Module covers and the Green correspondence

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Abstract

We prove that block induction and the categorical version of the Green correspondence are compatible with block covers and module covers.

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1. Introduction

It is well-known that the Green correspondence can be expressed as an equivalence between certain quotient categories of modules over group algebras. This result is due to J.A. Green [9], while M. Auslander and M. Kleiner [2] established a very general category equivalence using only the properties of adjoint functors.

Very recently, M.E. Harris [10] combined this approach with the Nagao-Green theorem on block induction, obtaining a version "with blocks" of the above mentioned equivalence. We briefly recall his result in Section 4.

In this note, we also deal with block covers and with module covers in the sense of Alperin [1]. To explain this, fix a finite group G, a normal subgroup N of G, and a complete discrete valuation ring $\mathcal{O}$ with residue field of characteristic $p > 0$. Our main result,

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stated in Theorem 4.3 below, says that if one starts with an indecomposable $\mathscr{O}G$-module V covering an indecomposable $\mathscr{O}N$-module U such that V and U are objects of appropriate quotient categories, then the images of V and U under Harris' functors also cover each other. Moreover, by restricting this correspondence to covering modules, one still gets a bijection.

In Section 2 we present some needed facts on block induction, while in Section 3 we discuss block induction in relation with module covers. The last section is devoted to the categorical Green correspondence and to the proof of the main result. Three corollaries are also derived, one of them being the well-known Harris-Knörr correspondence for covering blocks.

We denote by $\mathscr{S}(G)$ the set of subgroups of the finite group G, and by $\mathrm{Bl}(\mathscr{O}G)$ the set of block idempotents of the group algebra $\mathscr{O}G$. If V and W are $\mathscr{O}G$-modules, then $V|W$ means that V is a direct summand of W. We assume that all categories and functors are additive. Otherwise, our assumptions and notations are standard, and our general references are [5], [8] and [12].

2. On block induction

There exist several notions of block induction, and we refer the reader to [3] (and to the references given there) for a comparison between them. Here we use the block induction in the sense of Brauer, as in [8, Chapter 3, Section 9] and [12, Chapter 5, Section 3], but see also [5, Section 6.2]. We will freely use the fact that reduction modulo p induces a defect groups preserving bijection between the blocks of $\mathscr{O}G$ and the blocks of kG.

2.1. Let $\hat{k}$ be a finite extension of the field k, such that $\hat{k}$ is a splitting field for every subgroup of G. Let H be a subgroup of G and let $\tilde{B} \in \mathrm{Bl}(\mathscr{O}H)$. Denote by

$$w^*_{\tilde{B}} : \hat{k}H \to \hat{k}$$

the central character associated with $\tilde{B}$, and consider the k-linear map

$$s^*_H : Z(\hat{k}G) \to Z(\hat{k}H), \qquad \sum_{x\in C} x \mapsto \sum_{x\in C\cap H} x,$$

where C runs through the conjugacy classes of G.

Definition 2.2. The induced block $\tilde{B}^G$ is defined if the map $w^*_{\tilde{B}} \circ s^*_H : Z(kG) \to k$ is a central character.

2.3. If Q is a p-subgroup of G, we will also consider the Brauer map

$$\mathrm{Br}_Q : Z(kG) \to Z(kC_G(Q)).$$

Recall that if the subgroup H satisfies $QC_G(Q) \leq H \leq N_G(Q)$, then every idempotent in $Z(kH)$ belongs in fact to $kC_G(Q)$.

By [12, Chapter 5, Theorem 3.5] and [5, Chapter 6, Paragraph 6.2], we have the following important situation.

Lemma 2.1. *Assume that $QC_G(Q) \leq H \leq N_G(Q)$ for some p-subgroup of Q of G. Then the induction to G of any block of $\mathcal{O}H$ is defined. Moreover, if $\tilde{B}$ is a block of $\mathcal{O}H$, then the block $\tilde{B}^G =: B$ of $\mathcal{O}G$ is determined by the equality*

$$\tilde{B} = \tilde{B} \cdot \mathrm{Br}_Q(B).$$

We also need the following consequence of the transitivity property of block induction.

Lemma 2.2. *Assume that the subgroup H of G contains $QC_G(Q)$ for some p-subgroup Q of G. Then, for any block $b \in \mathrm{Bl}(\mathcal{O}N_H(Q))$, the blocks b^H, b^G and $(b^H)^G$ are defined and we have $b^G = (b^H)^G$.*

Proof. By our choice of H, we obtain $QC_G(Q) \leq N_H(Q) \leq N_G(Q)$, and also $QC_H(Q) \leq N_H(Q) \leq N_G(Q)$, since $C_G(Q) = C_H(Q)$. If b is block of $\mathcal{O}N_H(Q)$, then b^H and b^G are defined, hence by applying [12, Chapter 5, Lemma 3.4] to the situation $N_H(Q) \leq H \leq G$, the claim follows. $\square$

Remark 2.4. With the notations of the previous lemma, we see that the blocks of $\mathcal{O}H$ as well as the blocks of $\mathcal{O}G$, which do not lie in the kernel of the Brauer map, determine partitions of the set $\mathrm{Bl}(\mathcal{O}N_H(Q))$. Explicitly, for a fixed block $\tilde{B} \in \mathrm{Bl}(\mathcal{O}H)$, the blocks $b \in \mathrm{Bl}(\mathcal{O}N_H(Q))$ that satisfy $b^H = \tilde{B}$ are exactly the blocks that satisfy $\mathrm{Br}_Q(\tilde{B})b = b$. The same argument holds for the blocks of $\mathcal{O}G$.

Proposition 2.3. *Assume that $QC_G(Q) \leq H \leq G$ for a p-subgroup of G. The following hold:*

(1) If $B \in \mathrm{Bl}(\mathcal{O}G)$ with $\mathrm{Br}_Q(B) \neq 0$, then any block $\tilde{B} \in \mathrm{Bl}(\mathcal{O}H)$ with $\mathrm{Br}_Q(\tilde{B}) \neq 0$ verifies $\tilde{B}^G = B$ if and only if $\mathrm{Br}_Q(B)\mathrm{Br}_Q(\tilde{B}) \neq 0$.

(2) If $\tilde{B} \in \mathrm{Bl}(\mathcal{O}H)$ with $\mathrm{Br}_Q(\tilde{B}) \neq 0$, then there is a unique block $B \in \mathrm{Bl}(\mathcal{O}G)$ such that $\tilde{B}^G = B$, and B is given by $\mathrm{Br}_Q(B)\mathrm{Br}_Q(\tilde{B}) \neq 0$.

Proof. To prove (1), assume that

$$\mathrm{Br}_Q(B)\mathrm{Br}_Q(\tilde{B}) = \sum b_i,$$

where b_i are blocks of $kN_H(Q)$. Clearly, $\mathrm{Br}_Q(B)b_i = b_i$ and $\mathrm{Br}_Q(\tilde{B})b_i = b_i$ for all b_i. Then for all blocks appearing in the above decomposition we have $b_i^H = \tilde{B}$ and $b_i^G = B$, hence $((b_i)^H)^G = \tilde{B}^G = B$.

Conversely, for any block $\tilde{b}$ of $kN_G(Q)$ satisfying $\tilde{b} = \tilde{b}\mathrm{Br}_Q(\tilde{B})$, we have $\tilde{b}^H = \tilde{B}$, and $\tilde{b}^G$ is defined. This forces $\tilde{b}^G = B$, that is $\tilde{b} = \tilde{b}\mathrm{Br}_Q(B)$, and therefore $\mathrm{Br}_Q(B)\mathrm{Br}_Q(\tilde{B}) \neq 0$.

For the proof of (2), we observe that $\mathrm{Br}_Q(\tilde{B})b_i = b_i$ for some blocks b_i of $kN_H(Q)$. For any such b_i, the unique block $B := b_i^G$ is defined, as well as $b_i^H = \tilde{B}$. A similar argument already used gives $\tilde{B}^G = B$. □

Remark 2.5. Assume that in Proposition 2.3 we choose the subgroup H such that it contains $N_G(Q)$, for a p-subgroup Q of G. In this situation, any of the two assertions determine the Brauer correspondent of B or of $\tilde{B}$ provided that Q is a defect group of B or of $\tilde{B}$.

3. Block induction and normal subgroups

Let G be a finite group, let $B \in \mathrm{Bl}(\mathscr{O}G)$, and let N be a normal subgroup of G, and $b \in \mathrm{Bl}(\mathscr{O}N)$. Let V be an indecomposable $\mathscr{O}GB$-module, and let U be an indecomposable $\mathscr{O}Nb$-module. Recall that B is said to cover the block b of $\mathscr{O}N$, if $bB \neq 0$. Recall also the following definition due to Alperin [1].

Definition 3.1. The indecomposable $\mathscr{O}G$-module V *covers* the indecomposable $\mathscr{O}N$-module U if $U \mid V_N$, and V contains a vertex $R \leq G$ such that $R \cap N$ is a vertex of U.

The notions of module covering is clearly compatible with block covering.

Lemma 3.1. *With the above notations, if V covers U, then B covers b.*

Proof. Since V covers U, we have $bU = U \mid bV_N$, and assuming by contradiction that $bB = 0$, we would get $U = 0$. □

The main result of this section says that block induction is also compatible with coverings.

Theorem 3.2. *Let H be a subgroup of G and denote $H' := H \cap N$. Let $\beta' \in \mathrm{Bl}(\mathscr{O}H')$, and let W' be an indecomposable $\mathscr{O}H'\beta'$-module. Let $b \in \mathrm{Bl}(\mathscr{O}N)$, and let U be an indecomposable $\mathscr{O}Nb$-module. Let $\beta \in \mathrm{Bl}(\mathscr{O}H)$, and let W be an indecomposable $\mathscr{O}H\beta$-module. Finally, let $B \in \mathrm{Bl}(\mathscr{O}G)$, and let V be an indecomposable $\mathscr{O}GB$-module.*

Assume that $N_G(Q) \leq H$ for some vertex Q of W'. Then the following statements hold:

1. *If the module W covers W', then the block $\beta \in \mathrm{Bl}(\mathscr{O}H)$ covers β'.*
2. *If $W' \mid U_{H'}$, then the block $(\beta')^N$ is defined, and we have $(\beta')^N = b$.*
3. *If W covers W' and $W \mid V_H$, then the block β^G is defined, and $\beta^G = B$.*
4. *If the modules W', W, U and V satisfy all the above, then the $\mathscr{O}G$-module V covers U, and hence the block B covers b.*

Proof. Statement (1) follows immediately by Lemma 3.1.

For statement (2), observe that our assumptions on Q imply that H' contains $C_N(Q)$. Then we may readily apply [10, Theorem 1.14] (which is essentially the Nagao-Green theorem, see [12, Chapter 5, Theorem 3.12]), to conclude that $(\beta')^N$ is defined and that $(\beta')^N = b$.

For (3) we use again [10, Theorem 1.14], observing that the $\mathscr{O}H$-module W has a vertex $P \leq H$ such that $Q \leq P$, so it follows that $C_G(P) \leq C_G(Q) \leq H$.

For the last statement we use the Burry-Carlson-Puig Theorem, see [5, Theorem 3.12.3] or [4, Theorem 5]. Notice that since $W' \mid U_{H'}$ and $N_N(Q) \leq H'$, it follows that U has vertex Q. Similarly, since we have $W \mid V_H$ and $N_G(P) \leq H$, it follows that V has vertex P. Moreover, we have that $P \cap N = P \cap H' = Q$, since $P \leq N_G(Q)$ and $N_N(Q) \leq H'$ (because Q is normal in P). By the Burry-Carlson-Puig Theorem we also have that $U \mid V_N$, and finally we apply again Lemma 3.1. $\square$

4. Above the Green correspondence

Remark 4.1. Fix a p-subgroup R of G. Let H be a subgroup of G that contains $N_G(R)$, and let

$$\Delta := \{A \mid A \leq R\}.$$

Consider the sets

$$\mathfrak{X} := \{A \in \Delta \mid A \leq R \cap R^g \text{ for some } g \in G \setminus H\},$$

$$\mathfrak{Y} := \{A \in \Delta \mid A \leq H \cap R^g \text{ for some } g \in G \setminus H\}$$

and

$$\mathfrak{A} := \{A \in \Delta \mid A \notin_G \mathfrak{X}.\}$$

We have the following consequence of Theorem 3.2.

Proposition 4.1. *Let V be an indecomposable $\mathscr{O}GB$-module with vertex $T \in \mathfrak{A}$, and let W be its Green correspondent with respect to T and H. Then W lies in a block β of $\mathscr{O}H$ that satisfies $\beta^G = B$, $\mathrm{Br}_D(\beta) \neq 0$ and $\mathrm{Br}_D(B) \neq 0$ for some p-subgroup D of G that contains a vertex of V.*

Proof. Since $T \leq N_G(T) \leq H$, and T is also a vertex of W, it follows by Theorem 3.2 that $\beta^G = B$. Moreover, since $W \mid V_H$, we get $B\beta \neq 0$. Now the idempotent $B\beta$ of the subalgebra $\mathscr{O}G^H$ of H-fixed elements of $\mathscr{O}G$ admits a decomposition into pairwise orthogonal primitive idempotents

$$B\beta = \sum_j f_j.$$

For every j, denote by D_j a defect group in H of f_j. Since we have

$$W \mid \beta V_H = \beta B V_H,$$

we obtain $T \leq_H D_j$ for some j. Clearly, the equality $B\beta f_j = f_j$ implies that $\mathrm{Br}_{D_j}(B) \neq 0$ and $\mathrm{Br}_{D_j}(\beta) \neq 0$. $\square$

Remark 4.2. Let N be a normal subgroup of G, and fix the p-subgroup $P := R \cap N$ of N. Assume that the subgroup H of G contains $N_G(P)$ (so it also contains $N_G(R)$), and set $H' := H \cap N$, and

$$\Delta' := \{A \mid A \leq P\}.$$

Then we have $\Delta' = \Delta \cap \mathscr{S}(N)$, and denoting

$$\mathfrak{X}' := \{A \in \Delta' \mid A \leq P \cap P^g \text{ for some } g \in N \setminus H'\},$$

$$\mathfrak{Y}' := \{A \in \Delta' \mid A \leq H' \cap P^g \text{ for some } g \in N \setminus H'\},$$

and

$$\mathfrak{A}' := \{A \in \Delta' \mid A \notin_N \mathfrak{X}'\},$$

we obtain the inclusions $\mathfrak{X}' \subseteq \mathfrak{X}$ and $\mathfrak{Y}' \subseteq \mathfrak{Y}$.

We also assume that the following equalities hold:

$$\mathfrak{X}' = \mathfrak{X} \cap \mathscr{S}(N) \quad \text{and} \quad \mathfrak{A}' = \mathfrak{A} \cap \mathscr{S}(N). \tag{4.3.1}$$

Remark 4.3. We have the disjoint unions

$$\Delta' = \mathfrak{A}' \cup \mathfrak{X}' \subseteq \Delta = \mathfrak{A} \cup \mathfrak{X},$$

and note also that the equalities $\mathfrak{X}' = \mathfrak{X} \cap \mathscr{S}(N)$ and $\mathfrak{A}' = \mathfrak{A} \cap S(N)$ of (4.3.1) hold when R is a defect group of B and b is G-invariant. Moreover, because Δ' and $\mathfrak{X}'$ are $N_G(P)$-invariant, so is $\mathfrak{A}'$.

Proposition 4.2. *Assume that $H = N_G(P)$. Then the following statements hold.*

1) *Let W be an indecomposable $\mathscr{O}H$-module with vertex in $\mathfrak{A}$. Then any indecomposable $\mathscr{O}H'$-module W' covered by W has vertex in $\mathfrak{A}'$.*

2) *Let W' be an indecomposable $\mathscr{O}H'$-module with vertex in $\mathfrak{A}'$. Then any indecomposable $\mathscr{O}H$-module W that covers W' has vertex in $\mathfrak{A}$.*

Proof. 1) Since W covers W', it follows by Definition 3.1 that there is a vertex $T \leq H$ of W such that $T' := T \cap H'$ is a vertex of W'. There is $x \in N_G(P)$ such that $T^x = A \in \mathfrak{A}$, and then $A^{x^{-1}} \cap H' = (A \cap H')^{x^{-1}} = T' \in \mathfrak{A}'$, by Remark 4.3.

For the proof of 2) we argue as follows. There is a vertex $T \leq H$ of W such that $T \cap H'$ is a vertex of W'. Then, there is $y \in H'$ such that $A' := T^y \cap H' \in \mathfrak{A}'$ is a vertex of W', forcing T^y to be a vertex of W lying in $\mathfrak{A}$. $\square$

Remark 4.4. Let, as above, $H = N_G(P)$ and $H' = N_N(P)$, and we fix a block B of G covering some block b of N. Finally, we consider the sets of blocks

$$\mathscr{B} = \{\beta \mid \beta \in \mathrm{Bl}(\mathscr{O}H) \text{ with } \beta^G \text{ is defined and } \beta^G = B\}$$

and

$$\mathscr{B}' = \{\beta' \mid \beta' \in \mathrm{Bl}(\mathscr{O}H') \text{ with } (\beta')^N \text{ is defined and } (\beta')^N = b\}.$$

According to Harris [10, Theorem 2.7], there is an equivalence of categories, denoted by $\tilde{\Phi}_b$, between the quotient category of finite direct sums of indecomposable $\mathscr{O}Nb$-modules having vertices in Δ', denoted $b\mathrm{mod}(N,\Delta')/b\mathrm{mod}(N,\mathfrak{X}')$, and the direct product of quotient categories of $\mathscr{O}H'\beta'$-indecomposable modules

$$\mathscr{P}' := \prod_{\beta' \in \mathscr{B}'} \beta'\mathrm{mod}(H',\Delta' \cup \mathfrak{Y}')/\beta'\mathrm{mod}(H',\mathfrak{Y}').$$

Similarly, for the block $B \in \mathrm{Bl}(\mathscr{O}G)$, there is an equivalence of categories, denote by $\tilde{\Phi}_B$, between $B\mathrm{mod}(G,\Delta)/B\mathrm{mod}(G,\mathfrak{X})$ and the direct product

$$\mathscr{P} := \prod_{\beta \in \mathscr{B}} \beta\mathrm{mod}(H,\Delta \cup \mathfrak{Y})/\beta\mathrm{mod}(H,\mathfrak{Y}).$$

Note also that any non-zero indecomposable object of any of these four categories has a vertex which belongs to $\mathfrak{A}$ or to $\mathfrak{A}'$.

In order to formulate our main result in terms of equivalences of categories, we need to introduce two subcategories.

4.5. 1) Let $\mathscr{C}(B|b)$ be the full additive subcategory of the quotient category

$$B\mathrm{mod}(G,\Delta)/B\mathrm{mod}(G,\mathfrak{X})$$

consisting of those objects whose indecomposable summands cover some indecomposable $\mathscr{O}Nb$-module which is a nonzero object of $b\mathrm{mod}(N,\Delta')/b\mathrm{mod}(N,\mathfrak{X}')$.

2) Let $\mathscr{P}(\mathscr{B}|\mathscr{B}')$ be the full additive subcategory of $\mathscr{P}$ defined as follows. For each indecomposable object in $\mathscr{P}(\mathscr{B}|\mathscr{B}')$, there is an indecomposable $\mathscr{O}H\beta$-module for some block $\beta\in\mathscr{B}$, covering an indecomposable $\mathscr{O}H'\beta'$-module which is a nonzero object of the quotient category

$$\beta'\mathrm{mod}(H',\Delta'\cup\mathfrak{Y}')/\beta'\mathrm{mod}(H',\mathfrak{Y}')$$

for some block $\beta'\in\mathscr{B}'$ covered by the block β.

With the assumptions and notations of 4.1, 4.2, 4.4 and 4.5, we may state the main result of this paper.

Theorem 4.3. *The functor $\tilde{\Phi}_B$ induces an equivalence between $\mathscr{C}(B|b)$ and $\mathscr{P}(\mathscr{B}|\mathscr{B}')$*

Proof. Let V be a non-zero indecomposable object of $B\mathrm{mod}(G,\Delta)/B\mathrm{mod}(G,\mathfrak{X})$ covering a non-zero object U of $b\mathrm{mod}(N,\Delta')/b\mathrm{mod}(N,\mathscr{X}')$. By [10, Lemma 2.8], the indecomposable $\mathscr{O}H$-module $W:=\tilde{\Phi}_B(V)$ is a non-zero object of $\mathscr{P}$. Similarly, the indecomposable $\mathscr{O}H'$-module $W':=\tilde{\Phi}_b(U)$ is a non-zero object of $\mathscr{P}'$. If $A'\in\mathfrak{A}'$ is a vertex of U and of W' then, by Definition 3.1 and Remark 4.3, both V and W have a vertex $A\in\mathfrak{A}$ such that $A\cap N=A'$. We have that $W'\mid U_{H'}$, $U\mid V_N$ and $W\mid V_H$, which forces $W'\mid V_{H'}$. Assume that $V_H=W\oplus X$, so X is a direct sum of indecomposable $\mathfrak{Y}$-projective modules. Hence $W'\mid(W\oplus X)_{H'}$, and since W' has vertex $A'\in\mathfrak{A}'$, using Remark 4.3, we conclude that $W'\mid W_{H'}$ and $W\in\mathscr{P}(\mathscr{B}|\mathscr{B}')$. We have obtained a one-to-one map which we only need to show that it is surjective.

Since $\tilde{\Phi}_b$ is an equivalence of categories we may consider U and W' as before and choose $W\in\mathscr{P}(\mathscr{B}|\mathscr{B}')$, an indecomposable $\mathscr{O}H$-module that covers W'. Proposition 4.2 assures that W has vertex in $\mathfrak{A}$, implying that it is a non-zero object of $\mathscr{P}$. Since we have $A'\in\mathfrak{A}'$, for any $g\in N_G(A')\setminus N_G(P)$ we get $A'=(A')^g\le P\cap P^g\in\mathfrak{X}'$, which is a contradiction. Now Theorem 3.2 applies, and we obtain that $\tilde{\Phi}_B^{-1}(W)$ is an indecomposable $\mathscr{O}GB$-module that covers U. □

Corollary 4.4. *The functor $\tilde{\Phi}_B$ induces a bijection between the set of isomorphism classes of indecomposable $\mathscr{O}GB$-modules that are non-zero objects in*

$$B\mathrm{mod}(G,\Delta)/B\mathrm{mod}(G,\mathfrak{X})$$

and cover some non-zero object in $b\mathrm{mod}(N,\Delta')/b\mathrm{mod}(N,\mathfrak{X}')$, and the set of isomorphism classes of indecomposable modules that are non-zero objects of $\mathscr{P}$ and cover some non-zero object of $\mathscr{P}'$.

Corollary 4.5. *The correspondence of Theorem 4.3 determines the subset of $\mathscr{B}$ containing blocks that cover the blocks of $\mathscr{B}'$.*

Proof. It is clear that any block in $\mathscr{B}$, containing a module that corresponds to a module in B that covers a covers a non-zero object lying in b, covers a block in $\mathscr{B}'$. So all we need to prove is that for any $\beta' \in \mathscr{B}'$ there exists a block $\beta \in \mathscr{B}$ that covers it. Let W' be an indecomposable $\mathscr{O}H'\beta'$-module with vertex $A' \in \mathfrak{A}'$. Then $U := \tilde{\Phi}_b^{-1}(W')$ is an $\mathscr{O}Nb$-module with vertex A'. Now [7, Lemma 0.5] affirms that there is an $\mathscr{O}GB$-module V such that $U \mid V_N$. Remark 4.3 forces U to lie in the restriction to N of an indecomposable component X of V with vertex in $\mathfrak{A}$. By Theorem 3.2, $\tilde{\Phi}_B(X)$ lies in a block $\beta \in \mathscr{B}$ that covers β'. □

The main result of Harris and Knörr [11] also follows from the above discussion applied in a certain particular case.

We assume that the block b of $\mathscr{O}N$ has defect group P. Fix an indecomposable object $U \in b\mathrm{mod}(N,\Delta')/b\mathrm{mod}(N,\mathfrak{X}')$, where the sets $\mathfrak{X}'$ and Δ' are determined by the p-subgroup P, such that U has vertex P. As above we set $H' := N_N(P)$ and we assume that b is G-invariant. Denote by β' the block of $\mathscr{O}H'$ that contains the Green correspondent W' of U. In this situation Theorem 3.2 and Remark 2.5 show that the block β' is the Brauer correspondent of b. In this setting we have:

Corollary 4.6 (Harris-Knörr)**.** *There is a one-to-one correspondence between the blocks of G that cover b and the blocks of $H := N_G(P)$ that cover β', moreover this correspondence coincides with the Brauer correspondence for blocks.*

Proof. Let $B \in \mathrm{Bl}(\mathscr{O}G)$ covering b. Denote by R a defect group of B with $R \cap N = P$ and define the sets $\mathfrak{X}$, $\mathfrak{A}$ and Δ with respect to R. Note that condition (4.3.1) is satisfied. According to [7, Lemma 0.5] we have an $\mathscr{O}GB$-module V such that $U \mid V_N$, and then there is an indecomposable direct summand V_1 of V such that $U \mid (V_1)_N$. By Mackey decomposition we can choose a vertex T of V_1 such that $P \leq T \cap N$ and $T^g \leq R$, for some $g \in G$. Since $R \cap N = P$ we get that V_1 covers U and by our assumptions V_1 is a non-zero object of $B\mathrm{mod}(G,\Delta)/B\mathrm{mod}(G,\mathfrak{X})$. Moreover Remark 2.5 forces the Green correspondent of V_1 to lie in the Brauer correspondent of B which, according to Theorem 3.2 and Theorem 4.3, covers β'.

Conversely, if $\beta \in \mathrm{Bl}(\mathscr{O}H)$ covers β', arguing in a similar way as above we may choose an indecomposable $\mathscr{O}H\beta$-module W that covers W'. We redefine the sets $\mathfrak{X}$, $\mathfrak{A}$ and Δ with respect to a defect group of β that contains P. Then Theorem 4.3 assures that the non-zero object W of $\mathscr{P}$ has its Green correspondent lying in a block of $\mathscr{O}G$ that covers b.

□

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