MORITA EQUIVALENCES RELATED TO THE GLAUBERMAN CORRESPONDENCE

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Abstract. Starting with P-interior algebras, where P is a finite p-group, we prove two theorems establishing certain group graded Morita equivalences. These apply, in particular, to the case of blocks with normal defect groups, and defect zero blocks of normal subgroups, respectively.

1. Introduction

This paper is motivated by several results on the existence of Morita equivalences in the context of the Glauberman-Watanabe correspondence (see [10], [9], [7], [25], [21] and the references given there). In order to explain this, let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system, let G be a finite group and let A a solvable finite group acting on G such that G and A have coprime order. Let b be an A-invariant block of $\mathcal{O}G$. Under some additional conditions, there exists a Morita equivalence between $b\mathcal{O}G$ and $w(b)\mathcal{O}G^A$ induced by a $(b\mathcal{O}G, w(b)\mathcal{O}G^A)$-bimodule M with the property that regarded as a $G \times G^A$-module, M has a source which is an endopermutation module. Moreover, when we have a splitting p-modular system, this Morita equivalence induces a bijection

$$\pi(G, A) : \mathrm{Irr}_{\mathcal{K}}(G, b) \to \mathrm{Irr}_{\mathcal{K}}(G^A, w(b)),$$

which coincides with the Glauberman correspondence, where the block $w(b)$ of $\mathcal{O}G^A$ is the Watanabe correspondent of b (see [24]).

By induction, the problem of finding such a Morita equivalence is reduced to the case when we have blocks lying over a block of a normal p'-subgroup of G, actually of $O_{p'}(G)$. More generally, instead of a normal p'-subgroup, it is usefuel to consider blocks of defect zero of a normal subgroup (see [8]).

In this paper, we consider an even more general situation. We start with a strongly G-graded P-interior $\mathcal{O}$-algebra R whose identity component R_1 is an $\mathcal{O}$-simple algebra, where we assume that the p-group P is also a normal subgroup of G. Then the identity of R_1 has a defect group Q isomorphic to P, and the second G-graded algebra R' is constructed as a crossed product between the Brauer quotient $R'_1 := R_1(Q)$ and G. Theorem 4.7 below establishes a G-graded Morita equivalence over k between $k \otimes_{\mathcal{O}} R$ and R'. This generalizes the main result of Dade [2, (6)] on correspondences above the Glauberman correspondence. On this subject, we also refer to Turull [23] and Ladisch [13]), but our approach is more in the spirit of [2] and [3], by systematical use of Clifford extensions.

Our proof of Theorem 4.7 is inspired by the proof of [5, Theorem 3.4], and therefore, our first main result Theorem 3.9 is a generalization of Külshammer's theorem [11, Theorem A] on the structure of blocks with normal defect groups. Külshammer's theorem states that, in the situation of a splitting p-modular system, if e is a block of $\mathcal{O}G$ with normal defect group D, then the block algebra $e\mathcal{O}G$ is Morita equivalent to a twisted group algebra $\mathcal{O}_\beta DE$ of the semidirect product DE, where $E = G_e/DC_G(D)$, and G_e is the stabilizer of e in G. Other approaches and generalizations can be found in Puig [19, Proposition 14.6], Alperin, Linckelmann and Rouquier [1], Fan and Puig [6, Theorem 1.17]).

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In this paper, instead of starting with a block of the group algebra $\mathcal{O}G$, we only consider a separable algebra extension $\mathcal{O}P \to B$, where P is a finite p-group and B has finite $\mathcal{O}$-rank, and then we construct our Morita equivalent group graded algebras from this data. The connection with Külshammer's theorem is explained in 3.11. The techniques used in the proof of Theorem 3.9 are also used in the proof of Theorem 4.7.

Our general assumptions and notations are standard. In general, modules are left and finitely generated. We refer the reader to [22] and [20] for basic results on G-algebras, and to [14] for group graded algebras. Some other needed facts are recalled in the next section.

2. Modular group graded algebras

2.1. We consider a p-modular system $(\mathcal{K}, \mathcal{O}, k)$, where k is a perfect field. An important particular case is when $\mathcal{K} = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$ and $k = \mathbb{F}_p$.

Our main objects of study are G-graded crossed products $R = \bigoplus_{g\in G} R_g$, where R is assumed to be free of finite rank over $\mathcal{O}$, so G is a finite group. We have an exact sequence of groups

$$1 \to R_1^\times \to \mathrm{hU}(R) \to G \to 1,$$

where $\mathrm{hU}(R) := \bigcup(R^\times \cap R_g)$ is the group of homogeneous units of R. We usually denote $A := R_1$ and $kR := k \otimes_{\mathcal{O}} R$.

We will later consider the situation when A is an $\mathcal{O}$-simple algebra, that is, A is free as a $\mathcal{O}$-algebra and the k-algebra $kA := A/J(\mathcal{O})A$ is simple.

Example 2.2. Let K be a normal subgroup of H, let $G = H/K$ and let b be a block of $\mathcal{O}K$. The group H acts on $\mathcal{O}K$ by conjugation, while G acts on $Z(\mathcal{O}K)$, and denote by G_b the stabilizer of b in G. Then $R := b\mathcal{O}Hb$ is a G_b-graded crossed product with 1-component $R_1 = b\mathcal{O}K$. In this case we have that $R = b\mathcal{O}H_b$, and the algebra R is Morita equivalent to $\mathcal{O}Gb\mathcal{O}G$, so we can assume without much loss of generality that b is G-invariant.

Our intention is to no longer work with K and H, and refer only to R, G and to the defect groups of the block.

Example 2.3. We will frequently use the following construction from [20, Chapter 9] (see also [4, Section 2]), which gives a bijection between K-interior H-algebras and G-graded H-interior algebras, where H is a group, K is a normal subgroup of H, $G = H/K$, and $\mathcal{O}H$ is regarded as a G-graded $\mathcal{O}$-algebra in the obvious way.

A K-interior H-algebra is an $\mathcal{O}$-algebra A with group homomorphisms

$$\varphi : H \to \mathrm{Aut}(A) \quad \text{and} \quad \psi : K \to A^\times$$

such that, for any $x \in H$, $y \in K$ and $a \in A$, we have

$$(y \cdot a)^x = y^x \cdot a^x \quad \text{and} \quad a^y = y^{-1} \cdot a \cdot y,$$

where $y \cdot a$ and $a \cdot y$ denote $\psi(y)a$ and $a\psi(y)$ respectively, while $a^x := \varphi(x)^{-1}(a)$ and ${}^x a := a^{x^{-1}}$. Then A determines the G-graded $\mathcal{O}$-algebra $R := \bigoplus_{g\in G} R_g$ by letting

$$R := A \otimes_{\mathcal{O}K} \mathcal{O}H = \bigoplus_{x\in[H/K]} A \otimes x,$$

with multiplication defined by

$$(a \otimes x)(b \otimes y) = a({}^x b) \otimes xy$$

for all $a, b \in A$ and $x, y \in H$. In particular, we have $(1 \otimes x)(1 \otimes y) = 1 \otimes xy$, so there exists a homomorphism

$$\psi : \mathcal{O}H \to R, \qquad \psi(h) = 1 \otimes h$$

of G-graded algebras.

Conversely, if $\psi : \mathcal{O}H \to R$ is a homomorphism of G-graded $\mathcal{O}$-algebras, then $A := R_1$ is a K-interior H-algebra, where

$$\varphi : H \to \mathrm{Aut}(A), \qquad \varphi(h)(a) = \psi(h)a\psi(h)^{-1},$$

and $\psi : K \to A^\times$ is the restriction of ψ.

2.4. We denote by $J_{\mathrm{gr}}(R)$ the Jacobson radical of the crossed product R, and let $\bar{R} = R/J_{\mathrm{gr}}(R)$. Notice that the canonical ring epimorphism $R \to \bar{R}$ induces the commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & 1+J(A) \arrow[r, equal] \arrow[d] & 1+J(A) \arrow[d] & & \\
1 \arrow[r] & A^\times \arrow[r] \arrow[d] & \mathrm{hU}(R) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \bar{A}^\times \arrow[r] \arrow[d] & \mathrm{hU}(\bar{R}) \arrow[r] \arrow[d] & G \arrow[r] & 1 \\
 & 1 & 1 & & .
\end{tikzcd}$$

We need a connection between the splittings of the group extension $\mathrm{hU}(R)$ and the splittings of $\mathrm{hU}(\bar{R})$. The following theorem is a generalization of a theorem by E. Dade, proved in [14, Theorem 3.1.8].

Theorem 2.5. *Assume that*

(1) *the extension* $\mathrm{hU}(\bar{R})$ *of* $\bar{A}^\times$ *by* G *splits,*
(2) *there is* $\bar{a} \in \bar{A}$ *such that* $\mathrm{Tr}_1^G(\bar{a}) = 1$.

Then there is a bijection between the splittings of $\mathrm{hU}(\bar{R})$ *and the* $(1 + J(A))$*-conjugacy classes of splittings of* $\mathrm{hU}(R)$.

2.6. Let P be a finite p-group. Recall that a kP-module M is called *endopermutation* if $\mathrm{End}_k(M)$ has a P-stable basis. By [23, Theorem 3.3], if $\hat{k}/k$ is a field extension and M is an endopermutation $\hat{k}P$-module, then there is an endopermutation kP-module M_0 such that $M \simeq \hat{k} \otimes_k M_0$.

We will also need Dade's theorem [2], [3] on Clifford extensions of indecomposable endopermutation kP-modules (see also [17] and [23, Theorem 3.10] for alternative approaches).

Theorem 2.7. *Let R be a G-graded crossed product such that $R_1 = kP$, and let M be a G-invariant indecomposable endopermutation kP-module. Let $E = \mathrm{End}_R(R \otimes_{R_1} M)^{\mathrm{op}}$, and let $\bar{E} = E/J_{\mathrm{gr}}(R)$. Then the group extension*

$$1 \to \bar{E}_1^\times \to \mathrm{hU}(\bar{E}) \to G \to 1$$

splits.

3. The "normal defect group" situation

The main result of this section is a generalization of the structure theorem for blocks with normal defect group, originally due to Külshammer [11]. There is another approach in [1] using modules, and here we generalize the main result of [5], by establishing a Morita equivalence between strongly graded algebras. We lay down our assumptions in 3.1 and 3.2, while the algebras in discussion are defined after several steps in 3.5 and 3.8 below.

3.1. Let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system as in 2.1, and let D be a finite p-group. Let B be a $\mathcal{O}D$-interior $\mathcal{O}$-algebra, free of finite rank over $\mathcal{O}$, having a D-stable basis, and such that the modules ${}_{\mathcal{O}D}B$ and $B_{\mathcal{O}D}$ are projective (and actually free, because $\mathcal{O}D$ is a local ring). We suppose that the identity element 1 of B is a primitive idempotent in $Z(B)$ (so B is a block algebra), and that B has *defect group* D. By this we mean the following.

(1) The ring extension $\mathcal{O}D \to B$ is separable, that is, the homomorphism of (B, B)-bimodules
$$B \otimes_{\mathcal{O}D} B \xrightarrow{\mu} B$$
that takes $b \otimes b'$ to bb' for all $b, b' \in B$ splits, hence ${}_BB_B$ is a direct summand of $B \otimes_{\mathcal{O}D} B$ (see also [12] for other characterizations).

(2) The Brauer homomorphism $\mathrm{Br}_D : B^D \to B(D)$ satisfies $\mathrm{Br}_D(1) \neq 0$, where recall that
$$B(D) = k \otimes_{\mathcal{O}} (B^D / \sum_{Q<D} \mathrm{Tr}_Q^D(B^Q))$$
is the Brauer quotient of B (see [22, Section 11]).

3.2. Since B is indecomposable as a (B, B)-bimodule, there exists a primitive idempotent $i \in B^D$ such that $B \mid Bi \otimes_{\mathcal{O}D} iB$ as (B, B)-bimodules. Let
$$\gamma = \{aia^{-1} \mid a \in (B^D)^\times\}$$
the $(B^D)^\times$-conjugacy class of i. Then the pair $(D, \gamma) = D_\gamma$ is called a *defect pointed group of* B (a notion due to Puig, see [22]). Moreover, $\mathrm{Br}_D(i)$ is a primitive idempotent in $B(D)$, and $\mathrm{Br}_D(\gamma)$ is a point of $B(D)$, because $\mathrm{Br}_D : B^D \to B(D)$ is surjective.

There is a unique maximal ideal of B^D, denoted by $\mathfrak{m}_\gamma$, such that $\gamma \not\subset \mathfrak{m}_\gamma$, and a unique maximal ideal of $B(D)$, denoted by $\mathfrak{m}_{\mathrm{Br}_D(\gamma)}$, such that $\mathrm{Br}_D(\gamma) \not\subset \mathfrak{m}_{\mathrm{Br}_D(\gamma)}$. Moreover, we have that
$$B^D/\mathfrak{m}_\gamma \simeq B(D)/\mathfrak{m}_{\mathrm{Br}_D(\gamma)},$$
and we denote by S the simple k-algebra $B(D)/\mathfrak{m}_{\mathrm{Br}_D(\gamma)}$.

Let $\bar{V}$ be the unique (up to a isomorphism) simple S-module. There is a unique primitive idempotent $e_\gamma \in Z(B(D))$ such that $e_\gamma \mathrm{Br}_\gamma(i) \neq 0$, and then the image $\bar{e}_\gamma$ of e_γ via the canonical map $B(D) \to S$ is actually the identity of S.

We will assume that S has Schur index 1, that is, $\hat{k} := \mathrm{End}_S(\bar{V})$ is a field. In this case we have that $\hat{k} = Z(S)$, and moreover,
$$S \simeq \mathrm{End}_{\hat{k}}(\bar{V}) \simeq M_m(\hat{k}),$$
where $m := \dim_{\hat{k}} \bar{V}$.

3.3. Let $C_{B^\times}(D)$ and $N_{B^\times}(D)$ be the centralizer and the normalizer of D in $B^\times$, respectively. Then $C_{B^\times}(D)$ is a normal subgroup of $N_{B^\times}(D)$, and we denote
$$G := N_{B^\times}(D)/C_{B^\times}(D)$$
and
$$\bar{G} := N_{B^\times}(D)/DC_{B^\times}(D).$$
Note that $N_{B^\times}(D)$ acts on B^D and on $B(D)$ as algebra automorphisms. Moreover, we have the maps
$$C_{B^\times}(D) \hookrightarrow B^D \to B(D),$$
compatible with the action of $N_{B^\times}(D)$, so $B(D)$ is a $C_{B^\times}(D)$-interior $N_{B^\times}(D)$-algebra.

3.4. We consider the stabilizer
$$N_{B^\times}(D)_\gamma := \{a \in N_{B^\times}(D) \mid a\gamma a^{-1} = \gamma\}$$
of γ in $N_{B^\times}(D)$. Clearly, $C_{B^\times}(D) \subseteq N_{B^\times}(D)_\gamma$, and let
$$G_\gamma := N_{B^\times}(D)_\gamma / C_{B^\times}(D)$$
be the stabilizer of γ in G. Note that the stabilizer of e_γ in $N_{B^\times}(D)$ coincides with $N_{B^\times}(D)_\gamma$. We also have that $D \subseteq N_{B^\times}(D)_\gamma$, and we denote
$$\bar{G}_\gamma := N_{B^\times}(D)_\gamma / DC_{B^\times}(D)$$
and
$$\bar{D} := D/Z(D) \simeq DC_{B^\times}(D)/C_{B^\times}(D).$$

Since the modules ${}_{\mathcal{O}D}B$ and $B_{\mathcal{O}D}$ are free, the map $D \to B^\times$ is injective, hence we have that $Z(D) = D \cap C_{B^\times}(D)$, so $\bar{G}_\gamma \simeq G_\gamma/\bar{D}$.

3.5. Notice that all the homomorphisms in the diagram

$$\begin{tikzcd} B^D \arrow[r, "\mathrm{Br}_D"] \arrow[d] & B(D) \arrow[d] \\ B^D/\mathfrak{m}_\gamma \arrow[r] & B(D)/\mathfrak{m}_{\mathrm{Br}(\gamma)} = S \end{tikzcd}$$

are $N_{B^\times}(D)_\gamma$-algebra homomorphisms. Moreover $e_\gamma B(D)$ is a $C_{B^\times}(D)$-interior $N_{B^\times}(D)$-acted k-algebra. Then, as in 2.3 we may construct the strongly G_γ-graded crossed product

$$R := e_\gamma B(D) * G_\gamma,$$

with 1-component $R_1 = e_\gamma B(D)$.

In addition, since $\hat{k}$ is a perfect field and D is a p-group, there is a group homomorphism

$$\sigma : D \to S^\times$$

such that for all $u \in D$ and all $s \in S$ we have ${}^u s = \sigma(u) s \sigma(u)^{-1}$, hence the D-algebra S is actually a D-interior algebra. Consequently, S is a $DC_{B^\times}(D)$-interior $N_{B^\times}(D)_\gamma$-acted $\hat{k}$-algebra, and again as in 2.3, we can construct the $\bar{G}_\gamma$-graded crossed product k-algebra

$$\bar{R} := S * \bar{G}_\gamma.$$

3.6. We will assume that γ is G-invariant, that is, $G = G_\gamma$, because in general, passing from G to G_γ is a Morita equivalence, by making use of the Fong-Reynolds theorem. Summarizing, we have the commutative diagrams

$$\begin{tikzcd} 1 \arrow[r] & Z(D) \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & \bar{D} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & (e_\gamma B(D))^\times \arrow[r] & \mathrm{hU}(R) \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} & & \bar{D} \arrow[r, equal] \arrow[d] & \bar{D} \arrow[d] & \\ 1 \arrow[r] & S^\times \arrow[r] & \mathrm{hU}(\bar{R}) \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

where the lines are exact and the vertical maps are injective.

3.7. The group $N_{B^\times}(D)$ also acts by conjugation on D, and $C_{B^\times}(D)$ acts trivially, so we have the group homomorphisms

$$G \to \mathrm{Aut}(D) \quad \text{and} \quad \bar{G} \to \mathrm{Out}(D).$$

Notice that this implies that the group $\bar{G}$ (and hence G) is finite, because the map $\bar{G} \to \mathrm{Out}(D)$ is injective.

The group $\bar{G}$ acts on the field $\hat{k}$, because $N_{B^\times}(D)$ acts on S and on the center $Z(S) = \hat{k}$ of S. Moreover, $DC_{B^\times}(D)$ acts trivially on $\hat{k}$, so we have a group homomorphism

$$\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k).$$

Let $K \leq \bar{G}$ the kernel of θ. By hypothesis, the extension $\mathcal{O}D \to B$ is separable, so by an application of Mackey decomposition with respect to the pair (D, D) as in [22, Proposition 14.7], we deduce that the identity e_γ of $B(D)$ belongs to the image of $\mathrm{Tr}^G_{DC_{B^*}(D)}$, hence the identity $\bar{e}_\gamma$ of S belongs to $\mathrm{Tr}_1^{\bar{G}}(\hat{k})$. It follows that $\hat{k}$ is a projective module over the group algebra $\hat{k}K$, so by Maschke's theorem $p \nmid |K|$.

The action of $N_{B^\times}(D)$ on D and on $\hat{k}$ induces the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \bar{D} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \bar{G} \arrow[r] \arrow[d] \arrow[dl, dashed, "\sigma"'] & 1 \\
1 \arrow[r] & \mathrm{Int}(\hat{k}D) \arrow[r] & \mathrm{Aut}_k(\hat{k}D) \arrow[r] & \mathrm{Out}_k(\hat{k}D) \arrow[r] & 1,
\end{tikzcd}$$

By [6, Corollary 3.13], there exists a group homomorphism $\sigma : \bar{G} \to \mathrm{Aut}_k(\hat{k}D)$ that lifts the homomorphism $\bar{G} \to \mathrm{Out}_k(\hat{k}D)$.

3.8. Since we have assumed $\bar{V}$ to be $\bar{G}$-invariant, by Clifford theory we have the isomorphism of $\bar{G}$-graded algebras

$$\mathrm{End}_{\bar{R}}(\bar{R} \otimes_S \bar{V})^{\mathrm{op}} \simeq \hat{k}^{\theta}_{\beta}\bar{G},$$

where $\hat{k}^{\theta}_{\beta}\bar{G}$ is a $\bar{G}$-graded crossed product of $\hat{k}$ and $\bar{G}$ determined by the 2-cocycle $\beta : \bar{G} \times \bar{G} \to \hat{k}^\times$ and the action $\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k)$. In this case we have a $\bar{G}$-graded Morita equivalence between $\bar{R} = S * \bar{G}$ and $\hat{k}^{\theta}_{\beta}\bar{G}$, and moreover, we have the isomorphism

$$\bar{R} \simeq S \otimes_{\hat{k}} (\hat{k}^{\theta}_{\beta}\bar{G}), \qquad s\bar{g} \mapsto s \otimes_{\hat{k}} g$$

of $\bar{G}$-graded algebras.

By using the homomorphism $\sigma : \bar{G} \to \mathrm{Aut}_k(\hat{k}D)$, we can construct the strongly $\bar{G}$-graded crossed product $(\hat{k}D)^{\sigma}_{\beta}\bar{G}$, with $\bar{1}$-component $\hat{k}D$.

In fact, $(\hat{k}D)^{\sigma}_{\beta}\bar{G}$ can be viewed as a strongly G-graded algebra with the 1-component $\hat{k}Z(D)$, which means that we refine the grading by viewing $\hat{k}D$ as a $\bar{D}$-graded algebra. Let

$$R' := (\hat{k}D)^{\sigma}_{\beta}\bar{G},$$

viewed as a G-graded algebra, with 1-component $R'_1 = \hat{k}Z(D)$.

Now we are able to state one of the main theorems of this paper.

Theorem 3.9. *Let B be an $\mathcal{O}D$-interior algebra satisfying Assumption 3.1. There exists a G-graded Morita equivalence between $R = e_\gamma B(D) * G$ and $R' = (\hat{k}D)^{\sigma}_{\beta}\bar{G}$.*

Proof. The following diagram presents the steps of the proof, which is done by working from the bottom row upwards, where the horizontal relations are Morita equivalences induced by the respective bimodules.

$$\begin{tikzcd}
R \arrow[r, squiggly, no head, "W"] \arrow[d, dash, dashed] & R' \arrow[d, dash, dashed] & G\text{-graded and } \bar{G}\text{-graded} \\
R_{\bar{1}} = R_{\bar{D}} \arrow[r, squiggly, no head, "U"] \arrow[d, dash, dashed] & R'_{\bar{D}} = R_{\bar{1}} \arrow[d, dash, dashed] & \bar{D}\text{-graded} \\
R_1 \arrow[r, squiggly, no head, "V"] & R'_1. &
\end{tikzcd}$$

We first prove that there is a Morita equivalence between $R_1 = e_\gamma B(D)$ and $R'_1 = \hat{k}Z(D)$. We have that $\bar{V}$ is a $(S, \hat{k})$-bimodule, and here is a surjective k-algebra homomorphism

$$R_1 \otimes_k \hat{k} \to S \otimes_k \hat{k}.$$

Let V be the projective cover of $\bar{V}$ through this homomorphism, hence there exists a surjective $(R_1, \hat{k})$-bimodule homomorphism $V \to \bar{V}$. Obviously, V is an indecomposable $(R_1, \hat{k})$-bimodule. By construction, V is a generator of the category R_1-mod. Because $\hat{k}Z(D)$ is commutative, we can view V as a $(R_1, \hat{k}Z(D))$-bimodule, hence we have an algebra homomorphism $\hat{k}Z(D) \to$

$\mathrm{End}_{R_1}(V)^{\mathrm{op}}$. Because V is projective as a left R_1-module, and R_1 is a projective kD-module, this map is injective, as any $\hat{k}Z(D)$-module is free. Since

$$\mathrm{End}_{R_1}(V)/J(\mathrm{End}_{R_1}(V) \simeq \mathrm{End}_S(\bar{V}) \simeq \hat{k},$$

by Nakayama's lemma this map is also surjective. Hence, $\mathrm{End}_{R_1}(V)^{\mathrm{op}} \simeq \hat{k}Z(D) = R'_1$, and consequently, ${}_{R_1}V_{R'_1}$ induces a Morita equivalence between R_1 and R'_1.

For the second step, consider the $\bar{D}$-graded k-algebra

$$R_{\bar{D}} = e_\gamma B(D) * \bar{D}$$

obtained from R by truncation, that is, $R_{\bar{D}} = \bigoplus_{x\in\bar{D}} R_x$. Consider the $\bar{D}$-graded k-algebra

$$R'_{\bar{D}} := \hat{k}D.$$

We want to show that the Morita equivalence between R_1 and R'_1 induced by V lifts to a $\bar{D}$-graded Morita equivalence between $R_{\bar{D}}$ and $R'_{\bar{D}}$. For this we need to show that the (R_1, R'_1)-bimodule V extends to a $\Delta(R_{\bar{D}} \otimes_k R'^{\mathrm{op}}_{\bar{D}})$-module. We have the diagonal subgroup

$$\delta D = \{(u, u^\circ) \mid u \in D\}$$

of $D \times D^\circ$, where $u^\circ = u^{-1}$ regarded in D°, so that element $u \otimes_k u^\circ$ belongs to $\Delta(R_{\bar{D}} \otimes_k R'^{\mathrm{op}}_{\bar{D}})$. Notice that the diagonal subalgebra

$$\Delta(R_{\bar{D}} \otimes_k (\hat{k}D)^{\mathrm{op}}) = (B(D)e_\gamma \otimes_k 1)\hat{k}(\delta(D))$$

is a group algebra, hence there exists a surjective k-algebra homomorphism

$$\Delta(R_{\bar{D}} \otimes_k R'^{\mathrm{op}}_{\bar{D}}) \to B(D)e_\gamma.$$

This implies that V extends to $\Delta(R_{\bar{D}} \otimes_k R'^{\mathrm{op}}_{\bar{D}})$, where $u \otimes_k u^\circ \in \hat{k}(\delta(D))$ acts trivially on V. By [14, Theorem 5.1.2] the $\bar{D}$-graded $(R_{\bar{D}}, R'_{\bar{D}})$-bimodule

$$U := R_{\bar{D}} \otimes_{R_1} V \simeq V \otimes_{R'_1} R'_{\bar{D}} \simeq (R_{\bar{D}} \otimes_k R'^{\mathrm{op}}_{\bar{D}}) \otimes_{\Delta(R_{\bar{D}}\otimes_k R'^{\mathrm{op}}_{\bar{D}})} V$$

induces a $\bar{D}$-graded Morita equivalence between $R_{\bar{D}}$ and $R'_{\bar{D}}$.

For the third step of the proof, observe that because $\bar{D}$ is a normal subgroup of G, we can view R and R' as $\bar{G} = G/\bar{D}$-graded, such that

$$R_{\bar{1}} = D_{\bar{D}} = e_\gamma B(D) * \bar{D} \quad \text{and} \quad R'_{\bar{1}} = \hat{k}D.$$

Let

$$W := R \otimes_{R_{\bar{1}}} U \simeq R \otimes_{R_{\bar{1}}} (R_{\bar{1}} \otimes_{R_1} V) \simeq R \otimes_{R_1} V,$$

where 1 denotes the identity of G, and $\bar{1}$ the identity of $\bar{G}$. Then W is a $\bar{G}$-graded left R-module that can also be viewed as G-graded R-module. Moreover, $W_1 = V$, $W_{\bar{1}} = U$, and W is a $\bar{G}$-invariant $R_{\bar{1}}$-module, because V is a G-invariant R_1-module.

The tensor product $R\otimes_k R'^{\mathrm{op}}$ can be viewed as a $\bar{G}\times\bar{G}$-graded k-algebra. Consider its diagonal subalgebra

$$\Delta := \Delta(R \otimes_k R'^{\mathrm{op}}) = \bigoplus_{\bar{g}\in\bar{G}} R_{\bar{g}} \otimes_k R'^{\mathrm{op}}_{\bar{g}},$$

which is also $\bar{G}$-graded, with the $\bar{1}$-component

$$\Delta_{\bar{1}} = R_{\bar{1}} \otimes_k R'^{\mathrm{op}}_{\bar{1}} = (e_\gamma B(D) * \bar{D}) \otimes_k (\hat{k}D)^{\mathrm{op}}.$$

Then W is a Δ_1-module, and we claim that it extends to a Δ-module. Indeed, consider the $\bar{G}$-graded Jacobson radicals $J_{\mathrm{gr}}(R)$ and $J_{\mathrm{gr}}(R')$. We have that

$$R/J_{\mathrm{gr}}(R) \simeq S * \bar{G} = \bar{R}.$$

Let also

$$\bar{R}' := R'/J_{\mathrm{gr}}(R') \simeq \hat{k}^\theta_\beta \bar{G},$$

where $\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k)$ is defined in 3.8. Let

$$\bar{\Delta} := \Delta/J_{\mathrm{gr}}(\Delta) \simeq \Delta(\bar{R} \otimes_k \bar{R}'^{\mathrm{op}}),$$

with $\bar{1}$-component

$$\bar{\Delta}_{\bar{1}} = \Delta_{\bar{1}}/J(\Delta_{\bar{1}}) \simeq S \otimes_k \hat{k},$$

and notice that $W_{\bar{1}}/J(\Delta_{\bar{1}})W_{\bar{1}} \simeq \bar{V}$ is a simple Δ_1-module (and also a simple $\bar{\Delta}_1$-module). We consider the $\bar{G}$-graded endomorphism algebras

$$E := \mathrm{End}_{\Delta}(\Delta \otimes_{\Delta_1} W_1)^{\mathrm{op}}$$

and

$$\bar{E} := \mathrm{End}_{\bar{\Delta}}(\bar{\Delta} \otimes_{\bar{\Delta}_1} \bar{V})^{\mathrm{op}}.$$

By [15, Lemma 2.4] we have the isomorphism

$$E/J_{\mathrm{gr}}(E) \simeq \bar{E}$$

of $\bar{G}$-graded algebras. By construction, $\mathrm{End}_{\bar{R}}(\bar{R} \otimes_{\bar{R}_{\bar{1}}} \bar{V})^{\mathrm{op}} \simeq \bar{R}'$ as $\bar{G}$-graded algebras. This means that $\bar{V}$ extends to a $\bar{\Delta}$-module, hence the short exact sequence

$$1 \to \bar{E}_{\bar{1}}^{\times} \to \mathrm{hU}(\bar{E}) \to \bar{G} \to 1,$$

splits, where $\bar{E}_{\bar{1}}^{\times} = \hat{k}^{\times}$. We have the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & E_{\bar{1}}^{\times} \arrow[r] \arrow[d] & \mathrm{hU}(E) \arrow[r] \arrow[d] & \bar{G} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \bar{E}_{\bar{1}}^{\times} \arrow[r] & \mathrm{hU}(\bar{E}) \arrow[r] & \bar{G} \arrow[r] & 1.
\end{tikzcd}$$

By Theorem 2.5 we deduce that $\mathrm{hU}(E)$ is a split extension of $E_{\bar{1}}^{\times}$ by $\bar{G}$. Hence, $W_{\bar{1}} = U$ extends to Δ, and therefore the (R, R')-bimodule

$$W = R \otimes_{R_1} W_{\bar{1}} \simeq (R \otimes_k R'^{\mathrm{op}}) \otimes_{\Delta} U$$

induces a $\bar{G}$-graded Morita equivalence between R and R'. But since $W \simeq R \otimes_{R_1} V$ is in fact G-graded, we have that $R' \simeq \mathrm{End}_R(W)^{\mathrm{op}}$ as G-graded algebras, and hence W is a G-graded (R, R')-bimodule. This implies that there is a G-graded Morita equivalence between R and R'. □

Remark 3.10. Notice that since k is a perfect field and $\bar{D}$ a p-group, it follows by Green's theorem that U is an indecomposable $(R_{\bar{D}}, R'_{\bar{D}})$-bimodule. It is not difficult to see that W is also indecomposable as an (R, R')-bimodule, not only as a G-graded (R, R')-bimodule.

3.11. Next, we deduce Külshammer's main result [11, Theorem A] from Theorem 3.9. We will employ the notation introduced in this section.

Consider the finite group H and the block $B = b\mathcal{O}H$ of the group algebra $\mathcal{O}H$. Let D_γ be a defect pointed group $H_{\{b\}}$. Then there exist the algebra homomorphisms

$$\mathcal{O}C_H(D) \hookrightarrow \mathcal{O}H \xrightarrow{\mathrm{Br}_D} kC_H(D),$$

and we have that $e_\gamma B(D) = kC_H(D)e_\gamma$, where e_γ is, as in 3.2, the block with defect group $Z(D)$ of $kC_H(D)$ determined by D_γ. Moreover, e_γ is also a block of $kN_H(D)_\gamma$ with defect group D. With the notation of 3.4, the proof of [18] implies that $G_\gamma = N_H(D)_\gamma/C_H(D)$ and $\bar{G}_\gamma = N_H(D)_\gamma/DC_H(D)$.

By Theorem 3.9, there is a G_γ-graded Morita equivalence between $e_\gamma kN_H(D)_\gamma$ and $(\hat{k}D)^{\sigma}_{\beta}\bar{G}_\gamma$, where σ and β are defined in 3.7 and 3.8.

It is not difficult to show that this equivalence lifts to an equivalence over $\mathcal{O}$, and moreover, the proof of Theorem 3.9 gives the structure of blocks with normal defect groups in the form expressed by Külshammer (except that we do not assume here that the p-modular system $(\mathcal{K}, \mathcal{O}, k)$ is splitting).

Corollary 3.12. *There is an isomorphism of G_γ-graded $\mathcal{O}$ algebras between $\mathcal{O}N_H(D)_\gamma e_\gamma$ and $M_m(\mathcal{O}) \otimes_{\mathcal{O}} (\hat{\mathcal{O}}D)^\sigma_\beta \bar{G}_\gamma$, where $m = \dim_{\hat{k}} \bar{V}$ as in 3.2.*

4. G-graded $\mathcal{O}P$-interior algebras

4.1. Let R be a G-graded crossed product over $\mathcal{O}$, with 1-component $R_1 = A$ as in 2.1. The assumptions in this section are as follows.

(1) kA is a simple k-algebra, and denoting $\hat{k} := Z(A)$, $\hat{k}$ is a Galois extension of k, and kA has Schur index 1. Denote by V the unique (up to isomorphism) simple A-module. Thus we have the isomorphisms $\mathrm{End}_A(V) \simeq \hat{k}$ and $kA \simeq M_n(\hat{k})$, where $n = \dim_{\hat{k}} V$.

(2) Let P be a defect group of the pointed group $G_{\{1\}}$ on $Z(A)$. We assume that P is a normal subgroup of G, and denote $\bar{G} := G/P$.

(3) There exists a splitting $\varphi : P \to \mathrm{hU}(R_P)$ of the group extension

$$1 \to A^\times \to \mathrm{hU}(R_P) \to P \to 1,$$

and we denote $Q = \varphi(P)$.

(4) We consider the conjugation action of Q on A, and we assume that regarded as a central simple $\hat{k}$-algebra, kA is a *Dade Q-algebra*, that is, A has a Q-stable $\hat{k}$-basis containing the identity 1_A, and $A(Q) \neq 0$.

4.2. Note that because k is perfect, there exists a unique group homomorphism

$$\psi : Q \to kA^\times$$

such that $\det \psi(u) = 1$, and inducing the action of Q on A (that is, ${}^u a = \psi(u) a \psi(u)^{-1}$ for all $a \in A$ and $u \in Q$).

Remark 4.3. The group $\mathrm{hU}(R)$ acts on the simple algebra kA and on the center $\hat{k}$ of $Z(kA)$. Moreover, we have that $R_P = AQ$, $\mathrm{hU}(R_P) = A^\times Q$, thus $\bar{G} \simeq \mathrm{hU}(R)/\mathrm{hU}(R_P)$. Since $Q(kA)^\times$ acts trivially on $\hat{k}$, we have the group homomorphism

$$\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k),$$

and denote by K be the kernel of θ. By Assumption 4.1 (2) and [22, Lemma 14.1], there is an idempotent $i \in Z(A)^P$ and an element $a \in Z(A)^P$ such that $\mathrm{Tr}_P^G(ai) = 1$. In our case, $Z(kA) = \hat{k}$, so $i = 1$, and one easily deduces that $|K|$ is invertible in k.

Note that by [16], the surjectivity of the trace map

$$\mathrm{Tr}_P^G : Z(A)^P \to Z(A)^G$$

is equivalent to the separability of the algebra extension $R_P \to R$.

4.4. Consider the normalizers and centralizers $N_{A^\times}(Q)$, $N_{\mathrm{hU}(R)}(Q)$, $C_{A^\times}(Q)$ and $C_{\mathrm{hU}(R)}(Q)$, and observe that under our Assumption 4.1 (3), we have that

$$C_{A^\times}(Q) = N_{A^\times}(Q).$$

Denote

$$G' := N_{\mathrm{hU}(R)}(Q)/C_{A^\times}(Q),$$

so G' can be regarded as a subgroup of G, via the group homomorphism $\mathrm{hU}(R) \to G$.

Lemma 4.5. *The map $\mathrm{hU}(R) \to G$ induces an isomorphism from G' to G.*

Proof. We have to show that $\mathrm{hU}(R) = A^\times N_{\mathrm{hU}(R)}(Q)$. Indeed, let u be a homogeneous unit of R. Since P is normal in G, both Q and uQu^{-1} are subgroups of $\mathrm{hU}(R_P)$, and they define, via 4.2, R_P-module structures on the simple A-module V. But these R_P-modules are isomorphic, hence Q and uQu^{-1} are $A^\times$-conjugate, so there is $a \in A^\times$ such that au normalizes Q. □

4.6. There exists a group homomorphism

$$C_{A^\times}(Q) \to A(Q)^\times,$$

and moreover, this map, and the Brauer homomorphism $A^Q \to A(Q)$ are compatible with the conjugation action of $N_{\mathrm{hU}(R)}(Q)$ on these objects. This means that $A(Q)$ is a $C_{A^\times}(Q)$-interior $N_{\mathrm{hU}(R)}(Q)$-algebra, so we can now construct, as in 2.3, the G'-graded crossed product

$$R' := A(Q) * G',$$

with 1-component $A' := A(Q)$.

The next theorem is the second main result of this paper, and it is a generalization of the main result of Dade [2].

Theorem 4.7. *Let R be a G-graded crossed product satisfying Assumption 4.1. Then there is a G-graded Morita equivalence over k between kR and $R' = A(Q) * G'$.*

Proof. We have that V becomes a (simple) kR_P-module via the group homomorphism $\psi : Q \to kA^\times$. View kR as a G/P-graded algebra. Then $\mathrm{End}_{kR}(kR \otimes_{kR_P} V)^{\mathrm{op}}$ is a crossed product of

$$\mathrm{End}_{kR_1}(V)^{\mathrm{op}} \simeq \hat{k}$$

with $\bar{G}$, hence there exists a $\bar{G}$-graded k-algebra isomorphism

$$\mathrm{End}_{kR}(kR \otimes_{kR_P} V)^{\mathrm{op}} \simeq \hat{k}^{\theta}_{\alpha}\bar{G},$$

where $\alpha : \bar{G} \times \bar{G} \to \hat{k}$ is a 2-cocycle, and $\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k)$ is defined in 4.3.

Denote $P' := N_{\mathrm{hU}(R_P)}(Q)/C_{A^\times}(Q)$, which is a p-subgroup of G', isomorphic to P, so we have that $G'/P' \simeq \bar{G}$. Let V' be the unique (up to isomorphism) simple R'_1-module. As above, we have that V' extends to a $R'_{P'}$-module, and we have a G/P-graded k-algebra isomorphism

$$\mathrm{End}_{R'}(R' \otimes_{R'_{P'}} V')^{\mathrm{op}} \simeq \hat{k}^{\theta}_{\alpha'}(G'/P'),$$

where $\alpha' : G'/P' \times G'/P' \to \hat{k}$ is a 2-cocycle, and note that the action of G'/P' on $\hat{k}$ coincides with the action of $\bar{G}$ on $\hat{k}$.

The proof follows the following diagram, working from the bottom row upwards.

$$\begin{tikzcd}
kR \arrow[r, squiggly, "Z"] \arrow[d, dashed, no head] & R' \arrow[d, dashed, no head] & G\text{-graded and } \bar{G}\text{-graded} \\
kR_P \arrow[r, squiggly, "Y"] \arrow[d, dashed, no head] & R'_{P'} \arrow[d, dashed, no head] & P\text{-graded} \\
kA \arrow[r, squiggly, "X"] & A'. &
\end{tikzcd}$$

We know that kA and A' are central simple $\hat{k}$-algebras in the same Brauer class, and more precisely, we have a Morita equivalence

$$kA \sim A'$$

induced by the (kA, A')-bimodule $X := V \otimes_{\hat{k}} V'^{\vee}$, where $V'^{\vee}$ is the k-dual of V'.

For the second step, note that because V extends to kR_P, we have that

$$\mathrm{End}_{kR_P}(kR_P \otimes_{kA} V)^{\mathrm{op}} \simeq \hat{k}P,$$

and because V' extends to R'_P, we have that

$$\mathrm{End}_{R'_{P'}}(R'_{P'} \otimes_{A'} V')^{\mathrm{op}} \simeq \hat{k}P'.$$

But $P \simeq P'$, so this implies by [14, Section 5.1] that there exists a P-graded Morita equivalence between kR_P and $R'_{P'}$ induced by the bimodule

$$Y := kR_P \otimes_{kA} X \simeq X \otimes_{A'} R'_{P'}.$$

For the third step, we regard kR and R' as $\bar{G}$-graded algebras with $\bar{1}$-components $kR_{\bar{1}} = kR_P$ and $R'_{\bar{1}} = R'_P$. Consider the $\bar{G}$-graded algebra

$$\Delta := \Delta(kR \otimes_k R'^{\mathrm{op}}) = \bigoplus_{\bar{g}\in\bar{G}}(kR_{\bar{g}} \otimes_k R'^{\mathrm{op}}_{\bar{g}}).$$

In order to obtain a $\bar{G}$-graded equivalence between kR and R' we have to show that the $\Delta_{\bar{1}}$-module Y extends to a Δ-module. For this, by Dade's criterion (see [14, Theorem 3.1.1]), we must show that the extension

$$1 \to E_1^{\times} \to \mathrm{hU}(E) \to \bar{G} \to 1 \tag{1}$$

splits, where we have denoted

$$E = \mathrm{End}_{\Delta}(\Delta \otimes_{\Delta_1} Y)^{\mathrm{op}}.$$

We consider the $\bar{G}$-graded algebra

$$\bar{E} = E/J_{\mathrm{gr}}(E),$$

and let $\bar{\Delta} = \Delta/J_{\mathrm{gr}}(\Delta), k\bar{R} := kR/J_{\mathrm{gr}}(kR)$ and $\bar{R}' := R'/J_{\mathrm{gr}}(R')$, where Δ, R and R' are also regarded here as $\bar{G}$-graded algebras. As in the proof of Theorem 3.9, we have

$$\bar{\Delta} \simeq \Delta(k\bar{R} \otimes_k \bar{R}'^{\mathrm{op}}),$$

and because $X \simeq Y/J(\Delta_1)$, we have

$$\bar{E} \simeq \mathrm{End}_{\bar{\Delta}}(\bar{\Delta} \otimes_{\bar{\Delta}_{\bar{1}}} X).$$

By 4.3, p does not divide the order of the kernel K of the map $\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k)$. Therefore, again as in Theorem 3.9, it is enough to show that the extension

$$1 \to \bar{E}_1^{\times} \to \mathrm{hU}(\bar{E}) \to \bar{G} \to 1 \tag{2}$$

splits, and for this, we have to show that the 2-cocycles α and α' are cohomologous.

By Theorem 3.9, since P is a normal subgroup of hU(R), there is a P-graded Morita equivalence between kR_P and $\hat{k}P$, and a G-graded Morita equivalence between kR and $(\hat{k}P)^{\theta}_{\alpha}\bar{G}$. Similarly, there is a P'-graded Morita equivalence between $R'_{P'}$ and $\hat{k}P'$, and a G-graded Morita equivalence between R' and $(\hat{k}P')^{\theta}_{\alpha'}\bar{G}'$. We also have P-graded Morita equivalence between $\hat{k}P$ and $\hat{k}P'$ induced by an endopermutation $\hat{k}(P \times P')$-bimodule. By using the previous arguments for $(\hat{k}P)^{\theta}_{\alpha}\bar{G}$ and $(\hat{k}P')^{\theta}_{\alpha'}\bar{G}'$ instead of kR and R' respectively, we deduce by Dade's Theorem 2.7 that α and α' are indeed cohomologous. By 2.6, it also follows that the Morita equivalence between kR and R' is defined over k. □

Remark 4.8. Let us deduce Dade's result [2, (6)] from Theorem 4.7. We will follow our notations introduced in 4.1.

Let $A := e\mathcal{O}N$ be a block with defect zero of the normal subgroup N of the finite group H, where we assume that e is H-invariant. Let Q be a defect group in H of e, and assume that $M := QN$ is a normal subgroup of H. Denote $G := H/N$, $P := M/N$, $\bar{G} := H/M$, and $R := e\mathcal{O}H$, which is regarded as a G-graded algebra.

The Brauer correspondent of $e\mathcal{O}M$ in the block $e'\mathcal{O}N_M(Q)$ which covers the block $e'\mathcal{O}C_N(Q)$ of defect zero, and observe that the Brauer quotient $A(Q)$ of A is $\bar{e}'kC_N(Q)$. By Lemma 4.5, we have that $G \simeq N_H(H)/C_N(Q)$ and $\bar{G} \simeq N_H(H)/N_N(Q)$. By Theorem 4.7, there is a G-graded Morita equivalence between $kR = \bar{e}kH$ and $\bar{e}'kN_H(Q)$.

In this situation it is easy to see that this equivalence lifts to a G-graded Morita equivalence between $R = e\mathcal{O}H$ and $e'\mathcal{O}N_H(Q)$, and therefore it induces a perfect isometry between the ordinary characters of H lying over irreducible characters belonging to e and the ordinary characters of $N_H(Q)$ lying over irreducible characters belonging to e'.

Finally, note that e is a block with normal defect group of $\mathcal{O}M$, while e' is a block with normal defect group of $\mathcal{O}N_M(Q)$, so Külshammer's theorem applies to both blocks. The crucial fact in Dade's result is that the 2-cocycles given by Külshammer's theorem are cohomologous.

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