Tilting complexes for group graded algebras and Broué's abelian defect group conjecture

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- if D is abelian, Broué's conjecture predicts that there is a derived equivalence between the block algebras $A = kKb$ and $B = kN_K(D)c$ i.e. $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories;

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- moreover, such an equivalence should be compatible with p'-extensions, i.e. if $p \nmid |G|$, then the equivalence can be extended to a derived equivalence between the G-graded k-algebras $S = kHb$ and $R = kN_H(D)c$ induced by a bounded complex of G-graded (R, S)-bimodules.

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Denote $A = R_1$ and $B = S_1$. The diagonal subalgebra is

$$\Delta := \Delta(R \otimes_k S^{\mathrm{op}}) = \bigoplus_{g \in G} R_g \otimes_k S_{g^{-1}}.$$

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Graded endomorphism rings

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A complex $T \in \mathcal{H}(R\text{-}\mathrm{Gr})$ is called *G-invariant* if $T(g) \simeq T$ (in the category $\mathcal{H}(R\text{-}\mathrm{Gr})$) for all $g \in G$.

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The following statements are equivalent.
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(iii) *There are triangle equivalences $F : \mathcal{D}(S) \to \mathcal{D}(R)$ and $F^{\mathrm{gr}} : \mathcal{D}(S\text{-}\mathrm{Gr}) \to \mathcal{D}(R\text{-}\mathrm{Gr})$ such that F^{gr} is a G-graded functor and commutes with the ungrading functor.*

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(iv) (R and S strongly graded) There are (bounded) complexes X_1 of $\Delta(R \otimes_k S^{\mathrm{op}})$ modules and Y_1 of $\Delta(S \otimes_k R^{\mathrm{op}})$ modules, and isomorphisms

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a) T is a tilting complex for A if and only if $\tilde{T}$ is a G-graded tilting complex for R.
b) If T is a tilting complex for A and R is a finite dimensional symmetric crossed product, then S is a symmetric crossed product of $B := S_1 \simeq \mathrm{End}_{\mathcal{H}(A)}(T)^{\mathrm{op}}$ and G.

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Then $X_1^\bullet$ is a complex of Δ-modules, and also a two-sided tilting complex of (R_1, S_1)-bimodules.

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Then $X_1^\bullet$ is a complex of Δ-modules, and also a two-sided tilting complex of (R_1, S_1)-bimodules.
Let $Y_1^\bullet$ be a projective resolution of $X_1^\bullet$ as Δ-modules.

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Then $X_1^\bullet$ is a complex of Δ-modules, and also a two-sided tilting complex of (R_1, S_1)-bimodules.
Let $Y_1^\bullet$ be a projective resolution of $X_1^\bullet$ as Δ-modules.
It is possible (Rickard), to truncate $Y_1^\bullet$ and obtain a bounded complex

$$Z_1^\bullet := (\cdots \to 0 \to \mathrm{Ker} d^n \to Y_1^n \to Y_1^{n+1} \to \cdots),$$

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Then we have:
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Stable equivalences and Rickard equivalences

Definition (Rouquier)

The complex C of G-graded exact (R, S)-bimodules induces a G-graded stable equivalence between R and S if

$C \otimes_S C^\vee \simeq R \oplus Z, \quad C^\vee \otimes_R C \simeq S \oplus W$

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1) *$X=C\oplus P$, where P is a complex of G-graded projective bimodules;*
2) *X induces a G-graded Rickard equivalence between R and S;*
3) *In the derived category of G-graded (R,S)-bimodules, X is isomorphic to the composition between D and a G-graded Morita autoequivalence of R.*

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1. *$R^{(1)}$ is a G-graded crossed product, graded Morita equivalent to E.*
2. *There is a G-graded algebra map $S \to R^{(1)}$, and the $(S, R^{(1)})$-bimodule $R^{(1)}$ induces a graded stable equivalence of Morita type between S and $R^{(1)}$.*

On Okuyama's tilting complexes

Assume that the (A,B)-bimodule M induces a stable Morita equivalence between A and B. Consider the complex of (A,B)-bimodules

$P^\bullet(M,I_0)=(\cdots\to 0\to P\xrightarrow{\delta} M\to 0\to\cdots)$,

with P in degree -1 and M in degree 0, where $\delta : P\to M$ is a right minimal $((\bigoplus_{i\in I}P_i)\otimes_k B^{\mathrm{op}})$-approximation of M.

$A^{(1)}:=\mathrm{End}_{\mathcal{H}(A)}(P^\bullet(M,I_0))^{\mathrm{op}}$, $R^{(1)}:=\mathrm{End}_{\mathcal{H}(R)}(R\otimes_A P^\bullet(M,I_0))^{\mathrm{op}}$.

Regarded as a complex of A-modules, $P^\bullet(M,I_0))$ is a direct sum of complexes isomorphic to $P_i^\bullet$, $i\in I$.

Let $S_i^{(1)}$ be a simple $A^{(1)}$-modules corresponding to an indecomposable summand isomorphic to $P_i^\bullet$ of ${}_AP^\bullet(M,I_0)$.

Proposition

Assume that I_0 is a G-subset of I, and M is a $\Delta(R\otimes_k S^{\mathrm{op}})$-module.

1. *$R^{(1)}$ is a G-graded crossed product, graded Morita equivalent to E.*
2. *There is a G-graded algebra map $S\to R^{(1)}$, and the $(S,R^{(1)})$-bimodule $R^{(1)}$ induces a graded stable equivalence of Morita type between S and $R^{(1)}$.*
3. *There is an isomorphism of B-modules ${}_BS_{g_i}^{(1)}\simeq {}^gS_i^{(1)}$.*

Graded version of Okuyama's method

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Let $\{ T_i \mid i \in I \}$ be a set of representatives for the isomorphism classes of simple B-modules.

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Let $\{T_i \mid i \in I\}$ be a set of representatives for the isomorphism classes of simple B-modules.
Consider the A-modules $X_i = M \otimes_B T_i$, $i \in I$.
If all X_i are are simple A-modules, then by a theorem of Linckelmann, A and B are Morita equivalent, so if M is a Δ-module, then R and S are graded Morita equivalent.

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Otherwise, choose a subset I_0 of I and replace A by $A^{(1)} = \mathrm{End}_A(P^\bullet(M, I_0))^{\mathrm{op}}$ (which is Morita equivalent to $C = \mathrm{End}_A(P^\bullet(I_0))^{\mathrm{op}}$) and M by a $(B, A^{(1)})$-bimodule $M^{(1)}$ inducing a stable equivalence between B and $A^{(1)}$.

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This procedure continues until a stage t when simple B-modules will correspond to simple $A^{(t)}$-modules.

On Okuyama's tilting complexes

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This procedure continues until a stage t when simple B-modules will correspond to simple $A^{(t)}$-modules.
The point is that the G-invariance of a set I_s of simple $A^{(s)}$-modules can be established from the knowledge of the action of G on the simple A-modules.

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Let ${}_AM_B^\bullet = M^\bullet(I_0) = (\cdots \to 0 \to P \xrightarrow{\delta} M \to 0 \to \cdots)$, with M in degree 0. ${}_AM^\bullet$ is a tilting complex iff a certain condition is satisfied.

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Assume that ${}_AM^\bullet$ tilting complex, M Δ-module, I_0 G-subset of I. Then

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Let ${}_AM^\bullet_B = M^\bullet(I_0) = (\cdots \to 0 \to P \xrightarrow{\delta} M \to 0 \to \cdots)$, with M in degree 0. ${}_AM^\bullet$ is a tilting complex iff a certain condition is satisfied. Denote $C = \mathrm{End}_{\mathcal{H}(A)}(M^\bullet(I_0))^{\mathrm{op}}$ and $E = \mathrm{End}_{\mathcal{H}(R)}(R \otimes_A M^\bullet(I_0))^{\mathrm{op}}$.

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1. *$M^\bullet$ extends to a complex of Δ-modules.*
2. *E is a crossed product, and there is a G-graded stable Morita equivalence between E and S.*
3. *There is a complex $N^\bullet = N^\bullet(I_0)$ of $\Delta(R \otimes_k E^{\mathrm{op}})$-modules such that ${}_AN^\bullet_C$ is a tilting complex, and $N^\bullet$ is homotopy equivalent to $M^\bullet$ as complexes of Δ-modules.*

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Theorem

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Corollary

Let ${}_R M_S$ be inducing a G-graded Morita stable equivalence.

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Corollary

Let ${}_RM_S$ be inducing a G-graded Morita stable equivalence.
If in addition X_i is stably isomorphic to $M_1 \otimes_B T_i$, for all $i \in I$, then there is a G-graded derived equivalence between R and S.

Splendid stable and derived equivalences

Let $S = kHb$, $B = kKb$, $R = kN_H(D)c$, $A = N_K(D)$, $H' = N_H(D)$, $K' = N_K(D)$, and assume that $G = H/K$ is a p'-group.

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Assume that C induces a G-graded splendid stable equivalence, and X_i is stably isomorphic to $C_1 \otimes_B T_i$ for all $i \in I$.

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${}_R C_S^\bullet$ is *splendid*, if the indecomposable summands of C^i are $\delta(D)$-projective p-permutation $k(H' \times H)$-modules.

Corollary

Assume that C induces a G-graded splendid stable equivalence, and X_i is stably isomorphic to $C_1 \otimes_B T_i$ for all $i \in I$.
Then there is a complex X of G-graded (R, S)-bimodules such that:
1) *The image of X_1 in $\Delta\text{-stmod} \simeq \mathcal{D}^b(\Delta\text{-mod})/\mathcal{H}^b(\Delta\text{-proj})$ is C_1;*
2) *X induces a splendid derived equivalence between R and S;*
3) *$X_1 \otimes_B T_i \simeq X_i$ in $\mathcal{D}^b(A\text{-mod})$, for all $i \in I$.*

Example

a) T.I. situation: take $M_1 := {}_A A_B$ and $M = {}_R R_S$.

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a) T.I. situation: take $M_1 := {}_A A_B$ and $M = {}_R R_S$.
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Then there is a splendid complex of (A, B)-bimodules inducing a stable equivalence (Rouquier).

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Example

a) T.I. situation: take $M_1 := {}_AA_B$ and $M = {}_RR_S$.
b) Let D elementary abelian of order p^2, b the principal block of $\mathcal{O}K$. Then there is a splendid complex of (A, B)-bimodules inducing a stable equivalence (Rouquier). This applies to the examples considered by M. Holloway (5-blocks of $2.J_2$, $U_3(4)$ and $\mathrm{Sp}_4(4)$), and Y. Usami and N. Yoshida (principal 5-blocks of $G_2(2^n)$, $5 \mid 2^n + 1$, $25 \nmid 2^n + 1$).

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If M is an (R, S)-bimodule and $\hat{\rho} \in \hat{C}_n$, then the $\hat{\rho}$-th conjugate ${}^{\hat{\rho}}M$ of M is defined by ${}^{\hat{\rho}}M = (R \otimes_{\mathcal{O}} S^{\mathrm{op}})\hat{\rho} \otimes_{R \otimes_{\mathcal{O}} S^{\mathrm{op}}} M$.

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If ${}^{\hat{\rho}}Y \simeq Y$ as complexes of (R,S)-bimodules for all $\hat{\rho} \in \hat{C}_n$, then the block algebras $e\mathcal{O}G^+$ and $f\mathcal{O}H^+$ are Rickard equivalent.

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We use that the conjecture is known to hold for the symmetric group S_n by the work of J. Rickard, J. Chuang, R. Kessar and R. Rouquier, and we show how to "go down" to A_n, by using the above techniques.

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Let $p > 2$, $G = S_n$, $G^+ = A_n$, $\tilde{G} = \mathrm{Aut}(G^+)$, b^+ a block of $\mathcal{O}G^+$ with nontrivial abelian defect group D, $H^+ = N_{G^+}(D)$, and $c^+ \in \mathcal{O}H^+$ the Brauer correspondent of b^+.
Then there exists a splendid tilting complex of $\tilde{G}/G^+$-graded $(b^+\mathcal{O}\tilde{G}, c^+\mathcal{O}\tilde{H})$-bimodules.

We use that the conjecture is known to hold for the symmetric group S_n by the work of J. Rickard, J. Chuang, R. Kessar and R. Rouquier, and we show how to "go down" to A_n, by using the above techniques.
Related results:

Blocks of symmetric and alternating groups

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