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GROUP GRADED HECKE INTERIOR ALGEBRAS

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Abstract. We prove that when we consider blocks of normal subgroups of finite groups G and G', the $\mathcal{O}G$-interior Hecke algebra introduced by L. Puig [4, Section 4] has a natural group graded structure, and its alternative descriptions yield isomorphisms of group graded $\mathcal{O}G$-interior algebras.

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1. INTRODUCTION

The study of Morita and derived equivalences between blocks of group algebras is a central topic in modular representation theory. If b is a block of a finite group algebra $\mathcal{O}G$, and b' is a block of $\mathcal{O}G'$, then a a bimodule M inducing a Morita equivalence between $b\mathcal{O}G$ and $b'\mathcal{O}G'$ is an indecomposable $\mathcal{O}(G \times G')$-module. It therefore makes sense to investigate the local structure of the Morita equivalence by taking a vertex $\ddot{P} \leq G \times G'$ and an $\mathcal{O}\ddot{P}$-source $\ddot{N}$ of M, and look at the $\mathcal{O}(G \times G')$-interior algebra $\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\ddot{P}}^{G\times G'}(\ddot{N}))$. This is done in great detail in the book [4] by L. Puig. One of the important technical tools there is a description, given in [4, Theorem 4.4], of the so-called Hecke $\mathcal{O}G$-interior algebra $\mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B})^{1\times G'}$, where $\ddot{H}$ is a subgroup of $G\times G'$ and $\ddot{B}$ is an $\mathcal{O}\ddot{H}$-interior algebra (a particular case is when $\ddot{H} = \ddot{P}$ and $\ddot{B} = \mathrm{End}_{\mathcal{O}}(\ddot{N})$).

On the other hand, for inductive reasons, it is useful to consider, as in [2], normal subgroups N of G and N' of G' such that $G/N \simeq G'/N' \simeq \Gamma$, and such that b and b' are invariant blocks of $\mathcal{O}N$ and $\mathcal{O}N'$ respectively, while M is a Γ-graded bimodule. In this case, a vertex $\ddot{P}$ of the identity component of M is a subgroup of a certain Γ-diagonal subgroup of $G \times G'$. We prove in this paper that the above Hecke $\mathcal{O}G$-interior algebra has a natural Γ-graded structure.

Our main result, Theorem 7.5 below, states that moreover, the isomorphism of [4, Theorem 4.4] is an isomorphism of Γ-graded $\mathcal{O}G$-interior algebras. The starting point in this analysis is the observation that the endomorphism algebra $\mathrm{End}_{\mathcal{O}}(M)$ has a structure of Γ-graded $\mathcal{O}(G \times G')$-interior algebras, and its various fixed subalgebras are also Γ-graded. This is done in the next four sections, while that last two sections are devoted to the study of the Γ-graded $\mathcal{O}G$-interior Hecke algebra.

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Throughout the paper $\mathcal{O}$ is a commutative ring, and all modules over this ring are considered to be $\mathcal{O}$-free. For general notions and results on G-algebras and group graded algebras we refer to [4], [5] and [3].

2. PRELIMINARIES

2.1. We fix the finite groups G, G' and Γ, and the group epimorphisms $\gamma : G \to \Gamma$ and $\gamma' : G' \to \Gamma$, and we set $N := \mathrm{Ker}(\gamma)$ and $N' := \mathrm{Ker}(\gamma')$.

The group Γ is organized as a left $G \times G'$-set as follows. For any $x \in \Gamma$ and $(g, g') \in G \times G'$ we define

$$(g, g') \cdot x := \gamma(g)x\gamma'(g')^{-1}.$$

Let

$$K := \{(g, g') \in G \times G' \mid \gamma(g) = \gamma'(g)\}$$

be the stabilizer of the identity element of Γ. Obviously, $(G \times G')/K$ is a transitive left $G \times G'$-set via

$$(g, g') \cdot ((h, h')K) = (gh, g'h')K.$$

The following statements are easy to check.

LEMMA 2.2. 1) *The map*

$$(G \times G')/K \to \Gamma, \qquad (g, g')K \mapsto \gamma(g(g_1)^{-1}),$$

where $\gamma(g_1) = \gamma'(g')$, *is an isomorphism of left* $(G \times G')$*-sets.*

2) *The map*

$$(G \times G')/K \to \Gamma, \qquad (g, g')K \mapsto \gamma'(g'(g_1')^{-1})^{-1}) = \gamma'(g_1'(g')^{-1}),$$

where $\gamma(g) = \gamma'(g_1')$, *is an isomorphism of left* $(G \times G')$*-sets.*

2.3. Similarly, we may view Γ as a right $G \times G'$-set, where $x \cdot (g, g') = \gamma(g)^{-1}x\gamma'(g')$, for all $x \in \Gamma$, $g \in G$, $g' \in G'$. Then the maps

$$(G \times G')/K \to \Gamma, \qquad K(g, g') \mapsto \gamma(g^{-1}g_1),$$

where $\gamma(g_1) = \gamma'(g')$, and

$$(G \times G')/K \to \Gamma, \qquad K(g, g') \mapsto \gamma'((g_1')^{-1}g'),$$

where $\gamma(g) = \gamma'(g_1')$, are isomorphisms of $G \times G'$-sets.

3. BIGRADED $\mathcal{O}(G \times G')$-INTERIOR ALGEBRAS AND THE ASSOCIATED Γ-GRADED ALGEBRAS

3.1. Let A be an $\mathcal{O}$-algebra which is both an $\mathcal{O}G$-interior and an $\mathcal{O}G'$-interior algebra, that is, we are given two $\mathcal{O}$-algebra homomorphisms

$$\varphi : \mathcal{O}G \to A \ \text{ and } \ \varphi' : \mathcal{O}G' \to A.$$

We denote

$$g \cdot a := \varphi(g)a, \quad a \cdot g = a\varphi(g), \quad g' \cdot a := \varphi'(g')a, \quad a \cdot g' = a\varphi'(g'),$$

for all $g \in G$, $g' \in G'$ and $a \in A$. This equivalent to say that A is an $\mathcal{O}(G\times G')$-interior algebra via the map $\varphi \otimes \varphi' : \mathcal{O}(G \times G') \to A$.

Note also that the maps γ and γ' of 2.1 induce obvious Γ-gradings on $\mathcal{O}G$ and $\mathcal{O}G'$.

Definition 3.2. We say that the $\mathcal{O}(G\times G')$-interior algebra A is Γ-*bigraded*, if there is a decomposition

$$A = \bigoplus_{x,y\in\Gamma} {}_xA_y,$$

as $\mathcal{O}$-modules, and the following axioms hold.

(1) For all ${}_xa_y \in {}_xA_y$, ${}_za_t \in {}_zA_t$ we have ${}_xa_y \cdot {}_za_t \in {}_xA_t$ if $y = z$, and the product is 0 otherwise;
(2) $g \cdot {}_xa_y \in {}_{(g,1)\cdot x}A_y$,
(3) $g' \cdot {}_xa_y \in {}_{(1,g')\cdot x}A_y$,
(4) ${}_xa_y \cdot g \in {}_xA_{y\cdot(1,g)}$,
(5) ${}_xa_y \cdot g' \in {}_xA_{y\cdot(1,g')}$,

for all $(g, g') \in G \times G'$, $x, y \in \Gamma$. We will also denote $A = {}_\Gamma A_\Gamma$.

Remark 3.3. It is clear that that the axioms (2) – (5) can be replaced by the equivalent conditions

(2′) $(g, g') \cdot {}_xa_y \in {}_{(g,g')\cdot x}A_y$,
(3′) ${}_xa_y \cdot (g, g') \in {}_xA_{y\cdot(g,g')}$,

for all $(g, g') \in G \times G'$, $x, y \in \Gamma$.

Definition 3.4. Let $A = {}_\Gamma A_\Gamma = \bigoplus_{x,y\in\Gamma} {}_xA_y$ be a Γ-bigraded $\mathcal{O}(G \times G')$-interior algebra. We associate to A two Γ-graded algebras denoted ${}_\Gamma A = \bigoplus_{y\in\Gamma} {}_yA$ and $A_\Gamma = \bigoplus_{y\in\Gamma} A_y$.

Let $g \in G$ and $g' \in G'$, let $\gamma(g) =: y$ and $\gamma'(g') =: z$, and denote

$${}_yA := \bigoplus_{x\in\Gamma} {}_{(g,1)\cdot x}A_x = \bigoplus_{x\in\Gamma} {}_{yx}A_x$$

and

$$A_z := \bigoplus_{x\in\Gamma} {}_{(1,g')\cdot x}A_x = \bigoplus_{x\in\Gamma} {}_{xz^{-1}}A_x = \bigoplus_{t\in\Gamma} {}_tA_{tz}.$$

The proof of the following statement is left to the reader.

Proposition 3.5. *With the above notations, the following hold:*

(1) $A = \bigoplus_{y\in\Gamma} {}_yA$, ${}_xA \cdot {}_yA \subseteq {}_{xy}A$ *for all* $x, y \in \Gamma$, *and* $\phi : \mathcal{O}G \to {}_\Gamma A$ *is a homomorphism of* Γ*-graded algebras.*
(2) $A = \bigoplus_{y\in\Gamma} A_y$, $A_x \cdot A_y \subseteq A_{xy}$ *for all* $x, y \in \Gamma$, *and* $\phi : \mathcal{O}G' \to A_\Gamma$ *is a homomorphism of* Γ*-graded algebras.*

4. ACTIONS AND FIXED SUBALGEBRAS

4.1. Let A be an $\mathcal{O}(G\times G')$-interior Γ-bigraded algebra. As usual, if $(g,g')\in G\times G'$ and $a\in A$ we denote the action of (g,g') on a by

$$a^{(g,g')}:=(g,g')^{-1}\cdot a\cdot (g,g').$$

Next, for any $g\in G$ and $g'\in G'$, we set $a^g:=a^{(g,1)}$ and $a^{g'}:=a^{(1,g')}$.

For any subgroup H of G, any subgroup H' of G', and any subset $\tilde{A}$ of A that is acted by H and H', let

$$\tilde{A}^H:=\tilde{A}^{H\times 1}=\{a\in\tilde{A}\mid a^h=a \text{ for all } h\in H\}$$

and

$$\tilde{A}^{H'}:=\tilde{A}^{1\times H'}=\{a\in\tilde{A}\mid a^{h'}=a \text{ for all } h'\in H'\}.$$

The following result lists the properties of this action.

Proposition 4.2. *Let $g\in G$, $g'\in G'$, $x,y\in\Gamma$ and set $\gamma(g)=t$, $\gamma'(g')=z$. Then, for any ${}_xa_y\in {}_xA_y$, ${}_ya\in {}_yA$ and $a_y\in A_y$,*

(1) ${}_xa_y^g\in {}_{t^{-1}x}A_{t^{-1}y}$;
(2) ${}_xa_y^{g'}\in {}_{xz}A_{yz}$;
(3) ${}_ya^{g'}\in {}_yA$;
(4) $a_y^g\in A_y$;
(5) ${}_ya^g\in {}_{y^t}A$;
(6) $a_y^{g'}\in A_{y^z}$;

Proof. The first two assertions are clear. Let ${}_ya=\sum_{x\in\Gamma}{}_{yx}a_x$ and $a_y=\sum_{x\in\Gamma}{}_xa_{xy}$. Then we have

$$\begin{aligned}
{}_ya^{g'}&=\sum_{x\in\Gamma}{}_{yxz}a_{xz}=\sum_{w\in\Gamma}{}_{yw}a_w,\\
a_y^g&=\sum_{x\in\Gamma}{}_{t^{-1}x}a_{t^{-1}xy}=\sum_{w\in\Gamma}{}_wa_{wy},\\
{}_ya^g&=\sum_{x\in\Gamma}{}_{t^{-1}yx}a_{t^{-1}x}=\sum_{x\in\Gamma}{}_{t^{-1}ytt^{-1}x}a_{t^{-1}x}=\sum_{w\in\Gamma}{}_{y^tw}a_w,\\
a_y^{g'}&=\sum_{x\in\Gamma}{}_{xz}a_{xyz}=\sum_{x\in\Gamma}{}_{xz}a_{xzz^{-1}yz}=\sum_{w\in\Gamma}{}_wa_{wy^z},
\end{aligned}$$

and the proof is complete. □

Corollary 4.3. *Let H be a subgroup of G and H' a subgroup of G'. Then,*

$$(A_\Gamma)^H=\bigoplus_{y\in\Gamma}(A_y)^H=\bigoplus_{y\in\Gamma}(A^H)_y$$

is a Γ-graded subalgebra of A_Γ, and

$$({}_\Gamma A)^{H'}=\bigoplus_{y\in\Gamma}({}_yA)^{H'}=\bigoplus_{y\in\Gamma}{}_y(A^{H'})$$

is a Γ*-graded subalgebra of* ${}_\Gamma A$.

Moreover, ϕ *and* ϕ' *induce homomorphisms* $\mathcal{O}G \to ({}_\Gamma A)^{H'}$ *and* $\mathcal{O}G' \to (A_\Gamma)^H$ *of* Γ*-graded algebras.*

5. THE ENDOMORPHISM ALGEBRA OF A Γ-GRADED $\mathcal{O}(G\times G')$-MODULE

This section consists of an example of the above discussion.

5.1. We fix a Γ-graded $\mathcal{O}$-module $M=\bigoplus_{x\in\Gamma} M_x$, and set

$$A := \mathrm{End}_{\mathcal{O}}(M).$$

We also assume that M is an $\mathcal{O}(G\times G')$-module satisfying

$$(g,g')\cdot m_x \in M_{(g,g')\cdot x},$$

for any pair $(g,g')\in G\times G'$ and any $x\in\Gamma$. In other words, M is a Γ-graded $\mathcal{O}G\otimes(\mathcal{O}G')^{\mathrm{op}}$-bimodule via

$$(g,g')\cdot m_x = g\cdot m_x\cdot (g')^{-1}.$$

Then it is well-known that A is an $\mathcal{O}(G\times G')$-interior algebra via

$$G\times G' \ni (g,g')\mapsto \varphi_{(g,g')}\in A,$$

where

$$\varphi_{(g,g')}(m)=(g,g')\cdot m$$

for all $m\in M$. Note that $A^G=\mathrm{End}_{\mathcal{O}(G\times 1)}(M)$ and $A^{G'}=\mathrm{End}_{\mathcal{O}(1\times G')}(M)$.

The $\mathcal{O}$-algebra A carries a Γ-bigraded structure $A=\bigoplus_{x,y\in\Gamma} {}_xA_y$, defined as follows.

Definition 5.2. a) For any $x,y\in\Gamma$, let ${}_yA_x$ be the $\mathcal{O}$-submodule of A consisting of endomorphisms that send M_x to M_y and everything else to zero.

b) For any $a\in A$ and $x,y\in\Gamma$, define the endomorphism ${}_ya_x\in A$ by

$${}_ya_x(m_z)=\begin{cases} a(m_z)_y & \text{if } x=z\\ 0 & \text{if } x\neq z,\end{cases}$$

for any $m_z\in M_z$ and any $z\in\Gamma$.

5.3. It is not difficult to verify that ${}_ya_x\in {}_yA_x$ for all $x,y\in\Gamma$, $a=\sum_{x,y\in\Gamma} {}_ya_x$, and that we have the direct sum decomposition into $\mathcal{O}$-submodules $A=\bigoplus_{x,y\in\Gamma} {}_xA_y$.

We have seen in Proposition 4.2 and Corollary 4.3 that we have two possibilities to define a Γ-grading on A. The next remarks should be compared with [1, Section 3]

5.4. Consider first ${}_\Gamma A=\bigoplus_{y\in\Gamma} {}_yA$, where for each $y\in\Gamma$ we have ${}_yA=\bigoplus_{x\in\Gamma} {}_{yx}A_x$. In our situation one easily verifies that

$${}_yA=\{f\in A \mid f(M_x)\subseteq M_{yx} \text{ for all } x\in\Gamma\}.$$

Since A is an $\mathcal{O}(G\times G')$-interior algebra, it is also an $\mathcal{O}G$-interior algebra via

$$G \ni g \mapsto \varphi_g \in A,$$

where $\varphi_g(m) = g\cdot m$, for any $m\in M$. This satisfies

$$\varphi_g(m_x) = g\cdot m_x \in M_{yx},$$

for any $x\in\Gamma$, where $\gamma(g)=y$. Further, any element of $\gamma(g)=y$ maps to an element of ${}_yA$. Finally, since $g\cdot m_x = (g,1)\cdot m_x$, we get

$$(\varphi_g(m_x))^{g'} = (g')^{-1}\cdot((g,(g'))\cdot m_x) = g\cdot m_x,$$

for any $x\in\Gamma$ and any $g'\in G'$.

5.5. The other case is when we take $A_\Gamma = \bigoplus_{y\in\Gamma} A_y$, where

$$A_y = \bigoplus_{x\in\Gamma} {}_xA_{xy} = \bigoplus_{t\in\Gamma} {}_{ty^{-1}}A_t..$$

Then we have

$$A_y = \{f\in A \mid f(M_x)\subseteq M_{xy^{-1}} \text{ for all } x\in\Gamma\}.$$

Now A_Γ is an $\mathcal{O}G'$-interior algebra via

$$G'\ni g'\mapsto \varphi'_{g'}\in A_\Gamma,$$

where $\varphi'_{g'}(m) = g'\cdot m = (1,g')\cdot m$, for any $m\in M$. Then $\varphi'_{g'}(m_x)\in M_{xy^{-1}}$, where $\gamma'(g')=y$, for any $x\in\Gamma$ and any $m_x\in M_x$.

COROLLARY 5.6. *With the notations of this section we have:*

(1) *The endomorphism algebra* $A := \mathrm{End}_{\mathcal{O}}(M)$ *is a* Γ*-bigraded* $\mathcal{O}(G\times G')$*-interior algebra.*
(2) *The map* $g\mapsto\varphi_g$ *is a* Γ*-graded algebra homomorphism from* $\mathcal{O}G$ *to* ${}_\Gamma A^{G'} = \mathrm{End}_{\mathcal{O}(1\times G')}(M)$.
(3) *The map* $g'\mapsto\varphi'_{g'}$ *is a* Γ*-graded algebra homomorphism from* $\mathcal{O}G'$ *to* $A_\Gamma^G = \mathrm{End}_{\mathcal{O}(G\times 1)}(M)$.

6. Γ-GRADED $\mathcal{O}G$-INTERIOR HECKE ALGEBRAS

This section is devoted to the graded structure of the so-called Hecke $\mathcal{O}G$-interior algebras that were introduced in [4, Chapter 4]. We briefly recall the construction made there.

6.1. Let $\ddot{H}$ be a subgroup of K, let $\ddot{B}$ an $\mathcal{O}\ddot{H}$-interior algebra, and consider, as in [5, Section 16], the induced algebra

$$\ddot{A} := \mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B}) = \mathcal{O}(G\times G')\otimes_{\mathcal{O}\ddot{H}}\ddot{B}\otimes_{\mathcal{O}\ddot{H}}\mathcal{O}(G\times G').$$

Recall that the multiplication is given by

$$(g,g')\otimes\ddot{b}\otimes(h,h')\cdot(x,x')\otimes\ddot{b}'\otimes(y,y')$$
$$=\begin{cases}(g,g')\otimes\ddot{b}\cdot(h,h')(x,x')\cdot\ddot{b}'\otimes(y,y'), & \text{if } (h,h')(x,x')\in\ddot{H}\\ 0, & \text{if } (h,h')(x,x')\notin\ddot{H}\end{cases}.$$

Since $\mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B})$ is an $\mathcal{O}(G\times G')$-interior algebra it is also $\mathcal{O}G$-interior. Actually, it is easily checked that $\mathcal{O}G$ maps to $\mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B})^{1\times G'}$. We denote $\hat{B}:=\mathrm{Ind}_{\ddot{H}}^{K}(\ddot{B})$, and we begin with some properties of $\ddot{A}$.

Definition 6.2. The algebra

$$\hat{A}:=\ddot{A}^{1\times G'}=\mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B})^{1\times G'}\simeq\mathrm{Ind}_{K}^{G\times G'}(\hat{B})^{1\times G'}$$

is called the *Hecke $\mathcal{O}G$-interior algebra* determined by G', $\ddot{H}$ and $\ddot{B}$.

Proposition 6.3. *The $\mathcal{O}(G\times G')$-interior algebra $\ddot{A}$ is Γ-bigraded.*

Proof. Note that $\ddot{A}\simeq\mathrm{Ind}_{K}^{G\times G'}(\hat{B})$ as $\mathcal{O}(G\times G')$-interior algebras. Then there is the decomposition $\ddot{A}=\bigoplus_{x,y\in\Gamma}{}_y\ddot{A}_x$, where for all $y,x\in\Gamma$ the $\mathcal{O}$-module ${}_y\ddot{A}_x$ consists of sums of elements of the form $(g,g')\otimes\hat{b}\otimes(h,h')$ such that, according to Lemma 2.2 and Remark 2.3, $\gamma(gg_1^{-1})=y$ and $\gamma((h_1)^{-1}h)=x^{-1}$. Clearly $\gamma(g_1)=\gamma'(g')$ and $\gamma(h_1)=\gamma'(h')$. □

Remark 6.4. There are three other ways to express the grading on $\ddot{A}$. The monomial $(g,g')\otimes\hat{b}\otimes(h,h')$ belongs to the submodule ${}_y\ddot{A}_x$ if one of the following equivalent statements hold:

(1) $\gamma(gg_1^{-1})=y$ and $\gamma'((h_1')^{-1}h')=x^{-1}$, where $\gamma(g_1)=\gamma'(g')$ and $\gamma(h)=\gamma'(h_1')$;
(2) $\gamma'(g'(g_1')^{-1})=y$ and $\gamma((h_1)^{-1}h)=x^{-1}$, where $\gamma(g)=\gamma'(g_1')$ and $\gamma(h_1)=\gamma'(h')$;
(3) $\gamma'(g'(g_1')^{-1})=y$ and $\gamma'((h_1')^{-1}h')=x^{-1}$, where $\gamma(g)=\gamma'(g_1')$ and $\gamma(h)=\gamma'(h_1')$.

Proposition 6.5. *The following isomorphisms of Γ-bigraded algebras hold:*

$$\ddot{A}\simeq\mathcal{O}G\otimes_{\mathcal{O}N}\hat{B}\otimes_{\mathcal{O}N}\mathcal{O}G \tag{1}$$
$$\simeq\mathcal{O}G'\otimes_{\mathcal{O}N'}\hat{B}\otimes_{\mathcal{O}N'}\mathcal{O}G' \tag{2}$$
$$\simeq\mathcal{O}G'\otimes_{\mathcal{O}N'}\hat{B}\otimes_{\mathcal{O}N}\mathcal{O}G \tag{3}$$
$$\simeq\mathcal{O}G\otimes_{\mathcal{O}N}\hat{B}\otimes_{\mathcal{O}N'}\mathcal{O}G' \tag{4}$$

Proof. Let the monomial $g\otimes\hat{b}\otimes h\in\mathcal{O}G\otimes_{\mathcal{O}N}\hat{B}\otimes_{\mathcal{O}N}\mathcal{O}G$ correspond to $(g,1)\otimes\hat{b}\otimes(h,1)\in\ddot{A}$. One easily checks that this is an isomorphism of $\mathcal{O}$-algebras, and moreover, by Proposition 6.3 and Remark 6.4 it is a Γ-bigraded algebra homomorphism. The other isomorphisms can be similarly verified. □

REMARK 6.6. The above Remark 6.4 guarantees the fact that the elements of $\ddot{A}$ can be written by using the representatives of the classes of Γ. This means that it suffices to choose elements of the form $(g,1)\otimes\hat{b}\otimes(h,1)$ or $(1,g')^{-1}\otimes\hat{b}'\otimes(1,h')^{-1}$, and if these two coincide, we get by Proposition 2.2 and Remark 2.3 that $\gamma(g)=\gamma'(g')=y$ and $\gamma(h)=\gamma'(h')=x$, so that this element lies in ${}_y\ddot{A}_x$.

One may check that, fixing $y\in\Gamma$, the element $(g,1)\otimes\hat{b}\otimes(h,1)$ lies in ${}_y\ddot{A}$ if and only if $\gamma(g)\gamma(h)=y$. Similarly, the same element belongs to A_y if and only if $\gamma(g)^{-1}\gamma(h)^{-1}=y$. At the same time, due to the above discussion, we also have other characterizations of these gradings which can be considered.

We may connect these results with the previous section, as it is well-known that induced algebras as above occur as endomorphism algebras over $\mathcal{O}$ of modules induced from subgroups.

6.7. Let $\ddot{M}$ be an $\mathcal{O}\ddot{H}$-module and denote $\hat{M}:=\mathrm{Ind}_{\ddot{H}}^{K}(\ddot{M})$ and $M:=\mathrm{Ind}_{H}^{G\times G'}(\hat{M})$. Also let $\hat{B}:=\mathrm{End}_{\mathcal{O}}(\hat{M})$, and let $\ddot{A}:=\mathrm{Ind}_{K}^{G\times G'}(\hat{B})$ as in 6.1.

COROLLARY 6.8. *With the above setting, there is an isomorphism of* Γ-*bigraded* $\mathcal{O}(G\times G')$-*interior algebras*

$$\ddot{A}\simeq\mathrm{End}_{\mathcal{O}}(M).$$

Furthermore, this isomorphism restricts to an isomorphism of Γ-*graded* $\mathcal{O}G$-*interior algebras*

$$\hat{A}\simeq\mathrm{End}_{\mathcal{O}G'}(M),$$

and to an isomorphism of Γ-*graded* $\mathcal{O}G'$-*interior algebras*

$$\ddot{A}^{G}\simeq\mathrm{End}_{\mathcal{O}G}(M).$$

Proof. We only need to prove the first statement since the second and the third one follow from this and Corollary 4.3. First note that by [3, Lemma 1.6.3], we have the isomorphisms

$$M\simeq\mathrm{Ind}_{N}^{G}(\hat{M})\simeq\mathrm{Ind}_{N'}^{G'}(\hat{M}).$$

Here we choose to work with representatives of G/N. The well-known isomorphism given by [5, Example 16.4], sends an element $g\otimes f\otimes h\in\ddot{A}$ to the endomorphism of M defined by

$$(g\otimes f\otimes h)(t\otimes m)=\begin{cases} g\otimes f(ht\cdot m), & \text{if } ht\in K, \text{ or equivalently } ht\in N,\\ 0, & \text{if } ht\notin N.\end{cases}$$

for any $t\in G$ and $z\otimes m\in M_{\gamma(t)}:=(t,1)\otimes\hat{M}$. We denote $\gamma(g)=y$, $\gamma(h^{-1})=x$ and $\gamma(t)=z$, and setting ${}_ya_x:=g\otimes f\otimes h$, Definition 5.2 and Definition 3.2 imply the statement. □

7. THE Γ-GRADED ISOMORPHISM OF HECKE $\mathcal{O}G$-INTERIOR ALGEBRAS

7.1. We return to the notations introduced at the beginning of Section 6. Further denote by $\rho : \ddot{H} \to G$ and $\rho' : \ddot{H} \to G'$ respectively, the restriction to $\ddot{H}$ of the projections $G \times G' \to G$ and $G \times G' \to G'$. We also restrict these two projections to K, respectively denoted by π_K and π'_K. If $i : \ddot{H} \to K$ is the inclusion, then $\rho = \pi_K \circ i$ and $\rho' = \pi'_K \circ i$. Recall, see [4, Section 3], that

$$\mathrm{Ind}_\rho(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G')) = \mathrm{Ind}^G_{\rho(\ddot{H})}((\mathcal{O} \otimes_{\mathcal{O}(1\times T')} (\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G')))^{1\times T'}),$$

where $1 \times T' := \mathrm{Ker}(\rho) = \ddot{H} \cap 1 \times G' \leq 1 \times N'$.

7.2. We rearrange the algebra $\mathrm{Ind}_\rho(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G'))$ as follows. First note that, according to [4, Proposition 3.16], we have

$$\mathrm{Ind}^K_{\ddot{H}}(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G')) \simeq \mathrm{Ind}^K_{\ddot{H}}(\ddot{B}) \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G').$$

Then [4, Corollary 3.13] shows that

$$\begin{aligned}\mathrm{Ind}_\rho(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G')) &\simeq \mathrm{Ind}_{\pi_K \circ i}(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G'))\\ &\simeq \mathrm{Ind}_{\pi_K}(\mathrm{Ind}^K_{\ddot{H}}((\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G'))))\\ &\simeq \mathrm{Ind}_{\pi_K}(\mathrm{Ind}^K_{\ddot{H}}(\ddot{B}) \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G'))\\ &\simeq \mathrm{Ind}_{\pi_K}(\hat{B} \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G')).\end{aligned}$$

Now π_K is an epimorphism with kernel $1 \times N'$, and by using [4, Section 3] again, we identify the algebra $\mathrm{Ind}_\rho(\ddot{B} \otimes_\mathcal{O} \mathrm{Res}_{\rho'}(\mathcal{O}G'))$ with

$$\hat{C} := (\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G')))^{1\times N'},$$

as $\mathcal{O}G$-interior algebras.

PROPOSITION 7.3. *$\hat{C}$ is a Γ-graded $\mathcal{O}G$-interior algebra.*

Proof. Note that $\hat{C}$ is indeed an $\mathcal{O}G$-interior algebra via the map

$$g \mapsto g \cdot 1_{\hat{C}} = 1_{\hat{C}} \cdot g = 1 \otimes (g, g') \cdot 1_{\hat{B}} \otimes g',$$

where $g' \in G'$ satisfies $\gamma(g) = \gamma(g')$. It is clear that the definition of this homomorphism does not depend on the choice of $g' \in G'$ since, by [4, 3.2], the action of $1 \times N'$ on $\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G'))$ coincides with the right multiplication. We have

$$\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathrm{Res}_{\pi'_K}(\mathcal{O}G')) = \bigoplus_{g' \in [G'/N']} \mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathcal{O}g'N'),$$

as $\mathcal{O}$-modules. Moreover, for any $g' \in G'/N'$ the $\mathcal{O}$-module $\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathcal{O}g'N')$ is $(1 \times N')$-invariant. Hence we can view $\hat{C} = \bigoplus_{y\in\Gamma} {}_y\hat{C}$, where for any $y \in \Gamma$ we have

$${}_y\hat{C} := (\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_\mathcal{O} \mathcal{O}g'N'))^{1\times N'},$$

and $\gamma'(g') = y$. Now let $g \in G$ such that $\gamma(g) = y$ and let $g' \in G'$ such that $\gamma'(g') = \gamma(g)$, then by the above remark g, which lies in the y-component of $\mathcal{O}G$, maps to ${}_y\hat{C}$. □

7.4. We are now ready to point out the graded structure of the isomorphism given in [4, Theorem 4.4]. In the above setting, we recall that this theorem says that there is an isomorphism of $\mathcal{O}G$-interior algebras

$$\mathrm{Ind}_{\ddot{H}}^{G\times G'}(\ddot{B})^{1\times G'} \simeq \mathrm{Ind}_{\rho}(\ddot{B} \otimes_{\mathcal{O}} \mathrm{Res}_{\rho'}(\mathcal{O}G')),$$

mapping $\mathrm{Tr}_{1\times T'}^{1\times G'}((1,g')^{-1} \otimes \ddot{b} \otimes (1,1))$ to $1 \otimes (1 \otimes (\ddot{b} \otimes g')) \otimes 1$.

Theorem 7.5. *With the above notations, 7.4 defines an isomorphism of* Γ*-graded* $\mathcal{O}G$*-interior algebras from* $\hat{A}$ *to* $\hat{C}$*.*

Proof. Recall that we have identified $\hat{A}$ with $\mathrm{Ind}_{K}^{G\times G'}(\hat{B})^{1\times G'}$ and $\hat{C}$ with

$$(\mathcal{O} \otimes_{\mathcal{O}(1\times N')} (\hat{B} \otimes_{\mathcal{O}} \mathrm{Res}_{\pi'_K}(\mathcal{O}G')))^{1\times N'},$$

where $\hat{B} = \mathrm{Ind}_{\ddot{H}}^{K}(\ddot{B})$. Recall also that Corollary 4.3, Proposition 6.3 and Proposition 7.3 show that $\hat{A}$ and $\hat{C}$ are Γ-graded algebras. Moreover, the isomorphism 7.4 from $\hat{A}$ to $\hat{C}$ maps

$$\hat{a} := \mathrm{Tr}_{1\times N'}^{1\times G'}((1,g')^{-1} \otimes \hat{b} \otimes (1,1))$$

to $\hat{c} := 1 \otimes (\hat{b} \otimes g')$, for any $g' \in G'$ and $\hat{b} \in \hat{B}$. All we need to prove is that this isomorphism preserves the gradings. If $y := \gamma'(g')$, Remark 6.6 shows that $\hat{a}$ belongs to the y-component, and so does $\hat{c}$. □

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