then ${}_A\operatorname{HOM}(M, N)$ coincides with ${}_A\operatorname{Hom}(M, N)$.

Hopf bimodules

Proposition

If M is finitely generated projective as a left A-module, then ${}_A\operatorname{HOM}(M,N)$ coincides with ${}_A\operatorname{Hom}(M,N)$.
For $f \in {}_A\operatorname{HOM}(M,N)$, we have

$$\rho(f) = \sum_i m_i^* \cdot f(m_{i[0]})_{[0]} \otimes S^{-1}(m_{i[1]}) f(m_{i[0]})_{[1]},$$

where $\sum_i m_i^* \otimes_A m$ is a finite dual basis of $M \in {}_A\mathcal{M}$.

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Proposition

Let A, B, C be right H-comodule algebras.
If $M \in {}_A\mathcal{M}_B^H$ and $N \in {}_A\mathcal{M}_C^H$, then ${}_A\mathrm{HOM}(M,N) \in {}_B\mathcal{M}_C^H$.
We have a map

$$\beta : {}_A\mathrm{HOM}(M,N)^{\mathrm{co}H} \to {}_{A^{\mathrm{co}H}}\mathrm{Hom}(M^{\mathrm{co}H}, N^{\mathrm{co}H}).$$

If A is a faithfully flat H-Galois extension, then β is an isomorphism of left $B \square C^{\mathrm{op}}$-modules.

Corollary

Let $M_1 \in {}_{A\Box B^{\mathrm{op}}}\mathcal{M}$ *and* $N_1 \in {}_{A\Box C^{\mathrm{op}}}\mathcal{M}$*, and let*

$$M = (A \otimes B^{\mathrm{op}}) \otimes_{A\Box B^{\mathrm{op}}} M_1, \quad N = (A \otimes C^{\mathrm{op}}) \otimes_{A\Box C^{\mathrm{op}}} N_1.$$

If A *and* B *are faithfully flat* H*-Galois, then*

$${}_{A^{\mathrm{co}H}}\mathrm{Hom}(M_1, N_1) \cong {}_A\mathrm{HOM}(M, N)^{\mathrm{co}H} \in {}_{B\Box C^{\mathrm{op}}}\mathcal{M},$$

$${}_A\mathrm{HOM}(M, N) \cong (B \otimes C^{\mathrm{op}}) \otimes_{B\Box C^{\mathrm{op}}} {}_{A^{\mathrm{co}H}}\mathrm{Hom}(M_1, N_1).$$

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Proposition

Let $M \in {}_A\mathcal{M}_B^H$, $N \in {}_A\mathcal{M}_C^H$. *Then the evaluation map*

$$\varphi : \ M \otimes_B {}_A\mathrm{HOM}(M, N) \to N, \quad \varphi(m \otimes_B f) = f(m)$$

is in ${}_A\mathcal{M}_C^H$. *If* A *and* B *are faithfully flat* H*-Galois, then the evaluation*

$$M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} {}_{A^{\mathrm{co}H}}\mathrm{Hom}(M^{\mathrm{co}H}, N^{\mathrm{co}H}) \to N^{\mathrm{co}H}$$

is left $A\Box_H C^{\mathrm{op}}$*-linear.*

Hopf bimodules

Proposition

Let $M \in {}_A\mathcal{M}_B^H$. Then the map

$$\psi : \; B \to {}_A\mathrm{END}(M), \quad \psi(b)(m) = mb$$

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Proposition

Let $M \in {}_A\mathcal{M}_B^H$, $N \in {}_A\mathcal{M}_C^H$. *Then the map*

$$\mu : \ {}_A\mathrm{HOM}(M, A) \otimes_A N \to {}_A\mathrm{HOM}(M, N), \quad \mu(f \otimes n)(m) = f(m)n$$

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is left $B\Box_H C^{\mathrm{op}}$*-linear.*

Morita equivalences

In this section, we study Morita equivalences induced by two-sided relative Hopf modules.

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Definition

Let A and B be right H-comodule algebras.
An H-Morita context connecting A and B is a Morita context $(A, B, M, N, \alpha, \beta)$ such that $M \in {}_A\mathcal{M}_B^H$, $N \in {}_B\mathcal{M}_A^H$,

$$\alpha : \ M \otimes_B N \to A$$

is a morphism in ${}_A\mathcal{M}_A^H$, and

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is a morphism in ${}_B\mathcal{M}_B^H$.

Proposition

*Let $(A, B, M, N, \alpha, \beta)$ be a strict H-Morita context.
Then we have a pair of inverse equivalences $(M \otimes_B -, N \otimes_A -)$ between the categories ${}_B\mathcal{M}^H$ and ${}_B\mathcal{M}^H$.*

Morita equivalences

From now on that A and B are faithfully flat H-Galois extensions.
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Then $M^{\mathrm{co}H} \in {}_{A\Box_H B^{\mathrm{op}}}\mathcal{M}$, and $N^{\mathrm{co}H} \in {}_{B\Box_H A^{\mathrm{op}}}\mathcal{M}$.
It follows that we have a left $A\Box_H A^{\mathrm{op}}$-linear map

$$\alpha_1 = \alpha^{\mathrm{co}H} \circ f : \ M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}H} \to (M \otimes_B N)^{\mathrm{co}H} \to A^{\mathrm{co}H},$$

and a left $B\Box_H B^{\mathrm{op}}$-linear isomorphism

$$\beta_1 = \beta^{\mathrm{co}H} \circ f : \ N^{\mathrm{co}H} \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \to (N \otimes_A M)^{\mathrm{co}H} \to B^{\mathrm{co}H}.$$

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From the description of f, it follows that we have a commutative diagram of isomorphisms

$$\begin{array}{ccc}
M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}H} \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} & \longrightarrow & (M \otimes_B N)^{\mathrm{co}H} \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \\
\downarrow & & \downarrow \\
M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} (N \otimes_A M)^{\mathrm{co}H} & \longrightarrow & (M \otimes_B N \otimes_A M)^{\mathrm{co}H}
\end{array}$$

Morita equivalences

Now $\alpha \otimes_A M = M \otimes_B \beta$ implies $(\alpha \otimes_A M)^{\mathrm{co}H} = (M \otimes_B \beta)^{\mathrm{co}H}$.
It follows that

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In a similar way, we have that

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It follows that $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M^{\mathrm{co}H}, N^{\mathrm{co}H}, \alpha_1, \beta_1)$ is a Morita context. If $(A, B, M, N, \alpha, \beta)$ is strict, then $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M^{\mathrm{co}H}, N^{\mathrm{co}H}, \alpha_1, \beta_1)$ is also strict.

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This proves the following.

Proposition

Let $(A, B, M, N, \alpha, \beta)$ be a (strict) H-Morita context connecting the faithfully flat H-Galois extensions A and B.

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This proves the following.

Proposition

Let $(A, B, M, N, \alpha, \beta)$ be a (strict) H-Morita context connecting the faithfully flat H-Galois extensions A and B.
Then we have a (strict) Morita context $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M^{\mathrm{co}H}, N^{\mathrm{co}H}, \alpha_1, \beta_1)$, with $M^{\mathrm{co}H} \in {}_{A\square_H B^{\mathrm{op}}}\mathcal{M}$, $N^{\mathrm{co}H} \in {}_{B\square_H A^{\mathrm{op}}}\mathcal{M}$, $\alpha_1 \in {}_{A\square_H A^{\mathrm{op}}}\mathcal{M}$ and $\beta_1 \in {}_{B\square_H B^{\mathrm{op}}}\mathcal{M}$.

Morita equivalences

Now suppose that $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M_1, N_1, \alpha_1, \beta_1)$ is a *strict* Morita context. In our main result, we discuss when we can lift the Morita context to a strict H-Morita context connecting A and B.

It turns out that it is sufficient that M_1 is a left $A\Box B^{\mathrm{op}}$-module.

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Theorem

Let $M_1 \in {}_{A^{\mathrm{co}H}}\mathcal{M}_{B^{\mathrm{co}H}}$ and $N_1 \in {}_{B^{\mathrm{co}H}}\mathcal{M}_{A^{\mathrm{co}H}}$, *and consider*

$$M = A \otimes_{A^{\mathrm{co}H}} M_1 \in {}_A\mathcal{M}^H_{B^{\mathrm{co}H}} \quad \text{and} \quad N = B \otimes_{B^{\mathrm{co}H}} N_1 \in {}_B\mathcal{M}^H_{A^{\mathrm{co}H}}.$$

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Then the following statements are equivalent:

1. *The structure on M and N can be extended such that $M \in {}_A\mathcal{M}^H_B$ and $N \in {}_B\mathcal{M}^H_A$, and M and N induce a strict H-Morita context connecting A and B;*

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Theorem

Let $M_1 \in {}_{A^{\mathrm{co}H}}\mathcal{M}_{B^{\mathrm{co}H}}$ and $N_1 \in {}_{B^{\mathrm{co}H}}\mathcal{M}_{A^{\mathrm{co}H}}$, *and consider*

$$M = A \otimes_{A^{\mathrm{co}H}} M_1 \in {}_A\mathcal{M}^H_{B^{\mathrm{co}H}} \quad \text{and} \quad N = B \otimes_{B^{\mathrm{co}H}} N_1 \in {}_B\mathcal{M}^H_{A^{\mathrm{co}H}}.$$

Then the following statements are equivalent:

1. *The structure on* M *and* N *can be extended such that* $M \in {}_A\mathcal{M}^H_B$ *and* $N \in {}_B\mathcal{M}^H_A$, *and* M *and* N *induce a strict* H*-Morita context connecting* A *and* B*;*
2. *we have a strict Morita context* $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M_1, N_1, \alpha_1, \beta_1)$ *and the structure of* M_1 *can be extended to a structure of left* $A\Box B^{\mathrm{op}}$*-module.*

The Miyashita-Ulbrich action

Let A be a faithfully flat right H-Galois extension, and consider the map

$$\gamma_A = \mathrm{can}^{-1} \circ (\eta_A \otimes H) : \; H \to A \otimes_{A^{\mathrm{co}H}} A.$$

We use the notation

$$\gamma_A(h) = \sum_i l_i(h) \otimes_{A^{\mathrm{co}H}} r_i(h).$$

$\gamma_A(h)$ is then characterized by the property

$$\sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} = 1 \otimes h.$$

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Let M be an (A, A)-bimodule. On $M^{A^{\mathrm{co}H}}$, we can define a right H-action called the Miyashita-Ulbrich action. It is given by the formula

$$m \leftharpoonup h = \sum_i l_i(h) m r_i(h).$$

The Miyashita-Ulbrich action

In particular, for $X, Y \in \mathcal{M}_A$, $\mathrm{Hom}(X, Y) \in {}_A\mathcal{M}_A$, with left and right A-action given by

$$(a \cdot f \cdot a')(x) = f(xa)a'.$$

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It is easy to see that

$$\mathrm{Hom}(X,Y)^{A^{\mathrm{co}H}} = \mathrm{Hom}_{A^{\mathrm{co}H}}(X,Y),$$

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Lemma

Let A and B be faithfully flat right H-Galois extensions. For all $b \in B$, we have that

$$x := \gamma(S^{-1}(b_{[1]})) \otimes b_{[0]} \in A \otimes_{A^{\mathrm{co}H}} (A \square_H B^{\mathrm{op}}).$$

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Now we assume that $(A, B, M, N, \alpha, \beta)$ is a strict H-Morita context connecting the faithfully flat H-Galois extensions A and B.

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For $X \in \mathcal{M}_A$, we have the isomorphism

$$\varphi: \ X \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \cong X \otimes_A A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \xrightarrow{X \otimes_A \varepsilon_{1,M}} X \otimes_A M,$$

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We have that $X \otimes_A M \in \mathcal{M}_B$, and its right B-action can be transported to $X \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H}$.
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Lemma

The transported right B-action on $X \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H}$ is given by the formula

$$(x \otimes_{A^{\mathrm{co}H}} m) \cdot b = \sum_i x l_i(S^{-1}(b_{[1]})) \otimes_{A^{\mathrm{co}H}} (r_i(S^{-1}(b_{[1]})) \otimes b_{[0]}) m.$$

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Consider the setting of the main theorem:
$(A, B, M, N, \alpha, \beta)$ is a strict H-Morita context connecting the faithfully flat H-Galois extensions A and B, and $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, M^{\mathrm{co}H}, N^{\mathrm{co}H}, \alpha_1, \beta_1)$ is the corresponding Morita context connecting $A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}$.

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- For $X, Y \in \mathcal{M}_A$, we have an isomorphism

$$\phi: \ \mathrm{Hom}_{A^{\mathrm{co}H}}(X, Y) \to \mathrm{Hom}_{B^{\mathrm{co}H}}(X \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H}, Y \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H}),$$

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- We consider the Miyashita-Ulbrich action on $\mathrm{Hom}_{B^{\mathrm{co}H}}(X \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H}, Y \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H})$.

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Proposition

With notation as above, the map ϕ preserves the Miyashita-Ulbrich action.

Hopf subalgebras

H is a Hopf algebra with bijective antipode over a field k, and K is a Hopf subalgebra of H. We assume that the antipode of K is bijective, and that H is faithfully flat as a left K-module.

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- Let $K^+ = \mathrm{Ker}(\varepsilon_K)$. It is well-known that

  $$\overline{H} = H/HK^+ \cong H \otimes_K k$$

  is a left H-module coalgebra, with operations

  $$h \cdot \overline{l} = \overline{hl}, \quad \Delta_{\overline{H}}(\overline{h}) = \overline{h}_{(1)} \otimes \overline{h}_{(2)}, \quad \varepsilon_{\overline{H}}(\overline{h}) = \varepsilon(h).$$

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- A right H-comodule M is also a right $\overline{H}$-comodule, by corestriction of coscalars:

$$\rho_{\overline{H}}(m) = m_{[0]} \otimes \overline{m}_{[1]}.$$

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- The $\overline{H}$-coinvariants of $M \in \mathcal{M}^H$ are

  $$\begin{aligned} M^{\mathrm{co}\overline{H}} &= \{m \in M \mid m_{[0]} \otimes \overline{m}_{[1]} = m \otimes \overline{1}\} \\ &= \{m \in M \mid \rho(m) \in M \otimes K\} \cong M \Box_H K. \end{aligned}$$

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If A is a right H-comodule algebra, then $A^{\mathrm{co}\overline{H}}$ is a right K-comodule algebra, and $(A^{\mathrm{co}\overline{H}})^{\mathrm{co}K} = A^{\mathrm{co}H}$.

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Proposition (Schneider)

Assume that A is a faithfully flat right H-Galois extension. Then

1. *A is faithfully flat as a right $A^{\mathrm{co}\overline{H}}$-module, and*

$$\mathrm{can}: \ A \otimes_{A^{\mathrm{co}\overline{H}}} A \to A \otimes \overline{H}, \quad \mathrm{can}(a \otimes b) = ab_{[0]} \otimes \overline{b}_{[1]}$$

   is bijective.

2. *The functors $(A \otimes_{A^{\mathrm{co}\overline{H}}} -, (-)^{\mathrm{co}\overline{H}})$ form a pair of inverse equivalences between ${}_{A^{\mathrm{co}\overline{H}}}\mathcal{M}$ and ${}_A\mathcal{M}(H)^{\overline{H}}$.*

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Theorem

Assume that A is a faithfully flat right H-Galois extension.
Then $A^{\mathrm{co}\overline{H}}$ is a faithfully flat right K-Galois extension.

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We recall some elementary facts from category theory.

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- If (F_1, G_1) and (F_2, G_2) are adjoint pairs, respectively between $\mathcal{C}$ and $\mathcal{D}$, and between $\mathcal{D}$ and $\mathcal{E}$, then $(F = F_2 \circ F_1, G = G_1 \circ G_2)$ is an adjoint pair $\mathcal{C}$ and $\mathcal{E}$.
- If two of these three pairs are inverse equivalences, then the third one is also a pair of inverse equivalences.

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- We have two pairs of inverse equivalences:
  - $(F_1 = A \otimes_{A^{\mathrm{co}H}} -,\ G_1 = (-)^{\mathrm{co}H})$ between ${}_{A^{\mathrm{co}H}}\mathcal{M}$ and ${}_A\mathcal{M}^H$;
  - $(F_3 = A^{\mathrm{co}\overline{H}} \otimes_{A^{\mathrm{co}H}} -,\ G_3 = (-)^{\mathrm{co}K})$ between ${}_{A^{\mathrm{co}H}}\mathcal{M}$ and ${}_{A^{\mathrm{co}\overline{H}}}\mathcal{M}^K$.
- We have an adjoint pair $(F_4 = A \otimes_{A^{\mathrm{co}\overline{H}}} -, G_4 = (-)^{\mathrm{co}\overline{H}} \cong -\Box_H K)$ between ${}_{A^{\mathrm{co}\overline{H}}}\mathcal{M}^K$ and ${}_A\mathcal{M}^H$.
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Corollary

The adjoint pair $(F_4 = A \otimes_{A^{\mathrm{co}\overline{H}}} -, G_4 = (-)^{\mathrm{co}\overline{H}} \cong -\Box_H K)$ establishes a pair of inverse equivalences between the categories ${}_{A^{\mathrm{co}\overline{H}}}\mathcal{M}^K$ and ${}_A\mathcal{M}^H$.

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Theorem

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3. *the following diagram of categories and functors is commutative:*

$$
\begin{array}{ccc}
{}_A\mathcal{M}(H)^{\bar{H}} & \underset{M\otimes_B -}{\overset{N\otimes_A -}{\rightleftarrows}} & {}_B\mathcal{M}(H)^{\bar{H}} \\
A\otimes_{A^{\mathrm{co}\,\bar{H}}} - \uparrow \downarrow (-)^{\mathrm{co}\,\bar{H}} & & B\otimes_{B^{\mathrm{co}\,\bar{H}}} - \uparrow \downarrow (-)^{\mathrm{co}\,\bar{H}} \\
{}_{A^{\mathrm{co}\,\bar{H}}}\mathcal{M} & \underset{M^{\mathrm{co}\,\bar{H}}\otimes_{B^{\mathrm{co}\,\bar{H}}} -}{\overset{N^{\mathrm{co}\,\bar{H}}\otimes_{A^{\mathrm{co}\,\bar{H}}} -}{\rightleftarrows}} & {}_{B^{\mathrm{co}\,\bar{H}}}\mathcal{M}
\end{array}
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Recall that if the algebras A and B are Morita equivalent, then there is a Morita equivalence between $A \otimes A^{\mathrm{op}}$ and $B \otimes B^{\mathrm{op}}$ sending A to B.
In particular, this implies that the centers of A and B are isomorphic.

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Assume that the equivalent conditions of the main theorem hold.

1. *Let K and L be Hopf subalgebras of H with bijective antipodes, and assume that $H \otimes H$ is faithfully flat as a right $K \otimes L$-module.*
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2. *There is an isomorphism*

   $$C_A(A^{\mathrm{co}\,H}) \cong C_B(B^{\mathrm{co}\,H})$$

   of left H-module right H-comodule algebras,
   where $C_A(A^{\mathrm{co}\,H})$ denotes the centralizer in A of $A^{\mathrm{co}\,H}$.

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