**JÓZSEF SÁNDOR**

# THEORY OF MEANS AND THEIR INEQUALITIES

This book is dedicated to the memory of my parents:

ILKA SÁNDOR (1928-2010)

JÓZSEF SÁNDOR (1911-1981)

# Contents

# Preface

The aim of this book is to present short notes or articles, as well as studies on some topics of the Theory of means and their inequalities. This is mainly a subfield of Mathematical analysis, but one can find here also applications in various other fields as Number theory, Numerical analysis, Trigonometry, Networks, Information theory, etc. The material is divided into six chapters: Classical means; Logarithmic, identric and related means; Integral inequalities and means; Means and their Ky Fan type inequalities; Stolarsky and Gini means; and Sequential means.

Chapter 1 deals essentially with the classical means, including the arithmetic, geometric, harmonic means of two or more numbers and the relations connecting them. One can find here more new proofs of the classical arithmetic mean-geometric mean-harmonic mean inequality, as well as their weighted version. The famous Sierpinski inequality connecting these means, or applications of the simple Bernoulli inequality offer strong refinements of these results, including the important Popoviciu and Rado type inequalities. One of the subjects is the log-convexity properties of the power means, which is applied so frequently in mathematics. The chapter contains also applications of some results for certain arithmetic functions, the theory of Euler's constant $e$, or electrical network theory.

The largest chapter of the book is Chapter 2 on the identric, logarithmic and related means. These special means play a fundamental role in the study of many general means, including certain exponential, integral means, or Stolarsky and Gini means, etc. There are considered various identities or inequalities involving these means, along with strong refinements of results known in the literature. Monotonicity, convexity and subhomogeneity properties are also studied. Series representations, and their applications are also considered. Applications for certain interesting logarithmic inequalities, having importance in other fields of mathematics, or in the entropy theory are also provided.

Chapter 3 contains many important integral inequalities, as the Cauchy-Bouniakowski integral inequality, the Hadamard (or Hermite-Hadamard) integral inequalities, the Jensen integral inequality, etc. Refinements, as well as generalizations or extensions of these inequalities are considered. As the inequalities offer in fact results for the integral mean of a function, many consequences or applications for special means are obtained. The classical monotonicity notion and its extension of section 1 give in a surprising manner interesting and nice results for means. The Jensen functionals considered in the last section, offer very general results with many applications.

Chapter 4 studies the famous Ky Fan type inequalities. As this inequality contains in a limiting case the arithmetic mean-geometric mean inequality, these results are connected with Chapter 1. However, here one obtains a more detailed and complicated study of these inequalities, involving many new means. Section 1 presents one of the author's early results, rediscovered later by other authors, namely the equivalence of Ky Fan's inequality with Henrici's inequality. Extensions, converses, refinements are also provided. The related Wang-Wang inequality is also considered.

Ky Fan type inequalities will be considered also in Chapter 5, on the Stolarsky and Gini means. These general means are studied extensively in sections 7 and 9 of this chapter. Particular cases, as the generalized logarithmic means, and a particular Gini mean, which is also a weighted geometric mean, are included, too. This last mean is strongly related, by an identity, with the identric mean of Chapter 2, and has many connections with other means, so plays a central role in many results.

Finally, Chapter 6 deals with means, called by the author as "sequential means", which may be viewed essentially as the common limit of certain recurrent sequences. These are the famous arithmetic-geometric mean of Gauss, the Schwab-Borchardt mean, etc. Here the classical logarithmic mean is considered also as a such mean, and many other famous particular means, as the Seiffert means, or the Neuman-Sándor mean, are studied. Two new means of the author are introduced in the last section. These means have today a growing literature, and their importance is recognized by specialists of the field.

All sections in all chapters are based on the author's original papers, published in various national and international journals. We have included a "Final references" section with the titles of the most important publications of the author in the theory of means. The book is concluded with an author index, focused on articles (and not pages). A citation of type I.2(3) shows that the respective author is cited three times in section 2 of chapter 1.

Finally, we wish to mention the importance of this domain of mathematics. Usually, researchers encounter in their studies the need of certain bounds, which essentially may be reduced to a relation between means. They need urgently some exact informations (on bounds, inequalities, approximations, etc.) on these means. So, such a work could be an ideal place and reference for their needs. On the other hand, beginning researchers, students, teachers of colleges or high schools would find a clear introduction and explanation of methods and results. The primary audience for this work are the mathematicians working in mathematical analysis and its applications. Since this is a very extensive field, with many subfields, researchers working in theoretical or applied domains would be interested, too. Also, since this work contains material with historical themes, teachers and their students would benefit from the informations and methods used in this book. This is a very active research field, and here one can find the basis for further study. The author thinks that the main strengths of the work are the new and interesting results, published for the first time in a book form.

I wish to express my gratitude to a number of persons and organizations from where I received valuable advice and support in the preparation of this book. These are Mathematics and Informatics Departments of Babeş-Bolyai University of Cluj (Romania); the Domus Hungarica Foundation of Budapest (Hungary); the Sapientia Foundation (Cluj); and also to Professors H. Alzer, B.A. Bhayo, M. Bencze, S.S. Dragomir, E. Egri, M.V. Jovanović, R. Klén, R.-G. Oláh, J.E. Pečarić, T. Pogány, M.S. Pop, E. Neuman, I. Raşa, M. Räissouli, H.-J. Seiffert, V.E.S. Szabó, Gh. Toader, T. Trif and W. Wang, who were coauthors along the years, and who had an impact on the activity and realizations of the author.

The author

# Chapter 1

# Classical means

"Mathematics is concerned only with the enumeration and comparison of relations."

(C.-F. Gauss)

"Thus each truth discovered was a rule available in the discovery of subsequent ones."

(R. Descartes)

## 1.1 On the arithmetic mean – geometric mean inequality for $n$ numbers

This is an English version of our paper [2].

In paper [1] we have obtained a simple method of proof for the arithmetic-geometric inequality for three numbers. This method gave also a refinement. In what follows, we shall generalize the method for $n$ numbers. In [1] we used the following simple lemma:

$$x^3 + y^3 \geq xy \cdot (x + y), \ \ x, y > 0. \tag{1}$$

This may be generalized as follows:

$$x^n + y^n \geq xy \cdot (x^{n-2} + y^{n-2}) = x^{n-1}y + y^{n-1}x, \text{ for any } n \geq 2. \tag{2}$$

Indeed, (2) may be rewritten also as

$$(x^{n-1} - y^{n-1})(x - y) \geq 0,$$

which is trivial for any $x, y > 0$, $n \geq 2$.

Let now $x_i$ $(i = \overline{1, n})$ be positive numbers, and apply inequality (2). Let first $x = x_1$ and $y$ successively $x_2, x_3, \ldots, x_n$; then let $x = x_2$ and $y = x_3, x_4, \ldots, x_n$; finally let $x = x_{n-1}$, $y = x_n$. By adding the obtained inequalities, we can write

$$(x_1^n + x_2^n) + (x_1^n + x_3^n) + \ldots + (x_1^n + x_n^n)$$

$$\geq (x_1 x_2^{n-1} + x_2 x_1^{n-1}) + \ldots + (x_1 x_n^{n-1} + x_n x_1^{n-1})$$

$$(x_2^n + x_3^n) + \ldots + (x_2^n + x_n^n) \geq (x_2 x_3^{n-1} + x_3 x_2^{n-1}) + \ldots + (x_2 x_n^{n-1} + x_n x_2^{n-1})$$

$$\ldots\ldots\ldots\ldots$$

$$x_{n-1}^n + x_n^n \geq x_{n-1} x_n^{n-1} + x_n x_{n-1}^{n-1}$$

Adding again the inequalities, finally we get:

$$(n-1)(x_1^n + x_2^n + \ldots + x_n^n) \geq \sum_{i=1}^{n} x_i (x_1^{n-1} + \ldots + \widehat{x}_i^{n-1} + \ldots + x_n^{n-1}), \tag{3}$$

where we have used for simplification,

$$x_1^{n-1} + \ldots + \widehat{x}_i^{n-1} + \ldots + x_n^{n-1}$$

a sum, where the $i$th term is missing (thus $x_i^{n-1}$).

Now let us suppose inductively that, the arithmetic-geometric inequality holds true for any positive real numbers of $n-1$ terms. Then we have

$$x_1^{n-1} + \ldots + \widehat{x}_i^{n-1} + \ldots + x_n^{n-1} \geq (n-1)(x_1 \ldots \widehat{x}_i \ldots x_n),$$

so we get by using (4):

$$
\begin{aligned}
(n-1)(x_1^n + x_2^n + \ldots + x_n^n) &\geq \sum_{i=1}^{n} x_i (x_1^{n-1} + \ldots + \widehat{x}_i^{n-1} + \ldots + x_n^{n-1}) \\
&\geq (n-1) n x_1 \ldots x_n \qquad (4)
\end{aligned}
$$

This is the generalization of inequality $(***)$ from [1]. From (4) we get particularly that

$$x_1^n + \ldots + x_n^n \geq n x_1 \ldots x_n,$$

i.e. for the numbers $x_1, \ldots, x_n$ the arithmetic-geometric inequality holds true. By the principle of mathematical induction, the arithmetic-geometric inequality is proved. By letting $\sqrt[n]{x_i}$ in place of $x_i$, from (4) we get the following:

**Theorem.**

$$
\begin{aligned}
x_1 + \ldots + x_n &\geq \frac{1}{n-1} \sum_{i=1}^{n} \sqrt[n]{x_i} \left( x_1^{\frac{n-1}{n}} + \ldots + \widehat{x}_i^{\frac{n-1}{n}} + \ldots + x_n^{\frac{n-1}{n}} \right) \\
&\geq n \sqrt[n]{x_1 \ldots x_n}.
\end{aligned}
$$

For $n = 3$ we reobtains the result from [1].

# Bibliography


1. J. Sándor, *On the arithmetic-geometric mean inequalities in case of three numbers*, Matlap 2016/10, p. 370 (in Hungarian).

2. J. Sándor, *On the arithmetic mean – geometric mean inequality in case of $n$ numbers*, Matlap 2017/p, p. 254.

## 1.2 A refinement of arithmetic mean – geometric mean inequality

There exist many proof of the famous arithmetic mean – geometric mean inequality

$$G_n = \sqrt[n]{a_1 \dots a_n} \leq A_n = \frac{a_1 + \cdots + a_n}{n}, \tag{1}$$

where $a_i \geq 0 (i = 1, 2, \dots, n)$, see [2]. For a proof, based on an identity involving Riemann's integral, see [1].

In what follows, we will offer a new approach, which gives in fact infinitely many refinements.

**Theorem 1.** *For any $x \in [0, 1]$ one has*

$$1 \leq \frac{1}{n} \cdot \sum_{i=1}^{n} \left( \frac{a_i}{G_n} \right)^x \leq \frac{A_n}{G_n} \tag{2}$$

**Proof.** The function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sum_{i=1}^{n} \left( \frac{a_i}{G_n} \right)^x$ has the following property:

$$f''(x) = \sum_{i=1}^{n} \left( \frac{a_i}{G_n} \right)^x \cdot \log^2 \frac{a_i}{G_n} \geq 0.$$

This implies that

$$f'(x) = \sum_{i=1}^{n} \left( \frac{a_i}{G_n} \right)^x \cdot \log \frac{a_i}{G_n}$$

is an increasing function, yielding

$$f'(x) \geq f'(0) = \sum_{i=1}^{n} \log \frac{a_i}{G_n} = 0.$$

This in turn implies that $f(x)$ is increasing, so $f(0) \leq f(x) \leq f(1)$ for $x \in [0, 1]$, giving inequality (2).

Since $f''(x) = 0$ iff $\dfrac{a_1}{G_n} = \cdots = \dfrac{a_n}{G_n} = 1$, there is equality in each side of (2) only if $a_1 = \ldots = a_n$. $\square$

**Remark 1.** For $x = \dfrac{1}{2}$, from (2) we get the following simple refinement of (1):

$$1 \leq \frac{\sqrt{a_1} + \cdots + \sqrt{a_n}}{n\sqrt{G_n}} \leq \frac{A_n}{G_n} \tag{3}$$

## An application

Let $d_1, \ldots, d_n$ denote all distinct positive divisors of a positive integer $m \geq 1$. Let $a_1 = d_1^{2k}, \ldots, a_n = d_n^{2k}$ ($k$ fixed real number). As

$$\frac{m}{d_1} \cdot \ldots \cdot \frac{m}{d_n} = d_1 \cdot \ldots \cdot d_n,$$

we get $d_1 \cdot \ldots \cdot d_n = m^{\frac{n}{2}}$, so in our case we get $G_n = G_n(a_i) = m^k$. Let $\sigma_s(m)$ denote the sum of $s$th powers of divisors of $m$. From (3) we get immediately (with $\sigma(m) = \sigma_1(m)$ and $d(m) = \sigma_0(m)$)

$$1 \leq \frac{\sigma_k(m)}{d(m) \cdot m^{\frac{k}{2}}} \leq \frac{\sigma_{2k}(m)}{d(m) \cdot m^k} \tag{4}$$

We note that for $k \geq 1$, the left side of (4) was discovered by R. Sivaranakrishnan and C.S. Venkataraman [3]. The second inequality of (4) may be rewritten as

$$\sigma_{2k}(m) \leq m^{\frac{k}{2}} \cdot \sigma_k(m) \tag{5}$$

We note that in the left side of (4), as well as in (5), it is sufficient to consider $k > 0$. Indeed, if $k < 0$, put $k = -K$. Remarking that

$$\sigma_{-K}(m) = \frac{\sigma_K(m)}{m^K},$$

it is immediate that we obtain the same inequalities for $K$ as for $k$ in both of (4) and (5).

**Remark 2.** The weighted arithmetic mean – geometric mean inequality

$$A_{\alpha,n} = \alpha_1 a_1 + \cdots + \alpha_n a_n \geq a_1^{\alpha_1} \cdots a_n^{\alpha_n} = G_{\alpha,n}$$

(with $a_i \geq 0, \alpha_i \in [0,1]$, $\alpha_1 + \cdots + \alpha_n = 1$) can be proved in the similar manner, by considering the application

$$f_\alpha(x) = \sum_{i=1}^{n} \alpha_i \cdot \left(\frac{a_i}{G_{\alpha,n}}\right)^x$$

As

$$f'_\alpha(x) = \sum_{i=1}^{n} \alpha_i \left(\frac{a_i}{G_{\alpha,n}}\right)^x \cdot \log \frac{a_i}{G_{\alpha,n}}$$

and $f'_\alpha(0) = 0$, all can be repeated, and we get the inequality (for any $x \in [0,1]$)

$$1 \leq \sum_{i=1}^{n} \alpha_i \left(\frac{a_i}{G_{\alpha,n}}\right)^x \leq \frac{A_{\alpha,n}}{G_{\alpha,n}}, \tag{6}$$

which is an extension of (2), for $x = \frac{1}{2}$, we get an extension of (3):

$$1 \leq \frac{\alpha_1\sqrt{a_1} + \cdots + \alpha_n\sqrt{a_n}}{\sqrt{G_{\alpha,n}}} \leq \frac{A_{\alpha,n}}{G_{\alpha,n}} \tag{7}$$

# Bibliography


1. H. Alzer, *A proof of the arithmetic mean – geometric mean inequality*, Amer. Math. Monthly, **103**(1996), 585.

2. P.S. Bullen, *Handbook of means and their inequalities*, Kluwer Acad. Publ., 2003.

3. R. Sivaramakrishnan, C.S. Venkataraman, *Problem 5326*, Amer. Math. Monthly, **72**(1965), 915.

## 1.3 On Bernoulli's inequality

### 1

The famous Bernoulli inequality states that for each real number $x \geq -1$ and for a natural number $n \geq 1$ one has

$$(1+x)^n \geq 1+nx. \tag{1}$$

This simple relation has surprisingly many applications in different branches of Mathematics. In this Note we will obtain a new application with interesting consequences. For example, the well-known arithmetic-geometric inequality follows.

### 2

Let us write (1) firstly in the form (by using the substitution $y = x+1$)

$$y^n - 1 \geq n(y-1),\ y \geq 0. \tag{2}$$

For the sake of completeness, we shall give also the short proof of (2). By the algebraic identity

$$y^n - 1 = (y-1)(y^{n-1} + \ldots + y + 1),$$

the new form of (2) will be

$$(y-1)[(y^{n-1}-1) + (y^{n-2}-1) + \ldots + (y-1)] \geq 0. \tag{3}$$

The terms in the right parenthesis have the same sign as $y-1$ in all cases (i.e. if $y \geq 1$, or $0 \leq y-1$), thus yielding simply relation (3). One has equality only for $n=1$ or $y=1$.

**Remark.** Inequality (2) (or (1)) is valid also for all real numbers $n \geq 1$, but the proof in that case is not so simple as above. See e.g. [1].

## 3

Let us apply now (2) for

$$y = \left(\frac{u}{v}\right)^{\frac{1}{n-1}}, \text{ where } n \geq 2. \tag{4}$$

After some simple computations we get the following result:

**Theorem.** Let $u, v > 0$ be real numbers, and $n \geq 2$ a positive integer. Then one has the inequality

$$\left(\frac{nv - u}{n-1}\right)^{n-1} \leq \frac{v^n}{u}. \tag{5}$$

**Applications.** 1) As a first application, put

$$v = \frac{1}{n}V, \ u = U, \text{ where } U, V > 0.$$

One gets:

$$\left(\frac{V - U}{n-1}\right)^{n-1} \leq \frac{V^n}{U} \cdot \frac{1}{n^n}. \tag{6}$$

For $V = u + 1$ this yields that

$$\frac{(U+1)^n}{U} \geq \frac{n^n}{(n-1)^{n-1}}. \tag{7}$$

Inequality (7) can be obtained also by studying the function

$$U \to \frac{(U+1)^n}{U}$$

(by using derivatives), but it is more interesting in this case the simple way of obtaining this result. For $U = 1$, $V = n + 1$, relation (6) implies

$$\left(1 + \frac{1}{n-1}\right)^{n-1} \leq \left(1 + \frac{1}{n}\right)^n. \tag{8}$$

Thus the monotonicity of the Euler sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n.$$

(For $n > 1$ one has in fact strict inequality, see the proof of (2)).

2) Let now $a_i$ $(i = \overline{1,n})$ be positive real numbers, and

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}, \ G_n = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$$

their arithmetic, respectively geometric means. Apply (5) for $v = A_n$, $n = a_n$. Then clearly

$$nv - u = a_1 + \ldots + a_{n-1},$$

so one has

$$A_{n-1}^{n-1} \leq \frac{A_n^n}{a_n}. \tag{9}$$

Since $a_n = \dfrac{G_n^n}{G_{n-1}^{n-1}}$, (9) is equivalent to

$$\left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \leq \left(\frac{A_n}{G_n}\right)^n, \ n \geq 2. \tag{10}$$

As a corollary,

$$\left(\frac{A_1}{G_1}\right)^1 \leq \left(\frac{A_2}{G_2}\right)^2 \leq \ldots \leq \left(\frac{A_r}{G_r}\right)^r, \text{ for all } r \geq 1. \tag{11}$$

Here $\dfrac{A_1}{G_1} = 1$, so (11) gives, as a simple consequence that

$$A_r \geq G_r \tag{12}$$

i.e. the well-known arithmetic-geometric inequality.

For generalizations and other applications of Bernoulli's inequality we quote the monograph [1].

# Bibliography


1. D.S. Mitrinović (in coop. with P.M. Vasić), *Analytic Inequalities*, Springer Verlag Berlin, Heidelberg, New York, 1970.

## 1.4 A refinement of the harmonic – geometric inequality

Let $x = (x_1, \ldots, x_n)$, where $x_i > 0$, and put

$$A = A(x) = \frac{x_1 + \ldots + x_n}{n}, \quad G = G(x) = \sqrt{x_1 \ldots x_n},$$

$$H = H(x) = \frac{n}{\dfrac{1}{x_1} + \ldots + \dfrac{1}{x_n}}$$

for the arithmetic, geometric and harmonic means of $x_i$, $i = \overline{1,n}$. Put

$$M(x) = \frac{G(x) \sqrt[n]{A(x)}}{A\left(\sqrt[n]{x}\right)} \tag{1}$$

where $\sqrt[n]{x} = \left(\sqrt[n]{x_1}, \ldots, \sqrt[n]{x_n}\right)$. Since

$$G\left(\frac{1}{x}\right) = \frac{1}{G(x)}, \quad A\left(\frac{1}{x}\right) = \frac{1}{H(x)}, \text{ for } \frac{1}{x} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right),$$

clearly the inequality

$$G \leq M(x) \leq A, \tag{2}$$

is equivalent to

$$H \leq N(x) \leq G, \tag{3}$$

where

$$N(x) = \frac{G(x) \sqrt[n]{H(x)}}{H\left(\sqrt[n]{x}\right)}. \tag{4}$$

We will prove that (3) holds true (i.e. (2), too), even with a chain of improvements.

**Theorem.** *One has the inequalities*

$$H \leq N_2(x) \leq N_1(x) \leq N(x) \leq G \tag{5}$$

*where*

$$N_1(x) = \frac{H \sqrt[n]{A}}{H\left(\sqrt[n]{x}\right)}, \quad N_2(x) = \frac{H \cdot A\left(\sqrt[n]{x}\right)}{H\left(\sqrt[n]{x}\right)},$$

*with $H = H(x)$ etc., and $N(x)$ given by (4).*

**Proof.** We will apply the famous Sierpinski inequality (see [1], with a generalization), which can be written as follows:

$$H^{n-1}A \le G^n \le A^{n-1}H. \tag{6}$$

From the left side of (6) we get

$$\frac{H^n A}{H} \le G^n, \quad \text{i.e.} \quad H\sqrt[n]{A} \le G\sqrt[n]{H},$$

thus

$$\frac{H\sqrt[n]{H}}{H} \ge \sqrt[n]{A}. \tag{$*$}$$

Now, the inequality $\sqrt[n]{A(x)} \ge A(\sqrt[n]{x})$ is valid, being equivalent to

$$\sqrt[n]{\frac{1}{n}\sum_{k=1}^{n} x_k} \ge \frac{1}{n}\sum_{k=1}^{n} \sqrt[n]{x_k},$$

and this follows by Jensen's classical inequality, for the concave function

$$f(x) = \sqrt[n]{x}, \quad (x > 0,\ n \ge 1).$$

Now

$$A\left(\sqrt[n]{x}\right) \ge H\left(\sqrt[n]{x}\right),$$

so by ($*$), we get the relation

$$\frac{G\sqrt[n]{H}}{H} \ge \sqrt[n]{A} \ge A\left(\sqrt[n]{x}\right) \ge H\left(\sqrt[n]{x}\right) \tag{$**$}$$

Thus, we get

$$\frac{G\sqrt[n]{H}}{H\left(\sqrt[n]{x}\right)} \ge H,$$

but by ($**$), even two refinements of this inequalities, can be deduced. The right side of (2) is

$$\sqrt[n]{H(x)} \le H\left(\sqrt[n]{x}\right),$$

which with $x \to \dfrac{1}{x}$ becomes in fact

$$\sqrt[n]{A(x)} \geq A\left(\sqrt[n]{x}\right),$$

and this is true, as we have pointed out before. Thus the theorem follows.

# Bibliography


1. J. Sándor, *On inequality of Sierpinski*, Octogon Mathematical Magazine, Vol. 3, No. 1, April 1995, 21-22.

## 1.5 On certain conjectures on classical means

### 1

Let

$$A_n = A_n(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} x_i}{n},$$

$$G_n = G_n(x_1, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i},$$

$$H_n = H_n(x_1, \ldots, x_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}},$$

$$Q_n = Q_n(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

denote the classical means of positive real numbers $x_i > 0$, $i = \overline{1, n}$.

In [1], as well as in [2], the following conjectures are stated:

$$A_n \geq \frac{Q_n + G_n}{2} \tag{1}$$

(see Conjecture 1 of [1], and OQ. 1919 of [2]).

Another conjecture is (see Conjecture 4 of [1]):

$$G_n \leq \frac{aA_n + bH_n}{a + b} \tag{2}$$

for certain $a, b > 0$. Our aim in what follows is to show, that (1) is not true for all $n \geq 3$, and that (2) holds true e.g. with $a = \frac{n-1}{n}$, $b = \frac{1}{n}$.

## 2

Though, relation (1) is true for $n = 2$, we can show that even weaker inequality (by $\frac{Q_n + G_n}{2} \geq \sqrt{Q_n G_n}$)

$$A_n \geq \sqrt{Q_n G_n} \tag{3}$$

is not generally true for $n \geq 3$. For this purpose, select

$$x_1 = x_2 = \ldots = x_{n-1} = 1 \quad \text{and} \quad x_n = n - 1.$$

Then

$$A_n = \frac{2(n-1)}{n}, \quad G_n = \sqrt[n]{n-1}, \quad Q_n = \sqrt{\frac{n-1+(n-1)^2}{n}},$$

so (3) becomes in this case

$$\frac{4(n-1)^2}{n^2} \geq (n-1)^{\frac{1}{n}}(n-1)^{\frac{1}{2}} = (n-1)^{\frac{1}{n}+\frac{1}{2}}. \tag{4}$$

For $n = 2$, there is equality in (4). By supposing however, $n \geq 3$, then

$$4(n-1)^{3n-2} \geq n^{4n} = n^{4n-2} \cdot n^2$$

is not possible, as $n^2 > 4$ and $n^{4n-2} > (n-1)^{3n-2}$.

**Remark.** If (1) is not true, one may think that a weaker inequality

$$A_n \geq \frac{Q_n + H_n}{2} \tag{5}$$

is valid. And indeed, by quite complicated computations (e.g. by using computer algebra), Čerin, Gianella and Starc [3] have shown, that (5) holds for all $n \leq 4$. However, a counterexample shows that it is false for $n \geq 5$. In [3] are shown also the following facts:

$$G_n \leq \frac{H_n + Q_n}{2} \tag{6}$$

is true for all $n \leq 4$; false for $n \geq 5$;

$$G_n \leq \sqrt{H_n Q_n} \tag{7}$$

is true for $n \leq 2$; false for $n \geq 3$;

$$G_n \geq \frac{2}{\dfrac{1}{H_n} + \dfrac{1}{A_n}} \tag{8}$$

true for $n \leq 2$; false for $n \geq 3$;

$$A_n \geq \sqrt{\frac{H_n^2 + Q_n^2}{2}} \tag{9}$$

true for $n \leq 2$; false for $n \geq 3$;

$$G_n \leq \sqrt{\frac{H_n^2 + Q_n^2}{2}} \tag{10}$$

true for $n \leq 4$; false for $n \geq 8$.

They conjecture that (10) is valid also for $5 \leq n \leq 7$.

## 3

We now settle, in the affirmative, relation (2). The famous Sierpinski inequality (see e.g. [4] for a generalization) states that

$$H_n^{n-1} A_n \leq G_n^n \leq A_n^{n-1} H_n. \tag{11}$$

Now, the right side of (11) (which is equivalent, by a simple transformation, with the left side of (11)) implies

$$G_n \leq A_n^{\frac{n-1}{n}} H_n^{\frac{1}{n}} = A_n^\alpha H_n^\beta \leq \alpha A_n + \beta H_n, \text{ for } \alpha = \frac{n-1}{n}, \ \beta = \frac{1}{n},$$

by the classical Young inequality

$$x^\alpha y^\beta \leq \alpha x + \beta y \ (x, y > 0; \ \alpha, \beta > 0; \ \alpha + \beta = 1). \tag{12}$$

Therefore, we have proved (9), in the following refined form:

$$G_n \leq A_n^{\frac{n-1}{n}} H_n^{\frac{1}{n}} \leq \frac{(n-1)A_n + H_n}{n}. \tag{13}$$

The weaker form of inequality (13) for $n = 2$ and $n = 3$ is proved also in [1].

## 4

We want to point out now, weaker versions of (6)-(10), which are always true. First remark that

$$H_n \leq G_n \leq A_n \leq Q_n, \text{ for all } n \geq 2. \tag{14}$$

Now, by the first inequality of (13), and the last one of (14) one can write

$$H_n \leq Q_n^{\frac{n-1}{n}} H_n^{\frac{1}{n}} \leq \frac{(n-1)Q_n + H_n}{n},$$

by an application of (12). Thus, at another hand,

$$G_n \leq Q_n^{\frac{n-1}{n}} H_n^{\frac{1}{n}}, \ \forall\ n \geq 2 \tag{15}$$

which coincides for $n = 2$ with (7), but unlike (7), (15) is always true; and at another hand,

$$G_n \leq \frac{(n-1)Q_n + H_n}{n}, \tag{16}$$

which for $n = 2$ coincides with (6), and is true for all $n$. By (16), and the classical inequality

$$(x_1 + x_2 + \ldots + x_n)^2 \leq n(x_1^2 + \ldots + x_n^2),$$

applied the $x_1 = \ldots = x_{n-1} = Q_n$, $x_n = H_n$ one has

$$((n-1)Q_n + H_n)^2 \leq n((n-1)Q_n^2 + H_n^2),$$

so an extension of (10) has been found:

$$G_n \leq \sqrt{\frac{(n-1)Q_n^2 + H_n^2}{n}}. \tag{17}$$

Unlike (10), this is valid for all $n \geq 2$.

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## 1.6 On weighted arithmetic-geometric inequality

Let $x_i > 0$, $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$ $(i = 1, 2, \ldots, n,\ n \in \mathbb{N})$. Since the function $f(x) = \log(1+e^x)$ is convex for all $x > 0$, by Jensen's inequality

$$f(p_1a_1 + \ldots + p_na_n) \leq p_1f(a_1) + \ldots + p_nf(a_n),\ a_i \in \mathbb{R},$$

we get

$$\log(1 + e^{p_1a_1+\ldots+p_na_n}) \leq \log(1 + e^{a_1})^{p_1} \ldots (1 + e^{a_n})^{p_n}.$$

By letting $e^{a_i} = x_i > 0$, we obtain the following inequality:

$$1 + x_1^{p_1} \ldots x_n^{p_n} \leq (1 + x_1)^{p_1} \ldots (1 + x_n)^{p_n}. \tag{1}$$

This is in fact, an extended Chrystal inequality (see [1]). Now, let

$$g(x) = \log\log(1 + e^x).$$

It is immediate that $g$ is concave, which after some computations is equivalent to

$$\log(1 + e^x) < e^x.$$

Now, applying the some procedure as above, we can deduce the following relation:

$$\log(1 + x_1^{p_1} \ldots x_n^{p_n}) \geq \log(1 + x_1)^{p_1} \ldots \log(1 + x_n)^{p_n} \tag{2}$$

Now, by the weighted arithmetic-geometric mean inequality, it can be written that:

$$\begin{aligned}(1 + x_1)^{p_1} \ldots (1 + x_n)^{p_n} &\leq p_1(1 + x_1) + \ldots + p_n(1 + x_n)\\ &= 1 + p_1x_1 + \ldots + p_nx_n. \end{aligned} \tag{3}$$

By taking into account of (1), (2), (3), the following chain of inequalities holds true:

**Theorem.** $e^{\log(1+x_1)^{p_1}\dots\log(1+x_n)^{p_n}} - 1 \leq x_1^{p_1}\dots x_n^{p_n}$

$$\leq (1+x_1)^{p_1}\dots(1+x_n)^{p_n} - 1 \tag{4}$$

For $p_1 = p_2 = \ldots = p_n = \dfrac{1}{n}$, from (4) we get:

$$e^{\sqrt[n]{\log(1+x_1)\dots\log(1+x_n)}} - 1 \leq \sqrt[n]{x_1\dots x_n}$$

$$\leq \sqrt[n]{(1+x_1)\dots(1+x_n)} - 1 \leq \frac{x_1+x_2+\ldots+x_n}{n}, \tag{5}$$

which among other contains a refinement of the classical arithmetic-geometric inequality.

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# 1.7 A note on the inequality of means

## 1. Introduction

Let $\alpha_1, \ldots, \alpha_n > 0$, $\alpha_1 + \ldots + \alpha_n = 1$ and $x_1, \ldots, x_n > 0$. The weighted arithmetic mean-geometric mean inequality states that

$$G_\alpha = G(x, \alpha) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \leq \alpha_1 x_1 + \ldots + \alpha_n x_n = A(x, \alpha) = A_\alpha. \quad (1)$$

This is one of the most important inequalities, with applications and connections to many fields of Mathematics.

When $(\alpha) = (\alpha_1, \ldots, \alpha_n) = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, we get the classical inequality of means

$$G = \sqrt[n]{x_1 \ldots x_n} \leq \frac{x_1 + \ldots + x_n}{n} = A \quad (2)$$

In 1998 [2] the author introduced the so-called "tangential mean"

$$T(x) = \sqrt[n]{(1 + x_1) \ldots (1 + x_n)} - 1$$

and proved that

$$G \leq T \leq A \quad (3)$$

In fact, the method of proof immediately gives the more general inequality

$$G_\alpha \leq T_\alpha \leq A_\alpha, \quad (4)$$

where

$$T_\alpha = T_\alpha(x) = (1 + x_1)^{\alpha_1} \ldots (1 + x_n)^{\alpha_n} - 1 \quad (5)$$

An application of (3) to Wilker and Huygens type inequalities has been provided in our paper [3].

Recently, Y. Nakasuji and S.-E. Takahasi [1] have rediscovered relation (4), by an application of Jensen type inequalities in topological semigroups.

Let $a > 0$ be an arbitrary positive real number. We introduce a generalization of $T_\alpha$ of (5) by

$$T_\alpha^a = T_\alpha^a(x) = (a + x_1)^{\alpha_1} \dots (a + x_n)^{\alpha_n} - a \tag{6}$$

The aim of this note is to offer generalizations and refinements of inequality (4).

## 2. Main results

The following auxiliary results will be used:

**Lemma 1.** *If* $x_1, \dots, x_n > 0$ *and* $y_1, \dots, y_n > 0$. *Then*

$$(x_1 + y_1)^{\alpha_1} \dots (x_n + y_n)^{\alpha_n} \geq x_1^{\alpha_1} \dots x_n^{\alpha_n} + y_1^{\alpha_1} \dots y_n^{\alpha_n}, \tag{7}$$

*where* $(\alpha)$ *are as in the Introduction.*

**Lemma 2.** *The application* $f : (0, \infty) \to \mathbb{R}$ *given by* $f(a) = T_\alpha^a$ *is increasing.*

## 3. Proofs

**Lemma 1.** By inequality (1) applied to

$$x_1 := \frac{x_1}{x_1 + y_1}, \dots, x_n := \frac{x_n}{x_n + y_n}$$

we get

$$\left(\frac{x_1}{x_1 + y_1}\right)^{\alpha_1} \dots \left(\frac{x_n}{x_n + y_n}\right)^{\alpha_n} \leq \alpha_1 \cdot \frac{x_1}{x_1 + y_1} + \dots + \alpha_n \cdot \frac{x_n}{x_n + y_n} \tag{8}$$

Apply (1) in the same manner to $x_1 := \dfrac{y_1}{x_1 + y_1}, \dots, x_n := \dfrac{y_n}{x_n + y_n}$. We get

$$\left(\frac{x_1}{x_1 + y_1}\right)^{\alpha_1} \dots \left(\frac{y_n}{x_n + y_n}\right)^{\alpha_n} \leq \alpha_1 \cdot \frac{y_1}{x_1 + y_1} + \dots + \alpha_n \cdot \frac{y_n}{x_n + y_n} \tag{9}$$

Now, by a simple addition of (8) and (9), one gets (7).

**Lemma 2.** By computing the derivative $f'(a)$ one has

$$f'(a) = \alpha_1(a+x_1)^{\alpha_1-1}(a+x_2)^{\alpha_2}\dots(a+x_n)^{\alpha_n}$$

$$+\alpha_2(a+x_1)^{\alpha_1}(a+x_2)^{\alpha_2-1}\dots(a+x_n)^{\alpha_n}+\dots+$$

$$+\alpha_n(a+x_1)^{\alpha_1}\dots(a+x_{n-1})^{\alpha_{n-1}}(a+x_n)^{\alpha_n-1}-1.$$

Now applying inequality (1) to

$$x_1{:=}(a{+}x_1)^{\alpha_1-1}(a{+}x_2)^{\alpha_2}\dots(a{+}x_n)^{\alpha_n},\dots,x_n{:=}(a{+}x_1)^{\alpha_1}\dots(a{+}x_n)^{\alpha_n-1}$$

we get that

$$f'(a)\geq(x_1+a)^{\alpha_1(\alpha_1-1+\alpha_2+\dots+\alpha_n)}\dots(x_n+a)^{(\alpha_1+\alpha_2+\dots+\alpha_n-1)}-1=0$$

as $\alpha_1+\alpha_2+\dots+\alpha_n=1$.

The main result of this note is:

**Theorem.** *One has:*

$$G_\alpha\leq T_\alpha^a\leq A_\alpha,\ \textit{for}\ a>0 \tag{10}$$

$$G_\alpha\leq T_\alpha^a\leq T_\alpha^b\leq A_\alpha,\ \textit{when}\ 0<a\leq b \tag{11}$$

$$G_\alpha\leq T_\alpha^b\leq T_\alpha^a\leq A_\alpha,\ \textit{when}\ a\geq b. \tag{12}$$

**Proof.** By applying inequality (1) to $x_1:=x_1+a,\dots,x_n:=x_n+a$, we get

$$(x_1{+}a)^{\alpha_1}\dots(x_n{+}a)^{\alpha_n}\leq\alpha_1(x_1{+}a)\dots{+}\alpha_n(x_n{+}a)=\alpha_1x_1{+}\dots{+}\alpha_nx_n{+}a,$$

so the right side of (10) follows.

Applying now Lemma 1 for $y_1=\dots=y_n=a$, we get

$$(x_1+a)^{\alpha_1}\dots(x_n+a)^{\alpha_n}\geq x_1^{\alpha_1}\dots x_n^{\alpha_n}+a,$$

so the left side of (10) follows as well.

Relations (11) and (12) are consequences of Lemma 2, by remarking that $f(a) \leq f(1)$ for $a \leq 1$ and $f(a) \geq f(1)$ for $a \geq 1$.

**Remarks.** 1) Particularly, when $b = 1$ we get the following refinements of (10):

$$G_\alpha \leq T_\alpha^a \leq T_\alpha \leq A_\alpha, \text{ when } 0 < a \leq 1 \tag{13}$$

$$G_\alpha \leq T_\alpha \leq T_\alpha^a \leq A_\alpha, \text{ when } a \geq 1. \tag{14}$$

These offer infinitely (continuously) many refinements of inequalities (4) and (3).

**Remark.** An alternate proof for left side of (10) can be given by considering the application

$$g : \mathbb{R} \to \mathbb{R}, \ g(x) = \ln(a + e^x).$$

Since $g''(x) = \dfrac{ae^x}{(a+e^x)^2} > 0$, $g$ is strictly convex, so by Jensen's inequality

$$g(\alpha_1 y_1 + \ldots + \alpha_n y_n) \leq \alpha_1 g(y_1) + \ldots + \alpha_n g(y_n), \ y_1, \ldots, y_n \in \mathbb{R}$$

we get

$$(a + e^{y_1})^{\alpha_1} \ldots (a + e^{y_n})^{\alpha_n} \geq a + e^{\alpha_1 y_1 + \ldots + \alpha_n y_n} \tag{15}$$

Let now $y_i = \ln x_i$ $(i = 1, 2, \ldots, n)$, where $x_i > 0$ in (15). Then we get

$$(a + x_1)^{\alpha_1} \ldots (a + x_n)^{\alpha_n} \geq a + x_1^{\alpha_1} \ldots x_n^{y_n}, \tag{16}$$

so $G_\alpha \leq T_\alpha^a$ follows.

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## 1.8 On an inequality of Sierpinski

### 1

Let $x_i$, $i = \overline{1,n}$, be strictly positive numbers and denote their usual arithmetic, geometric and harmonic mean by

$$A_n(x) = \frac{\sum_{i=1}^{n} x_i}{n}, \quad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, \quad H_n(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}},$$

where $x = (x_1, \ldots, x_n)$.

In 1909 W. Sierpinski ([5]) discovered the following double-inequality:

$$(H_n(x))^{n-1} A_n(x) \leq (G_n(x))^n \leq (A_n(x))^{n-1} H_n(x). \tag{1}$$

The aim of this note is to obtain a very short proof of (1) (in fact, a generalization), by using Maclaurin's theorem for elementary symmetric functions. For another idea of proof for (1) (due to the present author), which leads also to a refinement of an inequality of Ky Fan, see [1]. For application of (1) see [4]. Now we state Maclaurin's theorem as the following:

**Lemma.** *Let $c_r$ be the $r$-th elementary symmetric function of $x$ (i.e. the sum of the products, $r$ at a time, of different $x_i$) and $p_r$ the average of these products, i.e.*

$$p_r = \frac{c_r}{\binom{n}{r}}.$$

*Then*

$$p_1 \geq p_2^{\frac{1}{2}} \geq p_3^{\frac{1}{3}} \geq \ldots \geq p_{n-k}^{\frac{1}{n-k}} \geq \ldots \geq p_n^{\frac{1}{n}}. \tag{2}$$

See [2], [3] for a proof and history of this result.

## 2

Our result is contained in the following

**Theorem.** *Let $k = 1, 2, \ldots$ and define the $k$-harmonic mean of $x$ by*

$$H_{n,k}(x) = \frac{\dbinom{n}{k}}{\displaystyle\sum_{\binom{n}{k}} \frac{1}{x_1 \ldots x_k}}.$$

*Then one has the inequalities*

$$(G_n(x))^n \leq (A_n(x))^{n-k} \cdot H_{n,k}(x) \tag{3}$$

$$(G_n(x))^n \geq (H_{n,k}(x))^{n-k} \cdot A_{n,k}(x), \tag{4}$$

*where*

$$A_{n,k}(x) = p_k = \frac{\sum x_1 \ldots x_k}{\dbinom{n}{k}}.$$

**Proof.** Apply $p_1 \geq p_{n-k}^{\frac{1}{n-k}}$ from (2), where

$$p_{n-k} = \sum \frac{x_1 \ldots x_{n-k}}{\dbinom{n}{n-k}} = \left(\prod_{i=1}^{n} x_i\right) \left(\frac{\sum \dfrac{1}{x_1 \ldots x_k}}{\dbinom{n}{k}}\right),$$

and we easily get (3).

For $k = 1$ one reobtains the right side of inequality (1). By replacing $x$ by

$$\frac{1}{x} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right),$$

and remarking that

$$G_n\left(\frac{1}{x}\right) = \frac{1}{G_n(x)}, \quad A_n\left(\frac{1}{x}\right) = \frac{1}{H_n(x)}, \quad H_n\left(\frac{1}{x}\right) = \frac{1}{A_n(x)},$$

we immediately get the left side of (1) from the right side of this relation. This finishes the proof of (3). Letting $\dfrac{1}{x}$ in place of $x$, we get (4).

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## 1.9 On certain inequalities for means of many arguments

### 1. Introduction

Let $a_i > 0$ $(i = 1, 2, \ldots, n)$ and introduce the well-known means of $n$ variables

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad G_n = \sqrt[n]{\prod_{i=1}^{n} a_i}, \quad Q_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}.$$

Recently, V.V. Lokot and S. Phenicheva [1] have proved the following inequality: For all $n \geq 3$,

$$nA_n^2 \geq (n-1)G_n^2 + Q_n^2. \tag{1}$$

The proof of (1) is based on mathematical induction, combined with the introduction and quite complicated study of an auxiliary function.

The aim of this note is to provide a very simple proof of (1). A related result, as well as an extension will be given, too.

### 2. The proof

Remark that (1) may be written equivalently also as

$$(a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) \geq n(n-1)G_n^2. \tag{2}$$

Now, the left side of (2) may be written also as $2\sum_{i<j} a_i a_j$, where the number of terms is $\dfrac{n(n-1)}{2}$. Apply now the arithmetic mean-geometric mean inequality

$$A_s \geq G_s \ (s \geq 1), \text{ where } A_s = \frac{x_1 + \ldots + x_s}{s}, \ G_s = \sqrt[s]{x_1 \ldots x_s} \tag{3}$$

with $s = \dfrac{n(n-1)}{2}$ and $x_1 = a_1 a_2, \ldots, x_s = a_{n-1} a_n$.

Then, as each term $a_i$ appears $n-1$ times, we get from (3):

$$\sum_{i<j} a_i a_j \le \frac{n(n-1)}{2}[(a_1 a_2 \dots a_n)^{n-1}]^{\frac{2}{n(n-1)}} = \frac{n(n-1)}{2} G_n^2.$$

This proves relation (2), and so relation (1), too.

**Remark.** As for $n=2$ there is equality in (1), this holds for all $n \ge 2$.

## 3. A related result

Apply now the inequality

$$A_s \le Q_s \ (s \ge 1), \text{ where } s = \frac{n(n-1)}{2}, \text{ etc.} \tag{4}$$

As $2\sum_{i<j}(a_i a_j)^2 = \left(\sum a_i^2\right)^2 - \sum a_i^4 = n^2 Q_n^4 - n R_n^4$, where

$$R_n = \sqrt[4]{\sum_{i=1}^{n} a_i^4 / n}, \tag{5}$$

we get by (4) the following inequality:

$$\frac{nA_n^2 - Q_n^2}{n-1} \le \sqrt{\frac{nQ_n^4 - R_n^4}{n-1}}, \ n \ge 2, \tag{6}$$

with $R_n$ defined as in (5). By (1) and (6), we get also:

$$G_n^2 \le \frac{nA_n^2 - Q_n^2}{n-1} \le \sqrt{\frac{nQ_n^4 - R_n^4}{n-1}}, \ n \ge 2 \tag{7}$$

## 4. An extension

In analogy with (5) let us define

$$R_{n,k} = \left(\frac{a_1^k + a_2^k + \ldots + a_n^k}{n}\right)^{1/k}. \tag{8}$$

Now we will prove the following:

**Theorem.** *For all $n, k \geq 2$ one has*

$$n^{k-1}A_n^k \geq (n^{k-1}-1)G_n^k + R_{n,k}^k.$$

*Proof.* We shall use the multinomial theorem, as follows:

$$(a_1+\ldots+a_n)^k = \sum_{\substack{i_1+i_2+\ldots+i_n=k \\ i_1,i_2,\ldots,i_n\geq 0}} \binom{k}{i_1,i_2,\ldots,i_n} a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n}, \tag{9}$$

where the multinomial coefficients

$$\binom{k}{i_1,i_2,\ldots,i_n} = \frac{k!}{i_1!i_2!\ldots i_n!}.$$

By letting $a_1 = a_2 = \ldots = a_n = 1$ in (9), we get that the sum of all multinomial coefficients is $n^k$:

$$\sum \binom{k}{i_1,i_2,\ldots,i_n} = n^k. \tag{10}$$

By letting $a_2 = \ldots = a_n = 1$, $a_1 = x$ in (9) and by taking a derivative upon $x$, one has

$$k(x+n-1)^{k-1} = \sum i_1 \binom{k}{i_1,i_2,\ldots,i_n} x^{i_1}, \tag{11}$$

and by putting $x = 1$ in (11) we get:

$$k \cdot n^{k-1} = \sum i_1 \binom{k}{i_1,i_2,\ldots,i_n}. \tag{12}$$

Now, remark that

$$(a_1+\ldots+a_n)^k - (a_1^k+\ldots+a_n^k) = n^k(A_n^k - R_{n,k}^k).$$

On the other hand, by using (9), we get

$$n^k(A_n^k - R_{n,k}^k) = \sum_{0\leq i_1\leq k-1,\ldots,0\leq i_n\leq k-1} \binom{k}{i_1,i_2,\ldots,i_n} a_1^{i_1}\ldots a_n^{i_n}. \tag{13}$$

On base of (10), the number of terms of the right hand side of (13) is $n^k - n$. Now write the sum on the right side of (13) as the sum $x_1 + x_2 + \ldots + x_s$, where $s = n^k - n$, where each term $x_k$ is of the form $a_1^{i_1} \ldots a_n^{i_n}$ (which appears in the sum $\binom{k}{i_1, \ldots, i_n}$ times), with $i_1 + \ldots + i_n = k$, $i_1 \neq k, \ldots, i_n \neq k$.

Applying inequality (3), we have to calculate the powers of $a_1$, $a_2, \ldots, a_n$. We will show that each such power is $kn^{k-1} - k$. First remark that in (9) in such a writing $a_1$ appears $\sum i_1 \binom{k}{i_1, i_2, \ldots, i_n}$ times, which is $kn^{k-1}$, by relation (12). But in (13) are missing the terms with $a_1^k$, which appears $k$ times. Thus in $x_1 x_2 \ldots x_s$ on the right side of (13), the power of $a_1$ will be $kn^{k-1} - k$. Clearly, the same is true for $a_2, \ldots, a_n$.

Now, by (3) we get

$$n^k (A_n^k - R_{n,k}^k) \geq (n^k - n)[(a_1 a_2 \ldots a_n)^{kn^{k-1}-k}]^{1/(n^k - n)}$$

$$= n(n^{k-1} - 1)(a_1 a_2 \ldots a_n)^{k/n} = n(n^{k-1} - 1) G_n^k.$$

By reducing with $n$, we get the Theorem.

**Remarks.** 1) For $k = 2$, we get relation (1). For $k = 3$ we get:

$$n^2 A_n^3 \geq (n^2 - 1) G_n^3 + R_{n,3}^3. \tag{14}$$

2) As by the inequality $(x_1 + \ldots + x_n)^2 \leq n(x_1^2 + \ldots + x_n^2)$ applied yo $x_1 = \ldots = x_{n-1} = G_n$, $x_n = Q_n$ implies

$$[(n-1)G_n + Q_n]^2 \leq n[(n-1)G_n^2 + Q_n^2] \leq n^2 A_n^2,$$

by (1); we get the most simple analogy to this relation:

$$nA_n \geq (n-1)G_n + Q_n. \tag{15}$$

Such inequalities involving $A_n$, $G_n$, $H_n$ (harmonic mean), or $G_n$, $Q_n$, $H_n$ are proved in [2].

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# 1.10 A note on log-convexity of power means

## 1. Introduction

Let $M_p(a,b) = (\frac{a^p+b^p}{2})^{1/p}$ $(p \neq 0), M_0(a,b) = \sqrt{ab}$ denote the power mean (or Hölder mean, see [2]) of two arguments $a, b > 0$. Recently A. Bege, J. Bukor and J. T. Tóth [1] have given a proof of the fact that for $a \neq b$, the application $p \to M_p$ is log-convex for $p \leq 0$ and log-concave for $p \geq 0$. They also proved that it is also convex for $p \leq 0$. We note that this last result follows immediately from the well-known convexity theorem, which states that all log-convex functions are convex, too (see e.g. [2]). The proof of authors is based on an earlier paper by T.J. Mildorf (see [1]).

In what follows, we will show that this result is well-known in the literature, even in a more general setting. A new proof will be offered, too.

## 2. Notes and results

In 1948 H. Shniad [6] studied the more general means

$$M_t(a,\xi) = \left(\sum_{i=1}^{n} \xi_i a_i^t\right)^{1/t} \quad (t \neq 0), \quad M_0(a,\xi) = \prod_{i=1}^{n} a_i^{\xi_i},$$

$$M_{-\infty}(a,\xi) = \min\{a_i : i = 1, \dots\},$$

$$M_{+\infty}(a,\xi) = \max\{a_i : i = 1, \dots\},$$

where $0 < a_i < a_{i+1}$ $(i = 1, \dots, n-1)$ are given positive real numbers, and $\xi_i (i = \overline{1,n})$ satisfy $\xi_i > 0$ and $\sum_{i=1}^{n} \xi_i = 1$.

Put $\Lambda(t) = \log M_t(a,\xi)$. Among other results, in [6] the following are proved:

**Theorem 2.1**

$$\text{If } \xi_1 \geq \frac{1}{2} \text{ then } \Lambda(t) \text{ is convex for all } t < 0; \tag{1}$$

$$\text{If } \xi_n \geq \frac{1}{2} \text{ then } \Lambda(t) \text{ is concave for all } t > 0. \tag{2}$$

Clearly, when $n = 2$, in case of $M_p$ one has $\xi_1 = \xi_2 = \frac{1}{2}$, so the result by Bege, Bukor and Tóth [1] follows by (1) and (2).

Another generalization of power mean of order two is offered by the Stolarsky means (see [7]) for $a, b > 0$ and $x, y \in \mathbb{R}$ define

$$D_{x,y}(a,b) = \begin{cases} \left[\dfrac{y(a^x - b^x)}{x(a^y - b^y)}\right]^{1/(x-y)}, & xy(x-y) \neq 0 \\ \exp\left(-\dfrac{1}{x} + \dfrac{a^x \ln a - b^x \ln b}{a^x - b^x}\right), & x = y \neq 0 \\ \left[\dfrac{a^x - b^x}{(\ln a - \ln b)}\right]^{1/x}, & x \neq 0, y = 0 \\ \sqrt{ab}, & x = y = 0 \end{cases}$$

The means $D_{x,y}$ are called som3times as the difference means, or extended means.

Let $I_x(a,b) = (I(a^x, b^x))^{1/x}$, where $I(a,b)$ denotes the identic mean (see [2], [4]) defined by

$$I(a,b) = D_{1,1}(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \ (a \neq b),$$

$$I(a,a) = a.$$

K. Stolarsky [7] proved also the following representation formula:

$$\log D_{x,y} = \frac{1}{y-x}\int_x^y \log I_t dt \text{ for } x \neq y. \tag{3}$$

Now, in 2001 the author [4] proved for the first time that the application $t \to \log I_t$ is convex for $t > 0$ and concave for $t < 0$.

This in turn implies immediately (see also [3]) the following fact:

**Theorem 2.2**

$$\begin{array}{l}\text{If } x>0 \text{ and } y>0, \text{ then } D_{x,y} \text{ is log-concave in both } x \text{ and } y. \\ \text{If } x<0 \text{ and } y<0, \text{ then } D_{x,y} \text{ is log-convexe in both } x \text{ and } y.\end{array} \tag{4}$$

Now, remark that

$$M_p(a,b) = D_{2p,p}(a,b) \tag{5}$$

so the log-convexity properties by H. Shniad are also particular cases of (4).

We note that an application of log-convexity of $M_p$ is given in [5].

## 3. A new elementary proof

We may assume (by homogeneity properties) that $b = 1$ and $a > 1$. Let

$$f(p) = \frac{\ln((a^p+1)/2)}{p},$$

and denote $x = a^p$. Then, as

$$x' = \frac{dx}{dp} = a^p \ln a = x \ln a,$$

from the identity

$$pf(p) = \ln(x+1)/2$$

we get by differentiation

$$f(p) + pf'(p) = \frac{x \ln a}{x+1} \tag{6}$$

By differentiating once again (6), we get

$$2f'(p) + pf''(p) = \frac{(x \ln^2 a)(x+1) - x^2 \ln^2 a}{(x+1)^2},$$

which implies, by definition of $f(p)$ and relation (6):

$$p^3 f''(p) = \frac{(x\ln^2 x)(x+1) - x^2\ln^2 x}{(x+1)^2} - \frac{2}{x+1}\left[x\ln x - (x+1)\ln\left(\frac{x+1}{2}\right)\right]$$

$$= \frac{x\ln^2 x + 2(x+1)^2\ln(x+1) - 2x(x+1)\ln x}{(x+1)^2}, \tag{7}$$

after some elementary computations, which we omit here.

Put

$$g(x) = x\ln^2 x + 2(x+1)^2\ln(\frac{x+1}{2}) - 2x(x+1)\ln x.$$

One has successively:

$$g'(x) = ln^2 x + 4(x+1)\ln\left(\frac{x+1}{2}\right) - 4x\ln x,$$

$$g''(x) = \frac{2\ln x}{x} + 4\ln\left(\frac{x+1}{2}\right) - 4\ln x,$$

$$g'''(x) = 2\left[\frac{1-\ln x}{x^2} - \frac{2}{x(x+1)}\right] = \frac{-2}{x^2(x+1)}[x - 1 + (x+1)\ln x].$$

Now, remark that for $x > 1$, clearly $g'''(x) < 0$, so $g''(x)$ is strictly decreasing, implying $g''(x) < g''(1) = 0$. Thus $g'(x) < g'(1) = 0$, giving $g(x) < g(1) = 0$. Finally, one gets $f''(p) < 0$, which shows that for $x > 1$ the function $f(p)$ is strictly concave function of $p$. As $x = a^p$ with $a > 1$, this happend only when $p > 0$.

For $x < 1$, remark that $x - 1 < 0$ and $\ln x < 0$, so $g'''(x) > 0$, and all above procedure may be repeted. This shows that $f(p)$ is a strictly convex function of $p$ for $p < 0$.

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## 1.11 A note on the $\varphi$ and $\psi$ functions

### 1. Introduction

In a recent paper [1] V. Kannan and R. Srikanth have stated the following inequality

$$\varphi(n)^{\varphi(n)} \cdot \psi(n)^{\psi(n)} \geq n^{2n\mu(n)}, \tag{1}$$

where $\varphi(n)$ and $\psi(n)$ are the Euler, resp. Dedekind arithmetic functions, while $\mu(n)$ is defined by

$$\mu(n) = \frac{1}{2}\left[\prod_{p|n}\left(1-\frac{1}{p}\right) + \prod_{p|n}\left(1+\frac{1}{p}\right)\right] \tag{2}$$

where $p$ runs through the prime divisors of $n$.

Recall that $\varphi(1) = \psi(1) = 1$, and one has

$$\varphi(n) = n \cdot \prod_{p|n}\left(1-\frac{1}{p}\right). \quad \psi(n) = n \cdot \prod_{p|n}\left(1+\frac{1}{p}\right). \tag{3}$$

The proof given in [1] depends on inequality

$$\prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\log\left(1-\frac{1}{p}\right) + \prod_{p|n}\left(1+\frac{1}{p}\right)\sum_{p|n}\log\left(1+\frac{1}{p}\right) \geq 0. \tag{4}$$

In what follows, we shall give another proof of inequality (1), and this proof offers in fact a stronger relation.

### 2. The proof

First remark that, by using (3), the expression $\mu(n)$ given by (2) may be written as

$$\mu(n) = \frac{\varphi(n)+\psi(n)}{2n}. \tag{5}$$

Therefore, the inequality may be rewritten as

$$\varphi(n)^{\varphi(n)} \cdot \psi(n)^{\psi(n)} \geq n^{\varphi(n)+\psi(n)}. \tag{6}$$

The following auxiliary results are needed:

**Lemma 1.** *For any $a, b > 0$ real numbers one has*

$$a^a \cdot b^b \geq \left(\frac{a+b}{2}\right)^{a+b}. \tag{7}$$

**Lemma 2.** *For any $n \geq 1$ one has*

$$\varphi(n) + \psi(n) \geq 2n. \tag{8}$$

Lemma 2 is well-known, see e.g. [2]. Lemma 1 is also well-known, but we shall give here a complete proof.

Let us consider the application $f(x) = x \log x$, $x > 0$. Since $f''(x) > 0$, $f$ is strictly convex, so by Jensen's inequality we can write

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}. \tag{9}$$

There is equality only for $a = b$. After simple computations, we get relation (7).

Another proof is based on the fact that the weighted geometric mean of $a$ and $b$ is greater than the weighted harmonic mean, i.e.

$$a^p \cdot b^q \geq \frac{1}{\frac{p}{a} + \frac{q}{b}}, \tag{10}$$

where $p, q > 0$, $p + q = 1$. Put $p = \dfrac{a}{a+b}$, $q = \dfrac{b}{a+b}$, and from (10) we get (7).

Now, for the proof of (6) apply Lemma 1 and Lemma 2 in order to deduce:

$$\varphi(n)^{\varphi(n)} \cdot \psi(n)^{\psi(n)} \geq \left(\frac{\varphi(n)+\psi(n)}{2}\right)^{\varphi(n)+\psi(n)} \geq n^{\varphi(n)+\psi(n)}. \tag{11}$$

Therefore, in fact a refinement of inequality (6) (and (1), too), is offered.

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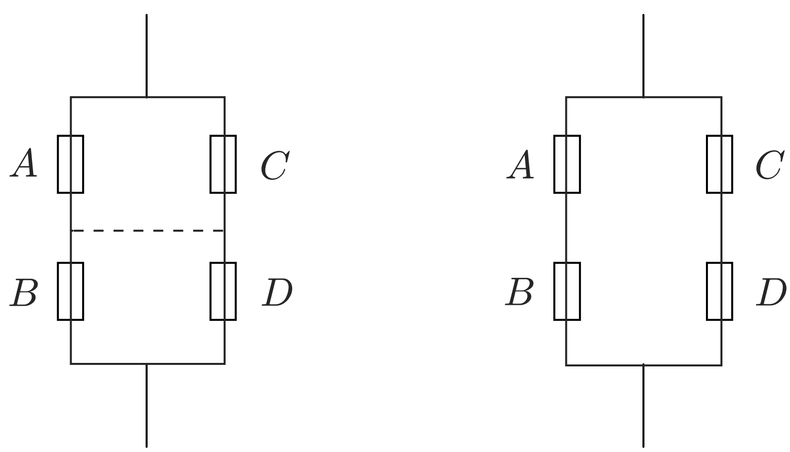
# 1.12 Generalizations of Lehman's inequality

## 1. Introduction

A. Lehman's inequality (see [6], [2]) (and also SIAM Review **4**(1962), 150-155), states that if $A, B, C, D$ are positive numbers, then

$$\frac{(A+B)(C+D)}{A+B+C+D} \geq \frac{AC}{A+C} + \frac{BD}{B+D}. \tag{1}$$

This was discovered as follows: interpret $A, B, C, D$ as resistances of an electrical network. It is well-known that if two resistances $R_1$ and $R_2$ are serially connected, then their compound resistance is $R = R_1 + R_2$, while in parallel connecting one has $1/R = 1/R_1 + 1/R_2$. Now consider two networks, as given in the following two figures:




$$R = \frac{(A+B)(C+D)}{A+B+C+D} \qquad\qquad R' = \frac{AC}{A+C} + \frac{BD}{B+D}$$

By Maxwell's principle, the current chooses a distribution such as to minimize the energy (or power), so clearly $R' \leq R$, i.e. Lehman's inequality (1).

In fact, the above construction may be repeated with $2n$ resistances, in order to obtain:

**Theorem 1.** *If* $a_i, b_i$ $(i = \overline{1,n})$ *are positive numbers, then*

$$\frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{a_1 + \cdots + a_n + b_1 + \cdots + b_n} \geq \frac{a_1 b_1}{a_1 + b_1} + \cdots + \frac{a_n b_n}{a_n + b_n} \tag{2}$$

*for any* $n \geq 2$.

**Remark.** Since $\dfrac{2ab}{a+b} = H(a,b)$ is in fact the harmonic mean of two positive numbers, Lehman's inequality (2) can be written also as

$$H(a_1 + \cdots + a_n, b_1 + \cdots + b_n) \geq H(a_1, b_1) + \cdots + H(a_n, b_n) \tag{3}$$

## 2. Two-variable generalization

In what follows, by using convexity methods, we shall extend (3) in various ways. First we introduce certain definitions. Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function with two arguments, where $A$ is a cone (e.g. $A = \mathbb{R}^2_+$). Let $k \in \mathbb{R}$ be a real number. Then we say that $f$ is $k$**-homogeneous**, if

$$f(rx, ry) = r^k f(x,y) \tag{4}$$

for any $r > 0$ and $x, y \in A$. When $k = 1$, we simply say that $f$ is **homogeneous**.

Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a function of an argument defined on an interval $I$. We say that $F$ is $k$**-convex** ($k$-concave), if

$$F(\lambda a + \mu b) \underset{(\geq)}{\leq} \lambda^k F(a) + \mu^k F(b), \tag{5}$$

for any $a, b \in I$, and any $\lambda, \mu > 0$, $\lambda + \mu = 1$. If $k = 1$, then $F$ will be called simply convex. For example, $F(t) = |t|^k$, $t \in \mathbb{R}$ is $k$-convex, for $k \geq 1$, since $|\lambda a + \mu b|^k \leq \lambda^k |a|^k + \mu^k |b|^k$ by $(u+v)^k \leq u^k + v^k$ $(u, v > 0)$, $k \geq 1$. On the other hand, the function $F(t) = |t|$, though is convex, is not 2-convex on $\mathbb{R}$.

The $k$-convex functions were introduced, for the first time, by W. W. Breckner [4]. See also [5] for other examples and results. A similar convexity notion, when in (5) one replaces $\lambda+\mu=1$ by $\lambda^k+\mu^k=1$, was introduced by W. Orlicz [12] (see also [8] for these convexities).

Now, let $A=(0,+\infty)\times(0,+\infty)=\mathbb{R}_+^2$ and $I=(0,+\infty)$. Define $F(t)=f(1,t)$ for $t\in I$.

**Theorem 2.** *If $f$ is $k$-homogeneous, and $F$ is $k$-convex ($k$-concave) then*

$$f(a_1+\cdots+a_n,b_1+\cdots+b_n)\underset{(\geq)}{\leq} f(a_1,b_1)+\cdots+f(a_n,b_n) \tag{6}$$

*for any $a_i,b_i\in A$ $(i=1,2,\ldots,n)$.*

**Proof.** First remark, that by (4) and the definition of $F$, one has

$$a^kF\left(\frac{b}{a}\right)=a^kf\left(1,\frac{b}{a}\right)=f(a,b) \tag{7}$$

On the other hand, by induction it can be proved the following Jensen-type inequality:

$$F(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)\underset{(\geq)}{\leq}\lambda_1^kF(x_1)+\lambda_2^kF(x_2)+\cdots+\lambda_n^kF(x_n), \tag{8}$$

for any $x_i\in I$, $\lambda_i>0$ $(i=\overline{1,n})$, $\lambda_1+\cdots+\lambda_n=1$.

E.g. for $n=3$, relation (8) can be proved as follows:

Put

$$a=\frac{\lambda_1}{\lambda_1+\lambda_2}x_1+\frac{\lambda_2}{\lambda_1+\lambda_2}x_2,\ b=x_3,\ \lambda=\lambda_1+\lambda_2,\ \mu=\lambda_3$$

in (5). Then, as $\lambda_1x_1+\lambda_2x_2+\lambda_3x_3=\lambda a+\mu b$, we have

$$F(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)\leq\lambda^kF(a)+\mu^kF(b)$$

$$\leq(\lambda_1+\lambda_2)^k\left[\frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k}F(x_1)+\frac{\lambda_2^k}{(\lambda_1+\lambda_2)^k}F(x_2)\right]+\lambda_3^kF(x_3)$$

$$=\lambda_1^kF(x_1)+\lambda_2^kF(x_2)+\lambda_3^kF(x_3).$$

The induction step from $n = m$ to $n = m + 1$ $(m \geq 2)$ follows on the same lines, by letting

$$a = \frac{\lambda_1}{\lambda} x_1 + \cdots + \frac{\lambda_m}{\lambda} x_m, \quad b = x_{m+1}, \quad \lambda = \lambda_1 + \cdots + \lambda_m, \quad \mu = \lambda_{m+1}$$

in (5).

Put now in (8)

$$x_1 = \frac{b_1}{a_1},\ x_2 = \frac{b_2}{a_2}, \ldots, x_n = \frac{b_n}{a_n},$$

$$\lambda_1 = \frac{a_1}{a_1 + \cdots + a_n},\ \lambda_2 = \frac{a_2}{a_1 + \cdots + a_n}, \ldots, \lambda_n = \frac{a_n}{a_1 + \cdots + a_n}$$

in order to obtain

$$F\left(\frac{b_1 + \cdots + b_n}{a_1 + \cdots + a_n}\right) \underset{(\geq)}{\leq} \frac{a_1^k F\left(\frac{b_1}{a_1}\right) + a_2^k F\left(\frac{b_2}{a_2}\right) + \cdots + a_n^k F\left(\frac{b_n}{a_n}\right)}{(a_1 + \cdots + a_n)^k} \tag{9}$$

Now, by (7) this gives

$$f(a_1 + \cdots + a_n, b_1 + \cdots + b_n) \leq f(a_1, b_1) + \cdots + f(a_n, b_n),$$

i.e. relation (6).

**Remark.** Let $f(a, b) = \dfrac{a + b}{ab}$. Then $f$ is homogeneous (i.e. $k = 1$), and

$$F(t) = f(1, t) = \frac{t + 1}{t}$$

is 1-convex (i.e., convex), since $F''(t) = 2/t^3 > 0$. Then relation (6) gives the following inequality:

$$\frac{1}{H(a_1 + \cdots + a_n, b_1 + \cdots + b_n)} \leq \frac{1}{H(a_1, b_1)} + \cdots + \frac{1}{H(a_n, b_n)}. \tag{10}$$

Let now $f(a, b) = \dfrac{ab}{a + b}$. Then $f$ is homogeneous, with $F(t) = \dfrac{t}{t + 1}$, which is concave. From (6) (with $\geq$ inequality), we recapture Lehman's inequality (3).

The following theorem has a similar proof:

**Theorem 3.** *Let $f$ be $k$-homogeneous, and suppose that $F$ is $l$-convex ($l$-concave) ($k, l \in \mathbb{R}$). Then*

$$(a_1 + \cdots + a_n)^{l-k} f(a_1 + \cdots + a_n, b_1 + \cdots + b_n)$$
$$\leq (\geq) a_1^{l-k} f(a_1, b_1) + \cdots + a_n^{l-k} f(a_n, b_n). \tag{11}$$

**Remarks.** For $k = l$, (11) gives (9).

For example, let $f(a, b) = \dfrac{a}{b}$, where $a, b \in (0, \infty) \times (0, \infty)$. Then $k = 0$ (i.e. $f$ is homogeneous of order 0), and $F(t) = \dfrac{1}{t}$, which is 1-convex, since

$$F''(t) = \frac{2}{t^3} > 0.$$

Thus $l = 1$, and relation (11) gives the inequality

$$\frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n} \leq \frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n} \tag{12}$$

Finally, we given another example of this type. Put

$$f(a, b) = \frac{a^2 + b^2}{a + b}.$$

Then $k = 1$. Since

$$F(t) = \frac{t^2 + 1}{t + 1},$$

after elementary computations,

$$F''(t) = 4/(t + 1)^3 > 0,$$

so $l = 1$, and (11) (or (9)) gives the relation

$$\frac{(a_1 + \cdots + a_n)^2 + (b_1 + \cdots + b_n)^2}{a_1 + \cdots + a_n + b_1 + \cdots + b_n} \leq \frac{a_1^2 + b_1^2}{a_1 + b_1} + \cdots + \frac{a_n^2 + b_n^2}{a_n + b_n} \tag{13}$$

Since $L_1(a,b) = \dfrac{a^2+b^2}{a+b}$ (more generally, $L_p(a,b) = \dfrac{a^{p+1}+b^{p+1}}{a^p+b^p}$) are the so-called "Lehmer means" [9], [7], [1] of $a, b > 0$, (13) can be written also as

$$L_1(a_1+\cdots+a_n, b_1+\cdots+b_n) \le L_1(a_1,b_1)+\cdots+L_1(a_n,b_n). \tag{14}$$

Clearly, one can obtain more general forms for $L_p$. For inequalities on more general means (e.g. Gini means), see [10], [11].

## 3. Hölder's inequality

As we have seen, there are many applications to Theorems 2 and 3. Here we wish to give an important application; namely a new proof of Hölder's inequality (one of the most important inequalities in Mathematics).

Let $f(a,b) = a^{1/p}b^{1/q}$, where $1/p + 1/q = 1$ ($p > 1$). Then clearly $f$ is homogeneous ($k = 1$), with $F(t) = t^{1/q}$. Since

$$F'(t) = \frac{1}{q}t^{-1/p}, \quad F''(t) = -\frac{1}{pq}t^{-(1/p)-1} < 0,$$

so by Theorem 2 one gets

$$(a_1+\cdots+a_n)^{1/p}(b_1+\cdots+b_n)^{1/q} \ge a_1^{1/p}b_1^{1/q}+\cdots+a_n^{1/p}b_n^{1/q} \tag{15}$$

Replace now $a_i = A_i^p$, $b_i = B_i^q$ ($i = \overline{1,n}$) in order to get

$$\sum_{i=1}^n A_iB_i \le \left(\sum_{i=1}^n A_i^p\right)^{1/p}\left(\sum_{i=1}^n B_i^q\right)^{1/q}, \tag{16}$$

which is the classical Hölder inequality.

## 4. Many-variables generalization

Let $f : A \subset \mathbb{R}_+^n \to \mathbb{R}$ be of $n$ arguments ($n \ge 2$). For simplicity, put

$$p = (x_1,\ldots,x_n),\ p' = (x_1',\ldots,x_n'),$$

when $p+p' = (x_1+x'_1, \ldots, x_n+x'_n)$ and $rp = (rx_1, \ldots, rx_n)$ for $r \in \mathbb{R}$. Then the definitions of $k$-homogeneity and $k$-convexity can be extended to this case, similarly to paragraph 2. If $A$ is a cone, then $f$ is $k$-homogeneous, if $f(rp) = r^k f(p)$ $(r > 0)$ and if $A$ is convex set then $f$ is $k$-**convex**, if $f(\lambda p + \mu p') \leq \lambda^k f(p) + \mu^k f(p')$ for any $p, p' \in A$, $\lambda, \mu > 0$, $\lambda + \mu = 1$. We say that $f$ is $k$-**Jensen convex**, if

$$f\left(\frac{p+p'}{2}\right) \leq \frac{f(p)+f(p')}{2^k}.$$

We say that $f$ is $r$-**subhomogeneous of order** $k$, if $f(rp) \leq r^k f(p)$. Particularly, if $k = 1$ (i.e. $f(rp) \leq rf(p)$), we say that $f$ is $r$-**subhomogeneous** (see e.g. [14], [15]). If $f$ is $r$-subhomogeneous of order $k$ for any $r > 1$, we say that $f$ is **subhomogeneous of order** $k$. For $k = 1$, see [13]. We say that $f$ is **subadditive** on $A$, if

$$f(p+p') \leq f(p) + f(p') \tag{17}$$

We note that in the particular case of $n = 2$, inequality (6) with "$\leq$" says exactly that $f(a, b)$ of two arguments is subadditive.

**Theorem 4.** *If $f$ is homogeneous of order $k$, then $f$ is subadditive if and only if it is $k$-Jensen convex.*

**Proof.** If $f$ is subadditive, i.e. $f(p+p') \leq f(p)+f(p')$ for any $p, p' \in A$, then

$$f\left(\frac{p+p'}{2}\right) = \frac{1}{2^k} f(p+p') \leq \frac{f(p)+f(p')}{2^k},$$

so $f$ is $k$-Jensen convex. Reciprocally, if $f$ is $k$-Jensen convex, then

$$f\left(\frac{p+p'}{2}\right) \leq \frac{f(p)+f(p')}{2^k},$$

so

$$f(p+p') = f\left[2\left(\frac{p+p'}{2}\right)\right] = 2^k f\left(\frac{p+p'}{2}\right) \leq f(p) + f(p'),$$

i.e. (17) follows.

**Remark.** Particularly, a homogeneous subadditive function is convex, a simple, but very useful result in the theory of convex bodies (e.g. "distance function", "supporting function", see e.g. [3], [16]).

**Theorem 5.** *If $f$ is 2-subhomogeneous of order $k$, and is $k$-Jensen convex, then it is subadditive.*

**Proof.** Since

$$f(p+p') = f\left(2\left(\frac{p+p'}{2}\right)\right) \le 2^k f\left(\frac{p+p'}{2}\right),$$

and

$$f\left(\frac{p+p'}{2}\right) \le \frac{f(p)+f(p')}{2^k},$$

we get $f(p+p') \le f(p)+f(p')$, so (17) follows.

**Remark.** Particularly, if $f$ is 2-subhomogeneous, and Jensen convex, then it is subadditive. (18)

It is well-known that a continuous Jensen convex function (defined on an open convex set $A \subset \mathbb{R}^n$) is convex. Similarly, for continuous $k$-Jensen convex functions, see [4].

To give an interesting example, connected with Lehman's inequality, let us consider $A = \mathbb{R}_+^n$, $f(p) = H(p) = n/\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)$.

Let

$$\frac{1}{g(p)} = \frac{1}{x_1}+\cdots+\frac{1}{x_n}.$$

Then

$$\frac{dg}{g^2} = \sum_{i=1}^n \frac{dx_i}{x_i^2}, \quad \frac{d^2g}{g^2} - 2\frac{dg^2}{g^3} = -2\sum_{i=1}^n \frac{dx_i^2}{x_i^3},$$

so

$$\frac{1}{2}\frac{d^2g}{g^3} = \left(\sum_{i=1}^n \frac{dx_i}{x_i^2}\right)^2 - \left(\sum_{i=1}^n \frac{1}{x_i}\right)\left(\sum_{i=1}^n \frac{dx_i^2}{x_i^3}\right).$$

(Here $d$ denotes a differential.) Now apply Hölder's inequality (16) for $p=q=2$ (i.e. Cauchy-Bunjakovski inequality),

$$A_i = 1/\sqrt{x_i},\ \ B_i = (1/x_i\sqrt{x_i})dx_i.$$

Then one obtains $\dfrac{d^2g}{g^3} \le 0$, and since $g > 0$, we get $d^2g \le 0$. It is well-known ([16]) that this implies the concavity of function $g(p) = H(p)/n$, so $-H(p)$ will be a convex function. By consequence (17) of Theorem 5, $H(p)$ is subadditive, i.e.

$$H(x_1 + x_1', x_2 + x_2', \ldots, x_n + x_n') \ge H(x_1, x_2, \ldots, x_n) +$$
$$+ H(x_1', x_2', \ldots, x_n'), \quad (x_i, x_i' > 0). \tag{19}$$

For $n = 2$ this coincides with (3), i.e. Lehman's inequality (1).

Finally, we prove a result, which is a sort of reciprocal to Theorem 5:

**Theorem 6.** *Let us suppose that $f$ is subadditive, and $k$-convex, where $k \ge 1$. Then $f$ is subhomogeneous of order $k$.*

**Proof.** For any $r > 1$ one can find a positive integer $n$ such that $r \in [n, n+1]$. Then $r$ can be written as a convex combination of $n$ and $n+1$: $r = n\lambda + (n+1)\mu$. By the $k$-convexity of $f$ one has

$$f(rp) = f(n\lambda p + (n+1)\mu p) \le \lambda^k f(np) + \mu^k f[(n+1)p].$$

Since $f$ is subadditive, from (17) it follows by induction that

$$f(np) \le n f(p),$$

so we get

$$f(rp) \le n\lambda^k f(p) + (n+1)\mu^k f(p) = [n\lambda^k + (n+1)\mu^k] f(p).$$

Now, since $k \ge 1$, it is well-known that

$$[\lambda n + (n+1)\mu]^k \ge (\lambda n)^k + ((n+1)\mu)^k.$$

But $(\lambda n)^k \ge n\lambda^k$ and $((n+1)\mu)^k \ge (n+1)\mu^k$, so finally we can write

$$f(rp) \le [\lambda n + (n+1)\mu]^k f(p) = r^k f(p),$$

which means that $f$ is subhomogeneous of order $k$.

**Remark.** For $k = 1$ Theorem 6 contains a result by R. A. Rosenbaum [13].

**Final remarks.** After completing this paper, we have discovered that Lehman's inequality (2) (or (3)) appears also as Theorem 67 in G. H. Hardy, J. E. Littlewood and G. Polya [Inequalities, Cambridge Univ. Press, 1964; see p.61], and is due to E. A. Milne [Note on Rosseland's integral for the stellar absorption coefficient, Monthly Notices, R.A.S. 85(1925), 979-984]. Though we are unable to read Milne's paper, perhaps we should call Lehman's inequality as the "Milne-Lehman inequality".

**Acknowledgments**. The author thanks Professors J. Peetre and H. Alzer for providing their reprints [2], resp. [1]. He is indebted to Professor F. A. Valentine for a copy of his book [16], and also to Professors W. W. Breckner of Cluj, and V. E. Szabó of Budapest for helpful discussions. He thanks also the Referee for some helpful remarks, which have improved the presentation of the paper.

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# Chapter 2

# Logarithmic, identric and related means

> "Mathematical discoveries, small or great are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labor, both conscious and subconscious."
>
> (H. Poincaré)

> "Try a hard problem. You may not
> solve it, but you will prove something else."
>
> (J.E. Littlewood)

## 2.1 On the identric and logarithmic means

### 1

Let $a, b > 0$ be positive real numbers. The identric mean $I(a, b)$ of $a$ and $b$ is defined by

$$I = I(a, b) = \frac{1}{e} \cdot \left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \text{ for } a \neq b, \quad I(a, a) = a,$$

while the logarithmic mean $L(a,b)$ of $a$ and $b$ is

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \text{ for } a \neq b, \quad L(a,a) = a.$$

In what follows it will be convenient to denote the arithmetic mean of $a$ and $b$ by

$$A = A(a,b) = \frac{a+b}{2}$$

and the geometric mean by

$$G = G(a,b) = \sqrt{ab}.$$

More generally, we will use also the mean

$$A(k) = A(k;a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}$$

where $k \neq 0$ is a real number. B. Ostle and H.L. Terwilliger [11] and B.C. Carlson [5], [6] have proved first that

$$G \leq L \leq A. \tag{1}$$

This result, or a part of it, has been rediscovered and reproved many times (see e.g. [10], [12], [13]).

In 1974 T.P. Lin [9] has obtained an important refinement of (1):

$$G \leq L \leq A(1/3) \leq A. \tag{2}$$

For new proofs see [12], [13]. We note that $A(k)$ is an increasing function of $k$, so

$$L \leq A(1/3) \leq A(k) \leq A \text{ for all } k \in [1/3, 1]$$

but, as Lin has showed, the number $1/3$ cannot be replaced by a smaller one.

For the identric mean, K.B. Stolarsky [17], [18] has proved that

$$L \leq I \leq A, \tag{3}$$

$$A(2/3) \le I \tag{4}$$

and that the constant $2/3$ in (4) is optimal.

Recently inequalities (1) and (3) appear also in a problem proposed by Z. Zaiming [19]. In [1] and [2] H. Alzer proved the following important inequalities:

$$\sqrt{G \cdot I} \le L \le \frac{1}{2}(G + I), \tag{5}$$

$$A \cdot G \le L \cdot I \quad \text{and} \quad A + G \ge L + I. \tag{6}$$

We notice that, in all inequalities (1)-(6), equality can occur only for $a = b$.

Very recently, H. Seiffert [16] has obtained the following result:

If $f : [a, b] \to \mathbb{R}$ is a strictly increasing function, having a logarithmically convex inverse function, then

$$\frac{1}{b-a}\int_a^b f(x)dx \le f(I(a, b)). \tag{7}$$

The aim of this note is to obtain some improvements and related results of type (1)-(7) as well as some new inequalities containing the identric and logarithmic means. We also define some new means and prove inequalities involving them.

## 2

We start with the relation

$$L(a, b) \le \left(\frac{\sqrt[3]{a} + \sqrt[3]{b}}{2}\right)^3$$

(see (2)), applied with $a \to a^2$, $b \to b^2$. Since

$$L(a^2, b^2) = A(a, b) \cdot L(a, b),$$

we get by a simple transformation, taking (4) into account, that

$$\sqrt{A \cdot L} \le A(2/3) \le I. \tag{8}$$

As for the inequality (5), one may ask (in view of (8)) whether

$$I \leq \frac{1}{2}(A+L)$$

holds always true. We shall prove that the reverse inequality

$$I > \frac{1}{2}(A+L) \quad \text{if} \quad a \neq b \tag{9}$$

is valid. For this purpose, let us divide by $a$ all terms of (9) and write

$$x = \frac{b}{a} > 1.$$

Then it is immediate that (9) becomes equivalent to

$$f(x) = \left(\frac{x-1}{\log x} + \frac{x+1}{2}\right)/g(x) < \frac{2}{e} \quad (x > 1), \tag{10}$$

where $g(x) = x^{x/(x-1)}$.

Since

$$g'(x) = g(x)\left[\frac{1}{x-1} - \frac{\log x}{(x-1)^2}\right],$$

an elementary calculation show that

$$2x(x-1)^2(\log x)^2 g(x) f'(x) = x(x+1)(\log x)^3 - 2(x-1)^3. \tag{11}$$

According to a result of E.B. Leach and M.C. Sholander [8], one has

$$\sqrt[3]{G^2(x,y) \cdot A(x,y)} < L(x,y), \ x \neq y. \tag{12}$$

Letting $y = 1$ in (12), one finds that the right side of (11) is strictly negative, that is, $f(x)$ is a strictly decreasing function for $x > 1$. Now, relation

$$\lim_{x \to 1+} f(x) = \frac{2}{e}$$

concludes the proof of (10).

**Remark.** Inequality (9) is better than the right side of (5). Indeed,

$$I > \frac{1}{2}(A+L) \quad \text{and} \quad \frac{1}{2}(A+L) > 2L - G \quad \text{by} \quad L < \frac{1}{3}(2G+A),$$

which is a known result proved by B.C. Carlson [6].

## 3

Let us first note that the proof of (7) (see [16]) shows that, if $f$ is strictly increasing and $f^{-1}$ logarithmically concave (which we abbreviate as "log conc" and analogously "log conv", for logarithmically convex), then (7) is valid with reversed sign of inequality (when $f$ is strictly decreasing and $f^{-1}$ log conc, then (7) is valid as it stands). It is easy to see that, for a twice differentiable function $f$ with $f'(x) > 0$, then $f^{-1}$ is log conv iff

$$f'(x) + xf''(x) \leq 0,$$

and $f^{-1}$ is log conc iff

$$f'(x) + xf''(x) > 0.$$

If $f'(x) < 0$ then these inequalities are reversed.

Take now $f(x) = x^s$ and notice that

$$f'(x) = sx^{s-1} > 0 \text{ for } s > 0$$

and

$$f'(x) < 0 \text{ for } s < 0.$$

In all cases,

$$f'(x) + xf''(x) = s^2x^{s-1} > 0.$$

Thus for $s > 0$ we can write

$$\frac{1}{b-a}\int_a^b x^s dx \geq (I(a,b))^s,$$

yielding (with the notation $s + 1 = t$)

$$\frac{t(b-a)}{b^t - a^t} < (I(a,b))^{1-t},\ t \gtrless 1. \tag{13}$$

Since, for $s < 0$, $f$ is strictly decreasing with $f^{-1}$ strictly log conv, (13) is true also for $t < 1$.

Using the method in [13], set $a = x^t$, $b = y^t$ ($x, y > 0$, $t \neq 0$) in (1). We get

$$(xy)^{t/2} \cdot \frac{t(y-x)}{y^t - x^t} < L(x,y) < \frac{x^t + y^t}{2} \cdot \frac{t(y-x)}{y^t - x^t}, \ t \neq 0, \tag{14}$$

a double inequality attributed to B.C. Carlson [6]. In view of (13) we obtain

$$L(a,b) \cdot (I(a,b))^{t-1} < L(a,b) \cdot \frac{b^t - a^t}{t(b-a)} < \frac{a^t + b^t}{2}, \ t \neq 0. \tag{15}$$

Some particular cases of (15) are of interest. For $t = -1$, $1$ see (1); for $t = 1/2$ one obtains (by (6))

$$I > \frac{A+G}{2} > \frac{L+I}{2}. \tag{16}$$

For $t = 2$ we get

$$\frac{a+b}{2}\sqrt{ab} < L(a,b) \cdot I(a,b) < \frac{a^2+b^2}{2}. \tag{17}$$

# 4

By simple calculation we can deduce the formulae

$$\log I(a,b) = \frac{b \log b - a \log a}{b-a} - 1$$

and

$$\begin{aligned} \log I\left(\sqrt{a}, \sqrt{b}\right) &= \frac{b \log b - a \log a}{2(b-a)} - 1 + \frac{\sqrt{ab}(\log b - \log a)}{2(b-a)} \\ &= \log \sqrt{I(a,b)} + \frac{G(a,b)}{2L(a,b)} - \frac{1}{2}. \end{aligned}$$

This relation implies, among others, that

$$I^2\left(\sqrt{a}, \sqrt{b}\right) \leq I(a,b) \tag{19}$$

since $G \leq L$. On the other hand, the inequality

$$L^2\left(\sqrt{a}, \sqrt{b}\right) \leq I^2\left(\sqrt{a}, \sqrt{b}\right)$$

is transformed, via (18), into

$$L^2 \leq I \cdot \left(\frac{A+G}{2}\right) e^{(G-L)/L}. \tag{20}$$

Now, Stolarsky's inequality (4), after replacing $a$ and $b$ by $\sqrt{a}$ and $\sqrt{b}$, respectively, and with Lin's inequality $L \leq A(1/3)$, easily implies

$$L(a,b) \leq \left(\frac{\sqrt[3]{a}+\sqrt[3]{b}}{2}\right)^3 \leq I^2\left(\sqrt{a}, \sqrt{b}\right) \leq I(a,b) \cdot e^{(G-L)/L} \leq I(a,b) \tag{21}$$

by (18), (19). This improves the inequality $L \leq I$.

## 5

Some interesting properties of the studied means follow from the following integral representations:

$$\log I(a,b) = \frac{1}{b-a}\int_a^b \log x dx, \tag{22}$$

$$\frac{1}{L(a,b)} = \frac{1}{b-a}\int_a^b \frac{1}{x} dx \tag{23}$$

$$A(a,b) = \frac{1}{b-a}\int_a^b x dx, \tag{24}$$

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{1}{x^2} dx, \tag{25}$$

where, in all cases, $0 < a < b$.

Using, in addition to (7), some integral inequalities, (22)-(25) give certain new relations involving the means $I, L, A, G$.

For $f(x) = 1/(x+m)$, $x \geq m$ we have

$$f'(x) < 0, \quad f'(x) + xf''(x) \geq 0,$$

thus $f^{-1}$ is log conv, so (7) implies

$$I(a,b) + m \geq L(a+m, b+m), \text{ for } \min(a,b) \geq m. \tag{26}$$

Analogously, letting $f(x) = \log(x+m)$, $m \geq 0$, one obtains

$$I(a+m, b+m) \geq I(a,b) + m, \ m \geq 0. \tag{27}$$

Notice that (26) and (27) written in a single line:

$$I(a+m, b+m) \geq I(a,b) + m \geq L(a+m, b+m) \tag{28}$$

for $\min(a,b) \geq m \geq 0$ improve also (in a certain sense) the inequality $I \geq L$.

The classical Jensen-Hadamard inequality ([10], [14]) states that if $f : [a,b] \to \mathbb{R}$ is continuous and convex, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{29}$$

For $f(x) = x \log x$, (29) gives

$$A^2 \leq I(a^2, b^2) \leq (a^a \cdot b^b)^{1/A} \tag{30}$$

If we observe that a simple integration by parts gives

$$\int_a^b x \log x dx = \frac{b^2 - a^2}{4} \log I(a^2, b^2). \tag{31}$$

Since $I^2(a,b) \leq A^2$, (30) refines relation (19):

$$I^2(a,b) \leq A^2 \leq I(a^2, b^2). \tag{32}$$

Let $[a,b] \subset \left[0, \frac{1}{2}\right)$ and $f : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \log(x/(1-x))$$

By $f''(x) = (2x-1)/x^2(1-x)^2 \leq 0$ and (29) we can derive

$$\frac{A}{A'} \geq \frac{I}{I'} \geq \frac{G}{G'} \text{ for } [a,b] \subset \left[0, \frac{1}{2}\right), \tag{33}$$

where we wrote

$$A' = A'(a,b) = A(1-a, 1-b),$$

$$I' = I'(a,b) = I(1-a, 1-b),$$

$$G' = G'(a,b) = G(1-a, 1-b).$$

This is a Ky Fan type inequality (see [3], [15]) for two numbers, in a stronger form (involving $I$ and $I'$).

Furthermore, amongst the many integral inequalities related to our situation, we mention two results. One is the classical Chebyshev inequality ([4], [7], [10]):

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \tag{34}$$

for $f, g$ having different types of monotonicity. Let

$$f(x) = \log x, \quad g(x) = \frac{1}{x \log x}.$$

Then (22), (23), (34) yield

$$L(\log a, \log b) \leq \log I(a,b) \tag{35}$$

if we notice that

$$\frac{\log\log b - \log\log a}{b-a} = \frac{1}{L(a,b)} \cdot \frac{1}{L(\log a, \log b)}.$$

The second result we deal with is contained in the following inequality. If $f : [a,b] \to \mathbb{R}$ is positive, continuous and convex (concave), then

$$\frac{1}{b-a}\int_a^b f^2(x)dx \underset{(\geq)}{\leq} \frac{1}{3}(f^2(a) + f(a)f(b) + f^2(b)) \tag{36}$$

with equality only for $f$ linear function.

For a proof of (36) denote by $K : [a, b] \to \mathbb{R}$ a linear function with the properties

$$K(a) = f(a), \quad K(b) = f(b).$$

Then

$$K(t) = \frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a),\ t \in [a, b].$$

Intuitively, the set $\{(t, z) :\ t \in [a, b],\ z = K(t)\}$ represents the line segment joining the points $(a, f(a))$, $(b, f(b))$ of the graph of $f$. The function $f$ being convex, we have

$$f(t) \leq K(t),\ t \in [a, b]$$

and, because $f$ is positive,

$$f^2(t) \leq K^2(t).$$

A simple computation gives

$$\int_a^b K^2(t)dt = (b-a) \cdot \frac{1}{3}(f^2(a) + f(a)f(b) + f^2(b)),$$

concluding the proof of (36).

Apply now (36) for $f(x) = 1/\sqrt{x}$, $0 < a < b$. We get the interesting inequality

$$\frac{3}{L} < \frac{1}{G} + \frac{2}{H} \tag{37}$$

where

$$H = H(a, b) = \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}$$

denotes the harmonic mean of $a$ and $b$. For another application choose

$$f(x) = \sqrt{\log x}$$

in (36). One obtains

$$I^3(a, b) > G^2(a, b) \cdot e^{\sqrt{\log a \cdot \log b}}. \tag{38}$$

**6**

Finally, we make two remarks. The mean

$$J = J(a,b) = \frac{1}{e} \cdot (b^a \cdot a^b)^{1/(b-a)}$$

is related to the mean $I$. In fact,

$$J(a,b) = \frac{1}{I\left(\frac{1}{a}, \frac{1}{b}\right)}. \tag{39}$$

Since $I(u,v) = I(v,u)$, $L(u,v) = L(v,u)$, by using certain inequalities for $I$, we can obtain information about $J$. Apply e.g.

$$I\left(\frac{1}{a}, \frac{1}{b}\right) > L\left(\frac{1}{a}, \frac{1}{b}\right)$$

and

$$I\left(\frac{1}{a}, \frac{1}{b}\right) < \frac{L^2\left(\frac{1}{a}, \frac{1}{b}\right)}{G\left(\frac{1}{a}, \frac{1}{b}\right)}$$

(see (3), (5)) in order to obtain

$$\frac{G^3}{L^2} < J < \frac{G^2}{L} \tag{40}$$

where we have used the fact that

$$L^2\left(\frac{1}{a}, \frac{1}{b}\right) = \frac{L^2(a,b)}{G^4(a,b)}, \quad G\left(\frac{1}{a}, \frac{1}{b}\right) = \frac{1}{G(a,b)}.$$

The second remark suggest a generalization of the studied means. Let $p : [a,b] \to \mathbb{R}$ be a strictly positive, integrable function and define

$$I_p(a,b) = \exp \frac{\int_a^b p(x) \log x dx}{\int_a^b p(x) dx} \tag{41}$$

$$A_p(a,b) = \frac{\displaystyle\int_a^b xp(x)dx}{\displaystyle\int_a^b p(x)dx} \tag{42}$$

$$\frac{1}{L_p(a,b)} = \frac{\displaystyle\int_a^b \frac{p(x)}{xdx}}{\displaystyle\int_a^b p(x)dx} \tag{43}$$

$$\frac{1}{G_p^2(a,b)} = \frac{\displaystyle\int_a^b \frac{p(x)}{x^2dx}}{\displaystyle\int_a^b p(x)dx}, \tag{44}$$

which reduce to (22)-(25) if $p(x) \equiv 1$, $x \in [a,b]$. By the well known Jensen inequality ([7], [10]):

$$\log \frac{\displaystyle\int_a^b f(x)p(x)dx}{\displaystyle\int_a^b p(x)dx} \geq \frac{\displaystyle\int_a^b p(x)\log f(x)dx}{\displaystyle\int_a^b p(x)dx} \tag{45}$$

applied to $f(x) = x$ and $f(x) = 1/x$, respectively, we get

$$L_p(a,b) \leq I_p(a,b) \leq A_p(a,b). \tag{46}$$

From the Cauchy-Schwarz inequality ([7]) we can easily obtain

$$\left(\int_a^b \left(\sqrt{p(x)}/x\right) \cdot \sqrt{p(x)}dx\right)^2 \leq \left(\int_a^b p(x)/x^2dx\right) \cdot \left(\int_a^b p(x)dx\right),$$

getting

$$L_p(a,b) \geq G_p(a,b). \tag{47}$$

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## 2.2 A note on some inequalities for means

### 1

The logarithmic and identric means of two positive numbers $a$ and $b$ are defined by

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \text{ for } a \neq b; \quad L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a$$

respectively.

Let

$$A = A(a,b) := \frac{a+b}{2} \quad \text{and} \quad G = G(a,b) := \sqrt{ab}$$

denote the arithmetic and geometric means of $a$ and $b$, respectively. For these means many interesting results, especially inequalities, have been proved (see e.g. [1], [2], [3], [5], [6], [7], [10]). Recently, in two interesting papers, H. Alzer [1], [2] has obtained the following inequalities:

$$A \cdot G \leq L \cdot I \quad \text{and} \quad L + I < A + G \tag{1}$$

$$\sqrt{G \cdot I} < L < \frac{1}{2}(G + I) \tag{2}$$

which hold true for all positive $a \neq b$.

The aim of this note is to prove that the left side of (1) is weaker than the left side of (2), while the right side of (1) is stronger than the right side of (2). Namely, we will prove:

$$A \cdot G/I < \sqrt{G \cdot I} < L \tag{3}$$

$$L < A + G - I < \frac{1}{2}(G + I). \tag{4}$$

## 2

The left side of (3) may be proved in different ways. Apply, e.g. the well-known Simpson quadrature formula ([4]):

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad (5)$$

$\xi \in (a,b)$, where $f : [a,b] \to \mathbb{R}$ has a continuous 4-th derivative on $(a,b)$, for $f(x) = -\ln x$. Since $f^{(4)}(x) > 0$, a simple derivation from (5) gives:

$$I^3 > A^2 \cdot G \quad (6)$$

i.e. the desired result. For the method, based on integral inequalities, see also [7], [8], [9].

A slightly stronger relation can be obtained by the following way:

T.P. Lin [6] and K.B. Stolarsky [10] have proved that for $a \neq b$ one has:

$$L(a,b) < \left(\frac{a^{1/3} + b^{1/3}}{2}\right)^3 \quad (7)$$

and

$$\left(\frac{a^{2/3} + b^{2/3}}{2}\right)^{3/2} < I(a,b), \quad (8)$$

respectively. Set $a = x^2$, $b = y^2$ in (7) and remark that

$$L(x^2, y^2) = A(x,y) \cdot L(x,y),$$

so via (8) we get:

$$\sqrt{A \cdot L} < I. \quad (9)$$

Now, it is easy to see that $\sqrt{A \cdot L} > \sqrt[3]{A^2 \cdot G}$, since this is exactly a result of B.C. Carlson [3]:

$$\sqrt[3]{G^2 \cdot A} < L. \quad (10)$$

## 3

For establish (1) and (2), H. Alzer [1], [2] has applied an ingenious method attributed to E.B. Leach and M.C. Sholander [5]. This can be summarized as follows:

Let (e.g. in (1)) $a = e^t$, $b = e^{-t}$, $t \in \mathbb{R}$, and prove (by using certain hyperbolic functions) the corresponding inequality. Then replace $t$ by $\frac{1}{2}\ln\frac{x}{y}$ and multiply both sides of the proved inequality by $\sqrt{xy}$.

In what follows we shall prove by a different argument the following:

**Theorem.** *For $a \neq b$ one has:*

$$I > \frac{2A+G}{3} > \frac{A+L}{2}. \tag{11}$$

**Proof.** First we note that the second inequality in (11), written in the form

$$L < \frac{2G+A}{3} \tag{12}$$

has been proved by E.B. Leach and M.C. Sholander [5].

For the first inequality divide all terms by $a < b$ and denote $x := \frac{b}{a} > 1$. Then the inequality to be proved is transformed into

$$\frac{x+1\sqrt{x}}{g(x)} < \frac{3}{e} \tag{13}$$

where

$$g(x) = x^{x/(x-1)},\ x > 1.$$

Introduce the function $f : [1, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{x+1+\sqrt{x}}{g(x)},\ x > 1; \quad f(1) = \lim_{x\to 1+} f(x) = \frac{3}{e}.$$

If we are able to prove that $f$ is strictly decreasing, then clearly (13) and (11) is proved. On has

$$g'(x) = g(x) \cdot \left[\frac{1}{x-1} - \frac{\ln x}{(x-1)^2}\right],$$

and after some elementary calculations, we can deduce:

$$\begin{aligned} & 2\sqrt{x} \cdot (x-1)^2 g(x) f'(x) \\ & = 2(\ln x)\sqrt{x}\left(x+1+\sqrt{x}\right) - 4\sqrt{x}(x-1) - (x^2-1). \end{aligned} \tag{14}$$

We now show that the right side of (14) is strictly negative, or equivalently

$$L > \frac{G \cdot (2A+G)}{A+2G} \tag{15}$$

where in our case $L = L(x,1)$, etc.

The obvious inequality $u/v > (2u^3+v^3)/(u^3+2v^3)$, for $u > v$ applied for $u = \sqrt[3]{A}$, $v = \sqrt[3]{G}$, leads to

$$G^2 A > \frac{G^3(2A+G)^3}{(A+2G)^3},$$

thus by (10), relation (15) is valid. This finishes the proof of the theorem. Since the right side of (4) is exactly the first part of (11), we have completed our aim stated at the beginning of the paper.

**Remark.** By $(A+L)/2 > \sqrt{A \cdot L}$, (11) gives a refinement and a new proof for (9).

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## 2.3 Refinements of certain inequalities for means

### 1

The logarithmic and identric means of two positive numbers $a$ and $b$ are defined by

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \text{ for } a \neq b; \quad L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a,$$

respectively.

Let

$$A = A(a,b) := \frac{a+b}{2} \quad \text{and} \quad G = G(a,b) := \sqrt{ab}$$

denote the arithmetic and geometric means of $a$ and $b$, respectively. For these means many interesting inequalities have been proved. For a survey of results, see [1] and [6].

The aim of this note is to indicate some connections between the following inequalities. B.C. Carlson [3] proved that

$$L < \frac{2G+A}{3} \tag{1}$$

(where, as in what follows, $L = L(a,b)$, etc., and $a \neq b$) while E.B. Leach and M.C. Sholander [4] showed that

$$L > \sqrt[3]{G^2 A} \tag{2}$$

These two inequalities appear in many proofs involving means. H. Alzer [1], [2] has obtained the following inequalities:

$$A \cdot G < L \cdot I \quad \text{and} \quad L + I < A + G \tag{3}$$

$$\sqrt{G \cdot I} < L < \frac{G+I}{2} \tag{4}$$

J. Sándor [7] has proved that the first inequality of (3) is weaker that the left side of (4), while the second inequality of (3) is stronger than the right side of (4). In fact, the above statement are consequences of

$$I > \sqrt[3]{A^2 G} \tag{5}$$

and

$$I > \frac{2A+G}{3}. \tag{6}$$

Clearly, (6) implies (5), but one can obtain different methods of proof for these results (see [7]). In [6] J. Sándor has proved (relation 21 in that paper) that

$$\ln \frac{I}{L} > 1 - \frac{G}{L}. \tag{7}$$

## 2

Particularly, as application of (7), one can deduce (1) and the right side of (4). First we note that (e.g. from (1), (2) and (5))

$$G < L < I < A. \tag{8}$$

Let $x > 1$. Then $L(x,1) > G(x,1)$ implies $\ln x < (x-1)/\sqrt{x}$ which applied to $x = \dfrac{I}{L} > 1$ gives, in view of (7):

$$(I-L)\sqrt{L} > (L-G)\sqrt{I}. \tag{9}$$

This inequality contains a refinement of the right side of (4), for if we put $a = \sqrt{I}/\sqrt{L} > 1$, (9) gives

$$L < \frac{I+aG}{1+a} < \frac{I+G}{2} \tag{10}$$

since the function $a \mapsto (I+aG)/(1+a)$ $(a \geq 1)$ is strictly decreasing. Now, inequality $L(x,1) < A(x,1)$ for $x > 1$ yields

$$\ln x > \frac{2(x-1)}{x+1}.$$

Since

$$\ln \frac{I}{G} = \frac{A - L}{L}$$

(which can be obtained immediately by simple computations) and

$$\ln \frac{I}{L} = \ln \frac{I}{G} - \ln \frac{L}{G},$$

from $\ln \frac{L}{G} > 2 \cdot \frac{L-G}{L+G}$ in (7) one obtains

$$2 \cdot \frac{L-G}{L+G} < \frac{A+G}{L} - 2 \tag{11}$$

By $L > G$ this refines Carlson's inequality (11), since by $L + G < 2L$ one has

$$\frac{2(L-G)}{L+G} > 1 - \frac{G}{L},$$

so by (11) one can derive

$$0 < \frac{(L-G)^2}{L(L+G)} < \frac{A+2G}{L} - 3. \tag{12}$$

## 3

Inequality (6) and (1) improves also the right side of (4). This follows by

$$I > \frac{2A+G}{3} > 2L - G \tag{13}$$

where the second relation is exactly (1). We note that

$$\sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A}$$

follows by (5), so from the left side of (4) one can write:

$$L > \sqrt{G \cdot I} > \sqrt[3]{G^2 \cdot A} \tag{14}$$

improving inequality (2). The left side of (4) can be sharpened also, if we use the second inequality of (3). Indeed, by the identity

$$\ln \frac{I}{G} = \frac{A-L}{L} \quad \text{and} \quad \ln x < \frac{x-1}{\sqrt{x}}$$

applied with $x = \dfrac{I}{G} > 1$ one can deduce

$$\sqrt{IG} < \frac{I-G}{A-L} \cdot L < L. \tag{15}$$

**Remark.** Inequality (6) with (1) can be written also as

$$I > \frac{2A+G}{3} > \frac{A+L}{2}. \tag{16}$$

Relation

$$I > \frac{A+L}{2} \tag{17}$$

appears also in [6], inequality (9). Since $\dfrac{A+L}{2} > \sqrt{AL}$, one has

$$I > \sqrt{AL}. \tag{18}$$

For a simple method of proof of (18), see [7]. As an application of (18) we note that in a recent paper M.K. Vamanamurthy and M. Vuorinen [11] have proved, among other results, that for the arithmetic-geometric mean $M$ of Gauss we have

$$M < \sqrt{AL} \tag{19}$$

$$M < I. \tag{20}$$

Now, by (18), relation (20) is a consequence of (19). In the above mentioned paper [11] the following open problem is stated:

Is it true that $I < \left(\dfrac{a^t + b^t}{2}\right)^{1/t} = S(t)$ for some $t \in (0, 1)$?

We note here that by a result of A.O. Pittinger [5] this is true for $t = \ln 2$. The reversed inequality $I > S(t)$ is valid for $t = \dfrac{2}{3}$ as has been proved by K.B. Stolarsky [10]. The values given by Pittinger and Stolarsky are best possible, so $I$ and $S(t)$ are not comparable for $t < \ln 2$ and $t > \dfrac{2}{3}$, respectively.

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## 2.4 On certain identities for means

### 1. Introduction

Let

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a \ (a,b > 0)$$

denote the identric mean of the positive real numbers $a$ and $b$. Similarly, consider the logarithmic mean

$$L = L(a,b) = (b-a)/\log(b/a) \text{ for } a \neq b; \quad L(a,a) = a.$$

Usually, the arithmetic and geometric means are denoted by

$$A = A(a,b) = \frac{a+b}{2} \quad \text{and} \quad G = G(a,b) = \sqrt{ab},$$

respectively. We shall consider also the exponential mean

$$E = E(a,b) = (ae^a - be^b)/(e^a - e^b) - 1 \text{ for } a \neq b; \quad E(a,a) = a.$$

These means are connected to each others by many relations, especially inequalities which are valid for them. For a survey of results, as well as an extended bibliography, see e.g. H. Alzer [1], J. Sándor [6], J. Sándor and Gh. Toader [8]. The aim of this paper is to prove certain identities for these means and to connect these identities with some known results. As it will be shown, exact identities give a powerful tool in proving inequalities. Such a method appears in [6] (Section 4 (page 265) and Section 6 (pp. 268-269)), where it is proved that

$$\log \frac{I^2\left(\sqrt{a}, \sqrt{b}\right)}{I(a,b)} = \frac{G-L}{L} \tag{1}$$

where $G = G(a,b)$ etc. This identity enabled the author to prove that (see [6], p. 265)

$$L^2 \leq I \cdot \left(\frac{a+G}{2}\right) \cdot e^{(G-L)/L} \tag{2}$$

and

$$L \leq I \cdot e^{(G-L)/L} \quad \left(\text{i.e. } \log \frac{I}{L} \geq 1 - \frac{G}{L}\right) \tag{3}$$

In a recent paper [9] it is shown how this inequality improves certain known results.

In [10] appears without proof the identity

$$\log \frac{I}{G} = \frac{A-L}{L} \tag{4}$$

We will prove that relations of type (1) and (4) have interesting consequences, giving sometimes short proofs for known results of refinements of these results.

## 2. Identities and inequalities

Identity (4) can be proved by a simple verification, it is more interesting the way of discovering it. By

$$\log I(a,b) = \frac{b\log b - a\log a}{b-a} - 1 = \frac{b(\log b - \log a)}{b-a} + \log a - 1 \tag{$*$}$$

it follows that

$$I(a,b) = \frac{b}{L(a,b)} + \log a - 1,$$

and by symmetry,

$$\log I(a,b) = \frac{a}{L(a,b)} + \log b - 1 \ (a \neq b)$$

i.e.

$$\log \frac{I}{a} = \frac{b}{L} - 1 \quad \text{and} \quad \log \frac{I}{b} = \frac{a}{L} - 1 \tag{5}$$

Now, by addition of the two identities from (5) we get relation (4). From (5), by multiplication it results:

$$\log \frac{I}{a} \cdot \log \frac{I}{b} = \frac{G^2}{L^2} - 2 \cdot \frac{A}{L} + 1 \tag{6}$$

and similarly

$$\log \frac{I}{a} / \log \frac{I}{b} = \frac{b-L}{a-L} \tag{7}$$

As analogous identity to (1) can be proved by considering the logarithm of identric mean. Indeed, apply the formula $(*)$ to $a \to \sqrt[3]{a}$, $b \to \sqrt[3]{b}$. After some elementary transformations, we arrive at:

$$\log \frac{I^3\left(\sqrt[3]{a}, \sqrt[3]{b}\right)}{I(a,b)} = \frac{2\sqrt[3]{G^2 \cdot M}}{L} - 2, \tag{8}$$

where

$$M = A_{1/3}(a,b) = \left(\frac{\sqrt[3]{a} + \sqrt[3]{b}}{2}\right)^3$$

denotes the power mean of order 1/3. More generally, one defines

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}.$$

Now, Lin's inequality states that

$$L(u,v) \le M(u,v)$$

(see [5]), and Stolarsky's inequality ([11]) that

$$I(u,v) \ge A_{2/3}(u,v).$$

Thus one has

$$I^3\left(\sqrt[3]{a}, \sqrt[3]{b}\right) \ge \left\{\left(\frac{(a^{2/3})^{1/3} + (b^{2/3})^{1/3}}{2}\right)^3\right\}^{3/2} \ge L^{3/2}(a^{2/3}, b^{2/3})$$

by the above inequalities applied to $u = a^{1/3}$, $v = b^{1/3}$ and $u = a^{2/3}$, $v = b^{2/3}$, respectively. Thus

$$I^3\left(\sqrt[3]{a}, \sqrt[3]{b}\right) \ge L^{3/2}(a^{2/3}, b^{2/3}). \tag{9}$$

This inequality, via (8) gives:

$$L^{3/2}(a^{2/3}, b^{2/3}) \le I(a,b) \cdot e^{2\sqrt[3]{G^2 \cdot M}/L - 2} \tag{10}$$

or

$$\log \frac{I}{L^{3/2}(a^{2/3}, b^{2/3})} \ge 2 - \frac{2\sqrt[3]{G^2 \cdot M}}{L}. \tag{11}$$

This is somewhat similar (but more complicated) to (3).

Finally, we will prove certain less known series representations of $\log \dfrac{A}{G}$ and $\log \dfrac{I}{G}$, with applications.

First, let us remark that

$$\log \frac{A(a,b)}{G(a,b)} = \log \frac{a+b}{2\sqrt{ab}} = \log \frac{1}{2}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right).$$

Put $z = \dfrac{b-a}{b+a}$ (with $b > a$), i.e. $t = \dfrac{1+z}{1-z}$, where $t = \dfrac{a}{b} \in (0,1)$. Since

$$\frac{1}{2}\left(\sqrt{\frac{1+z}{1-z}} + \sqrt{\frac{1-z}{1+z}}\right) = \frac{1}{\sqrt{1-z^2}}$$

and

$$\log 1/\left(\sqrt{1-z^2}\right) = -\frac{1}{2}\log(1-z^2) = \frac{1}{2}z^2 + \frac{1}{4}z^4 + \dots$$

(by $\log(1-u) = -u - \dfrac{u^2}{2} - \dfrac{u^3}{3} - \dots$), we have obtained:

$$\log \frac{A(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{b-a}{b+a}\right)^{2k}. \tag{12}$$

In a similar way, we have

$$\frac{A}{L} - 1 = \frac{a+b}{2}\left(\frac{\log b - \log a}{b-a}\right) - 1 = \frac{1}{2z}\log\frac{1+z}{1-z} - 1$$

$$= \frac{1}{z}\operatorname{arctanh} z - 1 = \frac{z^2}{3} + \frac{z^4}{5} + \dots,$$

implying, in view of (4),

$$\log \frac{I(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \cdot \left(\frac{b-a}{b+a}\right)^{2k} \tag{13}$$

The identities (12) and (13) have been transmitted (without proof) to the author by H.J. Seiffert (particular letter). We note that parts of these relations have appeared in other equivalent forms in a number of places. For (13) see e.g. [4]. (Nevertheless, (4) is not used, and the form is slightly different).

Clearly, (12) and (13) imply, in a simple manner, certain inequalities. By

$$\frac{z^2}{3} < \frac{z^2}{3} + \frac{z^4}{5} + \ldots < \frac{z^2}{3}(1 + z^2 + z^4 + \ldots) = \frac{z^2}{3} \cdot \frac{1}{1-z^2}$$

we get

$$1 + (b-a)^2/3(b+a)^2 < \log \frac{I}{G} < 1 + (b-a)^2/12ab \tag{14}$$

improving the inequality $I > G$. On the same lines, since

$$\frac{z^2}{2} < \frac{z^2}{2} + \frac{z^4}{4} + \ldots < \frac{1}{2}(z^2 + z^4 + \ldots) = \frac{z^2}{2} \cdot \frac{1}{1-z^2}$$

one obtains

$$\frac{1}{2}(b-a)^2/(b+a)^2 < \log \frac{A}{G} < (b-a)^2/8ab. \tag{15}$$

## 3. Applications

We now consider some new applications of the found identities.

a) Since it is well-known that $\log x < x - 1$ for all $x > 0$, by (4) we get

$$A \cdot G < L \cdot I \tag{16}$$

discovered by A. Alzer [1]. By considering the similar inequality

$$\log x > 1 - \frac{1}{x} \ (x > 0),$$

via (4) one obtains

$$\frac{A}{L} + \frac{G}{I} > 2 \tag{17}$$

due to H.J. Seiffert (particular letter).

b) The double inequality $G(x,1) < L(x,1) < A(x,1)$ for $x > 1$ (see the References from [5]) can be written as

$$2 \cdot \frac{x-1}{x+1} < \log x < \frac{x-1}{\sqrt{x}} \ (x > 1) \tag{18}$$

Let $x = \dfrac{I}{G} > 1$ in (18). By using (4) one obtains:

$$2 \cdot \frac{I-G}{I+G} < \frac{A-L}{L} < \frac{I-G}{\sqrt{IG}}. \tag{19}$$

These improve (16) and (17), since

$$\frac{2(I-G)}{I+G} > 1 - \frac{G}{I} \quad \text{and} \quad \frac{I-G}{\sqrt{IG}} < \frac{I}{G-1}.$$

Let us remark also that, since it is known that ([1])

$$I - G < A - L,$$

the right side of (19) implies

$$\sqrt{IG} < \frac{I-G}{A-L} \cdot L < L, \tag{20}$$

improving $\sqrt{IG} < L$ (see [2]).

c) For another improvement of (16), remark that the following elementary inequality is known:

$$e^x > 1 + x + \frac{x^2}{2} \ (x > 0) \tag{21}$$

This can be proved e.g. by the classical Taylor expansion of the exponential function. Now, let $x = A/L - 1$ in (21). By (4) one has

$$I = G \cdot e^{A/L-1} > G\left[\frac{A}{L} + \frac{1}{2}\left(\frac{A}{L} - 1\right)^2\right] = \frac{1}{2}G\left(1 + \frac{A^2}{L^2}\right).$$

Thus we have

$$G \cdot A < \frac{1}{2} \cdot \frac{G}{L}(L^2 + A^2) < L \cdot I, \tag{22}$$

since the left side is equivalent with $2LA < L^2 + A^2$. This result has been obtained in cooperation with H.J. Seiffert.

d) Let us remark that one has always

$$\log \frac{I}{a} \cdot \log \frac{I}{b} < 0,$$

since, when $a \neq b$, $I$ lies between $a$ and $b$. So, from (6) one gets

$$G^2 + L^2 < 2A \cdot L, \tag{23}$$

complementing the inequality $2A \cdot L < A^2 + L^2$.

e) By identities (1) and (4) one has

$$\frac{2G + A}{L} = 3 + \log \frac{I^4\left(\sqrt{a}, \sqrt{b}\right)}{I(a,b) \cdot G(a,b)}. \tag{24}$$

In what follows we shall prove that

$$\frac{I^4\left(\sqrt{a}, \sqrt{b}\right)}{I(a,b) \cdot G(a,b)} \geq 1, \tag{25}$$

thus (by (24)), obtaining the inequality

$$L \leq \frac{2G + A}{3} \tag{26}$$

due to B.C. Carlson [3]. In fact, as we will see, a refinement will be deduced.

Let us define a new mean, namely

$$S = S(a,b) = (a^a \cdot b^b)^{1/2A} = (a^a \cdot b^b)^{1/(a+b)} \tag{27}$$

which is indeed a mean, since if $a < b$, then $a < S < b$. First remark that in [6] (inequality (30)) it is proved that

$$A^2 \leq I(a^2, b^2) \leq S^2(a,b). \tag{28}$$

(However the mean $S$ is not used there). In order to improve (28), let us apply Simpson's quadrature formulas (as in [7])

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi),$$

$\xi \in (a,b)$, to the function $f(x) = x\log x$. Since $f^{(4)}(x) > 0$ and

$$\int_a^b x\log x dx = \frac{b^2-a^2}{4}\log I(a^2,b^2)$$

(see [6], relation (3.1)), we can deduce that

$$I^3(a^2,b^2) \le S^2 \cdot A^4. \tag{29}$$

Now, we note that for the mean $S$ the following representation is valid:

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}. \tag{30}$$

This can be discovered by the method presented in part 2 of this paper (see also [6]). By (29) and (30) one has

$$I(a^2,b^2) \le A^4/I^2(a,b) \tag{31}$$

which is stronger than relation (25). Indeed, we have

$$I(a^2,b^2) \le A^4/I^2(a,b) \le \frac{I^4(a,b)}{G^2(a,b)},$$

since this last inequality is

$$I^3 \ge A^2 \cdot G \tag{32}$$

due to the author [7]. Thus we have (by putting $a \to \sqrt{a}$, $b \to \sqrt{b}$ in (31))

$$\frac{I^4\left(\sqrt{a},\sqrt{b}\right)}{I(a,b)\cdot G(a,b)} \ge \frac{A^4\left(\sqrt{a},\sqrt{b}\right)}{I^2\left(\sqrt{a},\sqrt{b}\right) I(a,b)} \ge 1 \tag{33}$$

giving (by (1)):

$$\frac{2G+A}{L} \geq 3 + \log \frac{A^4\left(\sqrt{a},\sqrt{b}\right)}{I^2\left(\sqrt{a},\sqrt{b}\right)\cdot I(a,b)} \geq 3, \tag{34}$$

improving (26).

f) If $a < b$, then $a < I < b$ and the left side of (5), by taking into account of (18), implies

$$1 - \frac{a}{I} < 2 \cdot \left(\frac{I-A}{I+a}\right) < \frac{b-L}{L} < \frac{I-a}{\sqrt{aI}} < \frac{I}{a} - 1. \tag{35}$$

Remark that the weaker inequalities of (35) yields

$$\frac{b}{L} + \frac{a}{I} > 2. \tag{36}$$

Similarly, from (5) (right side) one obtains:

$$\frac{a}{L} + \frac{b}{I} > 2. \tag{37}$$

g) For the exponential mean $E$ a simple observation gives

$$\log I(e^a, e^b) = E(a,b),$$

so via (4) we have

$$E - A = \frac{A(e^a, e^b)}{L(e^a, e^b)} - 1. \tag{38}$$

Since $A > L$, this gives the inequality

$$E > A \tag{39}$$

due to Gh. Toader [11]. This simple proof explains in fact the meaning of (39). Since $I^3 > A^2G$ (see [6]), the following refinement is valid

$$E > \frac{A + 2\log A(e^a, e^b)}{3} > A, \tag{40}$$

where the last inequality holds by $(e^a + e^b)/2 > e^{(a+b)/2}$, i.e. the Jensen convexity of $e^x$.

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## 2.5 Inequalities for means

### 1

Let $a, b > 0$ be positive real numbers. The "identric mean" $I(a,b)$ of $a$ and $b$ is defined by

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b;\ I(a,a) = a;$$

while the "logarithmic mean" $L(a,b)$ of $a$ and $b$ is

$$L(a,b) = (b-a)/(\log b - \log a) \text{ for } a \neq b;\ L(a,a) = a.$$

Denote

$$M_t = M_t(a,b) = \left(\frac{a^t + b^t}{2}\right)^{1/t} \quad \text{for } t \neq 0$$

the root-power mean of $a$ and $b$. Plainly,

$$M_1(a,b) = A(a,b) = t,\ M_0(a,b) = \lim_{t\to 0} M_t(a,b) = G(a,b) = G$$

are the arithmetic and geometric means of $a$ and $b$, respectively.

B. Ostle and H.L. Terwilliger [8] and B.C. Carlson [3], [4] have proved first that

$$G \leq L \leq A. \tag{1}$$

This result, or a part of it, has been rediscovered and reproved many times (see e.g. [6], [9], [10]).

In 1974 T.P. Lin [6] has obtained an important refinement of (1):

$$G \leq L \leq M_{1/3} \leq A. \tag{2}$$

For new proofs, see [9], [10], [11]. We note that $M_k$ is an increasing function of $k$, so $L \leq M_{1/3} \leq M_k \leq A$ for all $k \in [1/3, 1]$ but, as Lin showed, the number $1/3$ cannot be replaced by a better one.

For the identric mean, K.B. Stolarsky [13], [14], has proved that

$$L \leq I \leq A. \tag{3}$$

$$M_{2/3} \leq I \tag{4}$$

and the constant 2/3 in (4) is optimal.

Recently, inequalities (1) and (3) appear also in a problem proposed by Z. Zaiming [16]. The relations $G \leq L \leq M_{1/2} \leq I \leq A$ have been proved also by Z.H. Yang [15].

In [1] and [2] H. Alzer proved the following inequalities:

$$\sqrt{G \cdot I} \leq L \leq \frac{1}{2}(G + I) \tag{5}$$

$$A \cdot G \leq L \cdot I \quad \text{and} \quad A + G \geq L + I. \tag{6}$$

We notice that, in all inequalities (1)-(6), equality can occur only for $a = b$.

In [12] the following integral inequality is proposed. If $f : [a, b] \to \mathbb{R}$ is a strictly increasing function, having a logarithmically convex inverse function, then

$$\frac{1}{b-a} \int_a^b f(x)dx \leq f(I(a, b)). \tag{7}$$

Finally, we recall that ([10], Theorem 2, $k = 2$), for a 4-times differentiable function, having a continuous 4-th derivative on $[a, b]$, with $f^{(4)}(x) > 0$, $x \in (a, b)$, one has

$$\frac{1}{b-a} \int_a^b f(x)dx > f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} \cdot f''\left(\frac{a+b}{2}\right). \tag{8}$$

This is a refinement of the famous Hadamard inequality, and has interesting applications for the exponential function and logarithmic means (see [10], [11]).

## 2

Applying (8) for $f(t) = -\log t$, $t > 0$, an easy calculation shows that

$$\log \frac{A(a, b)}{I(a, b)} > \frac{1}{6}\left(1 - \frac{a}{A(a, b)}\right)^2 > 0. \tag{9}$$

Remarking that

$$\int_a^b x \log x dx = \frac{1}{4}(b^2 - a^2) \log I(a^2, b^2),$$

and letting $f(x) = x \log x$ in (8), we get

$$\log \frac{A^2(a,b)}{I(a^2,b^2)} < -\frac{1}{3}\left(1 - \frac{a}{A(a,b)}\right)^2 < 0. \tag{10}$$

As a simple consequence of (10) and (1), (3), we note that:

$$I^2(a,b) < A^2(a,b) < I(a^2,b^2). \tag{11}$$

Inequality $I^2(a,b) < I(a^2,b^2)$ follows also from the representation

$$\log \frac{I^2\left(\sqrt{a}, \sqrt{b}\right)}{I(a,b)} = \frac{G(a,b) - L(a,b)}{L(a,b)} \tag{12}$$

which can be obtained e.g. by writing

$$\log I(a,b) = \frac{b \log b - a \log a}{b - a} - 1.$$

On the other hand, the inequality

$$L^2\left(\sqrt{a}, \sqrt{b}\right) < I^2\left(\sqrt{a}, \sqrt{b}\right)$$

is transformed via (12) into

$$L^2 < I \cdot \left(\frac{A+G}{2}\right) \cdot e^{(G-L)/L}, \tag{13}$$

where $L = L(a,b)$, etc.

## 3

Using the method of [10], set $a = x^t$, $b = y^t$ $(x, y > 0,\ t \neq 0)$ in (1) we get

$$\frac{(xy)^{t/2} t(y-x)}{y^t - x^t} < L(x,y) < \frac{t(y-x)}{y^t - x^t} \cdot \frac{x^t + y^t}{2} \quad (x \neq y) \tag{14}$$

a double-inequality attributed to B.C. Carlson [4].

Let now $t > 1$ and $f(x) = x^{t-1}$. Then $f$ is increasing with $f^{-1}$ log. conc. (i.e. logarithmically concave). The proof of (7) shows that (7) is valid also in this case, with reversed sign of inequality. So, on account of (14) we obtain

$$L(a,b)\cdot(I(a,b))^{t-1} < L(a,b)\cdot\frac{b^t-a^t}{t(b-a)} < \frac{a^t+b^t}{2},\ t \neq 0,1. \tag{15}$$

Since for $t < 1$, $f$ is strictly decreasing with $f^{-1}$ strictly log. convex, (15) is valid also for $t < 1$.

Some particular cases for (15) are of interest to note:

For $t = 1/2$ one obtains (by (6)):

$$I > \frac{A+G}{2} > \frac{L+I}{2}. \tag{16}$$

For $t = 2$ we have:

$$\frac{a+b}{2}\cdot\sqrt{ab} < L(a,b)\cdot I(a,b) < \frac{a^2+b^2}{2}. \tag{17}$$

## 4

Let us first remark that, when $f(x) > 0$ and $f$ has a second order derivative, a simple computation proves that $f^{-1}$ is log. conv. iff

$$f'(x)[f'(x) + xf''(x)] < 0$$

and $f^{-1}$ is log. conc. iff

$$f'(x)[f'(x) + xf''(x)] > 0.$$

The proof in [12] can entirely be repeated in order to see that, in (7) we have the sign of inequality:

$$\begin{aligned} &\leq,\ \text{if } f' > 0,\ f^{-1} \text{ log. conv. } \mathbf{or}\ f' < 0,\ f^{-1} \text{ log. conc.}\\ &\geq,\ \text{if } f' > 0,\ f^{-1} \text{ log. conc. } \mathbf{or}\ f' < 0,\ f^{-1} \text{ log. conv.} \end{aligned} \tag{18}$$

where the sign of inequality is strict whenever $f^{-1}$ is strictly log. conv. or log. conc.

Let $f(x) = (\log x)/x$. Then

$$f'(x) = \frac{1 - \log x}{x^2}, \quad f'(x) + xf''(x) = \frac{\log x - 2}{x^2}.$$

Using (18), we can derive the relations

$$\begin{aligned} &G^I > I^L \text{ for } a, b \in (e^2, \infty), \\ &G^I < I^L \text{ for } a, b \in (0, e^2) \end{aligned} \tag{19}$$

## 5

For a generalization of the studied means, let $p : [a, b] \to \mathbb{R}$ be a strictly positive, integrable function and define:

$$I_p(a, b) = \exp \frac{\displaystyle\int_a^b p(x) \log x dx}{\displaystyle\int_a^b p(x) dx} \tag{20}$$

$$A_p(a, b) = \frac{\displaystyle\int_a^b xp(x) dx}{\displaystyle\int_a^b p(x) dx} \tag{21}$$

$$1/L_p(a, b) = \frac{\displaystyle\int_a^b p(x)/x dx}{\displaystyle\int_a^b p(x) dx} \tag{22}$$

$$1/G_p^2(a, b) = \frac{\displaystyle\int_a^b p(x)/x^2 dx}{\displaystyle\int_a^b p(x) dx} \tag{23}$$

When $p(x) \equiv 1$, we get the classical means $I, A, L, G$. By the well-known Jensen inequality ([5], [7])

$$\log \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx} \geq \frac{\int_a^b p(x)\log f(x)dx}{\int_a^b p(x)dx} \tag{24}$$

applied to $f(x) = x$ and $f(x) = 1/x$ respectively, we obtain

$$L_p(a,b) \leq I_p(a,b) \leq A_p(a,b). \tag{25}$$

By the Cauchy-Schwarz inequality [5] we can find easily

$$\left(\int_a^b \sqrt{p(x)}/x \cdot \sqrt{p(x)}dx\right)^2 \leq \left(\int_a^b p(x)/x^2 dx\right)\left(\int_a^b p(x)dx\right),$$

getting

$$G_p(a,b) \leq L_p(a,b). \tag{26}$$

Finally, we note that these results have been obtain by the author in 1989 [17].

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## 2.6 Inequalities for means of two arguments

### 1

The logarithmic and the identric mean of two positive numbers $x$ and $y$ are defined by

$$\begin{aligned} L = L(x,y) &:= \frac{y-x}{\log y - \log x}, \quad \text{if} \quad x \neq y, \quad L(x,x) = x, \\ I = I(x,y) &:= \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}, \quad \text{if} \quad x \neq y, \quad I(x,x) = x, \end{aligned} \tag{1.1}$$

respectively.

Let $A = A(x,y) := (x+y)/2$ and $G = G(x,y) := \sqrt{xy}$ denote the arithmetic and geometric mans of $x$ and $y$, respectively. Many interesting results are known involving inequalities between these means. For a survey of results (cf. [1], [3], [4], [11], [13], [14]). Certain improvements are proved in [5], [7], while connections to other means are discussed, (cf. [6], [8], [9], [10], [15]). For identities involving various means we quote the papers [6], [12].

In [5], [8], the first author proved, among other relations, that

$$(A^2G)^{1/3} < I, \tag{1.2}$$

$$(U^3G)^{1/4} < I < \frac{U^2}{A}, \tag{1.3}$$

where

$$U = U(x,y) := \left(\frac{8A^2+G^2}{9}\right)^{1/2}. \tag{1.4}$$

We note that a stronger inequality than (1.2) is (cf. [5])

$$\frac{2A+G}{3} < I, \tag{1.5}$$

but the interesting proof of (1.2), as well as the left-hand side of (1.3), is based on certain quadrature formulas (namely Simpson's and Newton's quadrature formula, respectively). As a corollary of (1.3) and (1.5), the double inequality

$$4A^2 + 5G^2 < 9I^2 < 8A^2 + G^2 \tag{1.6}$$

can be derived (see [8]). Here and throughout the rest of the paper we assume that $x \neq y$.

The aim of this paper is twofold. First, by applying the method of quadrature formulas, we will obtain refinements of already known inequalities (e.g., of (1.2)). Second, by using certain identities on series expansions of the considered expressions, we will obtain the best possible inequalities in certain cases (e.g., for (1.6)).

## 2

**Theorem 2.1.** *If $x$ and $y$ are positive real numbers, then*

$$\exp\left(\frac{(x-y)^2}{24s^2}\right) < \frac{A}{I} < \exp\left(\frac{(x-y)^2}{24r^2}\right), \tag{2.1}$$

$$\exp\left(\frac{(x-y)^2}{12s^2}\right) < \frac{I}{G} < \exp\left(\frac{(x-y)^2}{12r^2}\right), \tag{2.2}$$

$$\exp\left(\frac{(x-y)^4}{480s^4}\right) < \frac{I}{(A^2G)^{1/3}} < \exp\left(\frac{(x-y)^4}{480r^4}\right), \tag{2.3}$$

$$\exp\left(\frac{(x-y)^2}{96s^2}\right) < \frac{\sqrt{3A^2+G^2}}{2I} < \exp\left(\frac{(x-y)^2}{96r^2}\right), \tag{2.4}$$

*where* $r = \min\{x, y\}$ *and* $s = \max\{x, y\}$.

**Proof.** Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(t) = \log(tx + (1-t)y).$$

Since

$$f''(t) = -\frac{(x-y)^2}{(tx+(1-t)y)^2}, \tag{2.5}$$

we have

$$
\begin{aligned}
m_2 &:= \min\{-f''(t) \mid 0 \le t \le 1\} = \frac{(x-y)^2}{s^2}, \\
M_2 &:= \max\{-f''(t) \mid 0 \le t \le 1\} = \frac{(x-y)^2}{r^2}.
\end{aligned} \tag{2.6}
$$

Applying the "composite midpoint rule" (cf. [2]) we get

$$
\int_0^1 f(t)dt = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right) + \frac{1}{24n^2} f''(\xi_n),\ 0 < \xi_n < 1. \tag{2.7}
$$

Remarking that

$$
I = \exp\left(\int_0^1 \log(tx + (1-t)y)dt\right),
$$

relation (2.7) via (2.6) gives

$$
\exp\left(\frac{m_2}{24n^2}\right) < \frac{\exp\left(1/n \sum_{i=1}^{n} f((2i-1)/2n)\right)}{I} < \exp\left(\frac{M_2}{24n^2}\right). \tag{2.8}
$$

Letting $n = 1$, we get the double inequality (2.1). For $n = 2$, after a simple computation we deduce (2.4).

In order to prove (2.2), we apply the "composite trapezoidal rule" (see [2]):

$$
\int_0^1 f(t)dt = \frac{1}{2}[f(0) + f(1)] - \frac{1}{12} f''(\eta),\ 0 < \eta < 1. \tag{2.9}
$$

As above, taking into account (2.6), relation (2.9) yields (2.2).

Finally, (2.3) follows as application of the "composite Simpson rule" (see [2], [5]):

$$
\int_0^1 f(t)dt = \frac{1}{6} f(0) + \frac{2}{3} f\left(\frac{1}{2}\right) + \frac{1}{6} f(1) - \frac{1}{2880} f^{(4)}(\zeta),\ 0<\zeta<1. \tag{2.10}
$$

We omit the details. $\square$

**Remarks.** Inequality (2.8) is a common generalization of (2.1) and (2.4). The left-hand side of (2.3) is a refinement of (1.2), while the left-hand side of (2.4) implies the inequality

$$4I^2 < 3A^2 + G^2, \tag{2.11}$$

which slightly improves the right-side of (1.6). However, the best inequality of this type will be obtained by other methods.

In [6] the following identities are proved:

$$\log \frac{I}{G} = \sum_{k=1}^{\infty} \frac{1}{2k+1} z^{2k}, \tag{2.12}$$

$$\log \frac{A}{G} = \sum_{k=1}^{\infty} \frac{1}{2k} z^{2k}, \tag{2.13}$$

$$\log \frac{I}{G} = \frac{A}{L} - 1, \tag{2.14}$$

where $z = (x-y)/(x+y)$.

Relation (2.14) is due to H.-J. Seiffert [11]. With the aid of these and similar identities, strong inequalities can be deduced. We first state the following.

**Theorem 2.2.** *The following inequalities are satisfied:*

$$\exp\left(\frac{1}{6}\left(\frac{x-y}{x+y}\right)^2\right) < \frac{A}{I} < \exp\left(\frac{(x-y)^2}{24xy}\right), \tag{2.15}$$

$$\exp\left(\frac{1}{3}\left(\frac{x-y}{x+y}\right)^2\right) < \frac{I}{G} < \exp\left(\frac{(x-y)^2}{12xy}\right), \tag{2.16}$$

$$\exp\left(\frac{1}{30}\left(\frac{x-y}{x+y}\right)^4\right) < \frac{I}{(A^2G)^{1/3}} < \exp\left(\frac{(x-y)^4}{120xy(x+y)^2}\right). \tag{2.17}$$

**Proof.** We note that (2.16) appears in [6], while the left-hand side of (2.15) has been considered in [12]. But proved first in 1989 by J. Sándor, Inequalities for means, Proc. Third Symp. Math. Appl., Timişoara, 3-4 nov. 1989, pp. 87-90. We give here a unitary proof for (2.15), (2.16) and (2.17), which in fact shows that much stronger approximations may be deduced, if we want.

We assume that $x > y$, that is, $0 < z < 1$. Taking into account that

$$\frac{z^2}{3} < \sum_{k=1}^{\infty} \frac{1}{2k+1} z^{2k} < \frac{z^2}{3}(1 + z^2 + z^4 + \ldots) = \frac{z^2}{3(1-z^2)}, \tag{2.18}$$

from (2.12) we obtain the double-inequality (2.16).

On the other hand, (2.12) and (2.13) yield

$$\log \frac{A}{I} = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} z^{2k}. \tag{2.19}$$

Since

$$\frac{z^2}{6} < \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} z^{2k} < \frac{z^2}{6}(1 + z^2 + z^4 + \ldots) = \frac{z^2}{6(1-z^2)}, \tag{2.20}$$

via (2.19) we get at once (2.15).

To prove (2.17), let us remark that from (2.12) and (2.19) we have

$$\frac{I}{A^{2/3}G^{1/3}} = \exp\left( \sum_{k=2}^{\infty} \frac{k-1}{3k(2k+1)} z^{2k} \right). \tag{2.21}$$

Since

$$\frac{k-1}{3k(2k+1)} \leq \frac{1}{30} \text{ for all integers } k \geq 2, \tag{2.22}$$

from (2.21) we get as above (2.17). □

**Remarks.** Inequalities (2.15), (2.16) and (2.17) improve (2.1), (2.2) and (2.3). From (2.14), taking account of (2.16), one can deduce that

$$\frac{4(x^2 + xy + y^2)}{3(x+y)^2} < \frac{A}{L} < \frac{x^2 + 10xy + y^2}{12xy}. \tag{2.23}$$

In [4] it is proved that

$$\log \frac{I}{L} > 1 - \frac{G}{L}. \tag{2.24}$$

Inequality (2.24) enabled the first author to obtain many refinements of known results (see [7]).

If one uses the estimations

$$\frac{z^2}{6}+\frac{z^4}{20}<\sum_{k=1}^{\infty}\frac{1}{2k(2k+1)}z^{2k}<\frac{z^2}{6}+\frac{z^4}{20}(1+z^2+z^4+\ldots)$$
$$=\frac{z^2}{6}+\frac{z^4}{20(1-z^2)}, \tag{2.25}$$

as well as

$$\frac{z^2}{3}+\frac{z^4}{5}<\sum_{k=1}^{\infty}\frac{1}{2k+1}z^{2k}<\frac{z^2}{3}+\frac{z^4}{5}(1+z^2+z^4+\ldots)$$
$$=\frac{z^2}{3}+\frac{z^4}{5(1-z^2)}, \tag{2.26}$$

one could deduce the following inequalities:

$$\begin{aligned}
&\exp\left(\frac{1}{6}\left(\frac{x-y}{x+y}\right)^2+\frac{1}{20}\left(\frac{x-y}{x+y}\right)^4\right)<\frac{A}{I}\\
&\quad<\exp\left(\frac{1}{6}\left(\frac{x-y}{x+y}\right)^2+\frac{(x-y)^4}{80xy(x+y)^2}\right),\\
&\exp\left(\frac{1}{3}\left(\frac{x-y}{x+y}\right)^2+\frac{1}{5}\left(\frac{x-y}{x+y}\right)^4\right)<\frac{I}{G}\\
&\quad<\exp\left(\frac{1}{3}\left(\frac{x-y}{x+y}\right)^2+\frac{(x-y)^4}{20xy(x+y)^2}\right).
\end{aligned} \tag{2.27}$$

The next theorem provides a generalization of (2.17).

**Theorem 2.3.** *If $p$ and $q$ are positive real numbers with $2q\ge p$, then*

$$\exp\left(\frac{2q-p}{6}\left(\frac{x-y}{x+y}\right)^2+\frac{4q-p}{20}\left(\frac{x-y}{x+y}\right)^4\right)<\frac{I^{p+q}}{A^pG^q}$$
$$<\exp\left(\frac{2q-p}{6}\left(\frac{x-y}{x+y}\right)^2+\frac{4q-p}{80}\cdot\frac{(x-y)^4}{xy(x+y)^2}\right). \tag{2.28}$$

**Proof.** We assume that $x > y$, that is, $0 < z < 1$. From (2.12) and (2.19) we can deduce the following generalization of (2.21):

$$\frac{I^{p+q}}{A^p G^q} = \exp\left(\sum_{k=1}^{\infty} \frac{2kq - p}{2k(2k+1)} z^{2k}\right). \tag{2.29}$$

Since

$$\frac{2kq - p}{2k(2k+1)} \leq \frac{4q - p}{20} \text{ for all integers } k \geq 2, \tag{2.30}$$

we have

$$\begin{aligned} \frac{2q-p}{6} z^2 + \frac{4q-p}{20} z^4 &< \sum_{k=1}^{\infty} \frac{2kq - p}{2k(2k+1)} z^{2k} \\ &< \frac{2q-p}{6} z^2 + \frac{4q-p}{20} \cdot \frac{z^4}{1 - z^2}. \end{aligned} \tag{2.31}$$

The above estimation together with (2.29) yields (2.28). $\square$

**Remark 2.4.** For $p = 2/3$ and $q = 1/3$, (2.28) gives (2.17), while for $p = q = 1/2$ we get

$$\begin{aligned} \exp\left(\frac{1}{12}\left(\frac{x-y}{x+y}\right)^2 + \frac{3}{40}\left(\frac{x-y}{x+y}\right)^4\right) &< \frac{I}{\sqrt{AG}} \\ &< \exp\left(\frac{1}{12}\left(\frac{x-y}{x+y}\right)^2 + \frac{3}{160} \cdot \frac{(x-y)^4}{xy(x+y)^2}\right). \end{aligned} \tag{2.32}$$

**Theorem 2.5.** *If $x$ and $y$ are positive real numbers, then*

$$\exp\left(\frac{1}{45}\left(\frac{x-y}{x+y}\right)^4\right) < \frac{\sqrt{2A^2 + G^2}}{\sqrt{3}I} < \exp\left(\frac{1}{180} \cdot \frac{(x-y)^4}{xy(x+y)^2}\right). \tag{2.33}$$

**Proof.** Assume that $x > y$, that is, $0 < z < 1$. We prove first the following identity:

$$\log \frac{\sqrt{\alpha A^2 + G^2}}{\sqrt{\alpha + 1} I} = \sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{1}{2k+1} - \frac{1}{(\alpha+1)^k}\right) z^{2k}, \tag{2.34}$$

for all positive real numbers $\alpha$. Indeed, since

$$\log\sqrt{\alpha A^2+G^2}=\log\sqrt{xy}+\log\sqrt{1+\frac{\alpha}{4}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)^2}, \tag{2.35}$$

letting $z=(x-y)/(x+y)$ we obtain

$$\begin{aligned}\log\sqrt{\alpha A^2+G^2}&=\log G+\frac{1}{2}\log\left(1+\frac{\alpha}{1-z^2}\right)\\&=\log G+\frac{1}{2}\log(1+\alpha-z^2)-\frac{1}{2}\log(1-z^2).\end{aligned} \tag{2.36}$$

By the well-known formula

$$\log(1-u)=-\sum_{k=1}^{\infty}\frac{u^k}{k},\ 0<u<1, \tag{2.37}$$

we can deduce

$$\begin{aligned}&\log(1+\alpha-z^2)=\log(1+\alpha)-\textstyle\sum_{k=1}^{\infty}\dfrac{z^{2k}}{k(\alpha+1)^k},\\&\log(1-z^2)=-\sum_{k=1}^{\infty}\frac{z^{2k}}{k}.\end{aligned} \tag{2.38}$$

Thus

$$\log\frac{\sqrt{\alpha A^2+G^2}}{\sqrt{\alpha+1}G}=\sum_{k=1}^{\infty}\frac{1}{2k}\left(1-\frac{1}{(\alpha+1)^k}\right)z^{2k}. \tag{2.39}$$

This identity combined with (2.12) ensures the validity of (2.34).

For $\alpha=2$, (2.34) yields

$$\log\frac{\sqrt{2A^2+G^2}}{\sqrt{3}I}=\sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{1}{2k+1}-\frac{1}{3^k}\right)z^{2k}. \tag{2.40}$$

Since

$$\frac{1}{2k}\left(\frac{1}{2k+1}-\frac{1}{3^k}\right)\leq\frac{1}{45}\text{ for all integers }k\geq 2, \tag{2.41}$$

we have

$$\frac{z^4}{45}<\sum_{k=2}^{\infty}\frac{1}{2k}\left(\frac{1}{2k+1}-\frac{1}{3^k}\right)z^{2k}<\frac{z^4}{45(1-z^2)}. \tag{2.42}$$

This estimation together with (2.40) gives (2.33). $\square$

**Remarks.** From (2.33) it follows that

$$3I^2 < 2A^2 + G^2. \tag{2.43}$$

This inequality refines (2.11) and it is the best inequality of the type

$$I^2 < \frac{\alpha}{\alpha+1}A^2 + \frac{1}{\alpha+1}G^2. \tag{2.44}$$

Indeed, the function $f : ]0, \infty[ \to \mathbb{R}$ defined by

$$f(\alpha) = \frac{\alpha}{\alpha+1}A^2 + \frac{1}{\alpha+1}G^2$$

is increasing because $A > G$. Taking into account (2.43) we get

$$I^2 < \frac{2}{3}A^2 + \frac{1}{3}G^2 < \frac{\alpha}{\alpha+1}A^2 + \frac{1}{\alpha+1}G^2 \tag{2.45}$$

whenever $\alpha > 2$. On the other hand, if $0 < \alpha < 2$, from (2.34) it follows that (2.44) cannot be true for all positive numbers $x \neq y$.

The fact that (2.43) is the best inequality of the type (2.44) can be proved also by elementary methods, without resorting to series expansion (2.12). Indeed, letting $t = (1/2)(x/y - 1)$, and assuming that $x > y$, it is easily seen that (2.44) is equivalent to

$$0 < 2t - (1+2t)\log(1+2t) + t\log\left(1 + 2t + \frac{\alpha}{\alpha+1}t^2\right) \tag{2.46}$$

whenever $t > 0$. Let $g_\alpha : ]0, \infty[ \to \mathbb{R}$ be the function defined by

$$g_\alpha(t) = 2t - (1+2t)\log(1+2t) + t\log\left(1 + 2t + \frac{\alpha}{\alpha+1}t^2\right). \tag{2.47}$$

We set, for convenience, $g_2 := g$. Easy computations give

$$\begin{aligned} g'(t) &= \frac{2t + (4/3)t^2}{1 + 2t + (2/3)t^2} + \log\left(1 + 2t + \frac{2}{3}t^2\right) - 2\log(1+2t), \\ g''(t) &= \frac{8t^3}{9(1+2t)(1+2t+(2/3)t^2)^2}. \end{aligned} \tag{2.48}$$

Since $g''(t) > 0$ for all $t > 0$, $g'$ must be increasing. Therefore, $g'(t) > 0$ for $t > 0$, because $g'(0) = 0$. Consequently $g$ is increasing, too. Hence $g(t) > 0$ whenever $t > 0$, because $g(0) = 0$. This guarantees the validity of (2.46) for $\alpha = 2$. Thus (2.43) is proved.

On the other hand, since

$$\begin{aligned} &\log(1+2t) = 2t - 2t^2 + \frac{8t^3}{3} + o(t^3), \\ &\log\left(1+2t+\frac{\alpha}{\alpha+1}t^2\right) = 2t + \frac{\alpha}{\alpha+1}t^2 - \frac{1}{2}\left(2t+\frac{\alpha}{\alpha+1}t^2\right) + o(t^2), \end{aligned} \tag{2.49}$$

it follows that

$$g_\alpha(t) = \left(\frac{\alpha}{\alpha+1} - \frac{2}{3}\right)t^3 + o(t^3). \tag{2.50}$$

Therefore (2.46) cannot be true for all positive real numbers $t$ if $0<\alpha<2$.

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## 2.7 An application of Gauss' quadrature formula

### 1

The aim of this note is to point out a new proof of a result from [6] on the theory of means, by application of Gauss' quadrature formula with two nodes. The fact that quadrature formulae are of interest in the theory of means has been first shown by the author in [4], where

- Simpson's quadrature formula:

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880}\cdot f^{(4)}(\xi) \tag{1}$$

with $\xi \in [a,b]$, has been applied.

- The Newton quadrature formula:

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{648}\cdot f^{(4)}(\xi) \tag{2}$$

has been applied in [5]. Further, (1) has been applied, in [7], too. We now offer an application to the

- Gauss quadrature formula, with two nodes (see e.g. [1], [2], [3])

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f\left(\frac{a+b}{2}-\frac{b-a}{6}\cdot\sqrt{3}\right)+f\left(\frac{a+b}{2}+\frac{b-a}{6}\cdot\sqrt{3}\right)\right] + \frac{1}{4320}\cdot f^{(4)}(\xi). \tag{3}$$

Here

$$x_1 = \frac{a+b}{2}+\frac{b-a}{6}\cdot\sqrt{3} = \frac{a\left(3-\sqrt{3}\right)}{6}+\frac{b\left(3+\sqrt{3}\right)}{6}$$

and

$$x_2 = \frac{a+b}{2} - \frac{b-a}{6} \cdot \sqrt{3} = \frac{a\left(3+\sqrt{3}\right)}{6} + \frac{b\left(3-\sqrt{3}\right)}{6}$$

are the roots of the corresponding Lagrange polynomial of order 2.

## 2

Let

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)} \ (a \neq b), \quad I(a,b) = a$$

denote the identric mean of $a, b > 0$; and put, as usually,

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab}$$

for the arithmetic, respectively geometric means of $a$ and $b$. The following result appears in [6].

**Theorem.** *For $a \neq b$ one has*

$$3I^2 < 2A^2 + G^2. \tag{4}$$

We offer a new proof to (4). Put $f(x) = -\log x$ in (3). As

$$f^{(4)}(\xi) > 0 \quad \text{and} \quad \frac{1}{b-a}\int_a^b \log x dx = \log I(a,b),$$

we get

$$I^2 < \frac{1}{6}a^2 + \frac{12+6\sqrt{3}}{36}ab + \frac{12-6\sqrt{3}}{36}\sqrt{ab} + \frac{1}{6}b^2 = \frac{a^2+b^2+4ab}{6}$$
$$= \frac{(a+b)^2+2ab}{6} = \frac{4A^2+2G^2}{6} = \frac{2A^2+G^2}{3},$$

i.e. relation (4).

**Remark 1.** Inequality (4) is the best possible relation of type

$$I^2 < \frac{\alpha}{\alpha+1} \cdot A^2 + \frac{1}{\alpha+1} \cdot G^2, \tag{5}$$

where $\alpha > 0$, see [6].

**Remark 2.** An improvement of another type for (4) follows by the series representation (see [6])

$$\log \frac{\sqrt{2A^2 + G^2}}{I\sqrt{3}} = \sum_{k=2}^{\infty} \frac{1}{2k} \left( \frac{1}{2k+1} - \frac{1}{3^k} \right) \left( \frac{a-b}{a+b} \right)^{2k} \tag{6}$$

**Remark 3.** In [4] it is shown also that

$$I > \frac{2A + G}{3}.$$

Together with (4) this implies

$$\frac{4A^2 + 4AG + G^2}{9} < I^2 < \frac{2A^2 + G^2}{3}. \tag{7}$$

The two extrem sides of (7) give a best possible inequality, namely

$$(A - G)^2 > 0.$$

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## 2.8 On certain subhomogeneous means

A mean $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called subhomogeneous (of order one) when

$$M(tx, ty) \leq tM(x, y),$$

for all $t \in (0, 1]$ and $x, y > 0$. Similarly, $M$ is log-subhomogenous when the property

$$M(x^t, y^t) \leq M^t(x, y)$$

holds true for all $t \in (0, 1]$ and $x, y > 0$. We say that $M$ is additively subhomogeneous if the inequality

$$M(x + t, y + t) \leq t + M(x, y),$$

is valid for all $t \geq 0$ and $x, y > 0$. In this paper we shall study the subhomogeneity properties of certain special means, related to the logarithmic, identric and exponential means.

### 1. Introduction

A mean of two positive real numbers is defined as a function

$$M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$$

(where $\mathbb{R}_+ = (0, \infty)$) with the property:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \text{ for all } x, y \in \mathbb{R}_+. \tag{1}$$

Clearly, it follows that $M(x, x) = x$. The most common example of a mean is the power mean $A_p$, defined by

$$A_p(x, y) = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}}, \text{ for } p \neq 0;$$

$$A_0(x, y) = \sqrt{xy} = G(x, y) \quad \text{(the geometric mean).}$$

We have:

$$A_1(x,y) = A(x,y) \quad \text{(arithmetic mean)},$$

$$A_{-1}(x,y) = H(x,y) \quad \text{(harmonic mean)},$$

and, as limit cases:

$$A_{-\infty}(x,y) = \min\{x,y\}, \quad A_{+\infty}(x,y) = \max\{x,y\}.$$

The logarithmic mean is defined by

$$L(x,x) = x, \quad L(x,y) = \frac{x-y}{\log x - \log y}, \text{ for } x \neq y,$$

and the identric mean by

$$I(x,x) = x; \quad I(x,y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, \text{ for } x \neq y.$$

For early result, extensions, improvements and references, see [1], [2], [10], [11]. In [22] and [13] the following exponential mean has been studied:

$$E(x,x) = x \quad \text{and} \quad E(x,y) = \frac{xe^x - ye^y}{e^x - e^y} - 1 \text{ for } x \neq y.$$

Most of the used means are homogeneous (of order $p$), i.e.

$$M(tx,ty) = t^p M(x,y) \text{ for } t > 0.$$

For example $A, H$ and $I$ are homogeneous of order $p = 1$, while $L$ is homogeneous of order $p = 0$. There are also log-homogeneous means:

$$M(x^t, y^t) = M^t(x,y), \ t > 0.$$

For example, the mean $G$ is log-homogeneous. In [10] it is proved that

$$I(x^2, y^2) \geq I^2(x,y)$$

and in [13] that

$$E\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{1}{2}E(x,y).$$

These relations suggest the study of a notion of subhomogeneity. In [13] a mean $M$ is called $t$-subhomogeneous when

$$M(tx,ty) \le tM(x,y),\ x,y>0,$$

holds true, and it is shown that for $M=E$, this holds true for $t=\dfrac{2}{\log 8}$.

## 2. Subhomogeneity and log-subhomogeneity

The mean $A_p$ is clearly log-subhomogeneous, since it is well known that $A_p < A_q$ for $p<q$. We now prove the following result:

**Theorem 1.** *The means $L, I$ and $\dfrac{L^2}{I}$ are log-subhomogeneous.*

**Proof.** First we note that the log-subhomogenity of $L$ and $I$ follows from a monotonicity property of Leach and Sholander [7] on the general class of means

$$S_{a,b}(x,y)=\left(\frac{x^a-y^a}{a}\cdot\frac{b}{x^b-y^b}\right)^{\frac{1}{a-b}},$$

if $a,b\in\mathbb{R}$, $x,y>0$ and $ab(a-b)(x-y)\neq 0$. It is known that $S$ can be extended continuously to the domain $\{(a,b;x,y):\ a,b\in\mathbb{R},\ x,y>0\}$ and that

$$L(x,y)=S_{1,0},\quad I(x,y)=S_{1,1}(x,y).$$

Then the log-subhomogeneity of $L$ and $I$ is equivalent to

$$S_{1,0}(x,y)\le S_{t,0}(x,y)\quad\text{and}\quad S_{1,1}(x,y)\le S_{t,t}(x,y)\text{ for }t\ge 1.$$

However, this result cannot be applied to the mean $\dfrac{L^2}{I}$. of $L$, for $x\neq y$ we have

$$L(x^t,y^t)=\frac{x^t-y^t}{t(\log x-\log y)}=L(x,y)\frac{x^t-y^t}{t(x-y)}.$$

Since in [10] (relation (13)) it is proved that

$$\frac{x^t-y^t}{t(x-y)}>I^{t-1}(x,y)\text{ for }t>1,$$

by the above identity, we get

$$L(x^t, y^t) > L(x, y)I^{t-1}(x, y) > L^t(x, y),$$

since

$$I > L \tag{$*$}$$

Thus, we have obtained (in a stronger form) the inequality

$$L(x^t, y^t) \geq L^t(x, y) \text{ for } t \geq 1. \tag{2}$$

Another proof of (2) is based on the formula

$$\frac{d}{dt}\left(\frac{\log L(x^t, y^t)}{t}\right) = \frac{1}{t^2}\log\frac{I(x^t, y^t)}{L(x^t, y^t)},\ x \neq y,$$

which can be deduced after certain elementary computations. Since $I > L$, the function

$$t \to \frac{\log L(x^t, y^t)}{t},\ t > 0,$$

is strictly increasing, implying

$$\frac{\log L(x^t, y^t)}{t} > \log L(x, y) \text{ for } t > 1,$$

giving relation (2). A similar simple formula can be deduce for the mean $I$, too, namely

$$\frac{d}{dt}\left(\frac{\log I(x^t, y^t)}{t}\right) = \frac{1}{t^2}\left(1 - \frac{G^2(x^t, y^t)}{L^2(x^t, y^t)}\right) > 0 \quad \text{by} \quad L > G.$$

Thus

$$I(x^t, y^t) \geq I^t(x, y) \text{ for all } t \geq 1. \tag{3}$$

we now study the mean $\frac{L^2}{I}$. This is indeed a mean, since by $L \leq I$, we have $\frac{L^2}{I} \leq I$. On the other hand, it is known that $\frac{L^2}{I} \geq G$. Thus

$$\min\{a, b\} \leq G(a, b) \leq \frac{L^2(a, b)}{I(a, b)} \leq I(a, b) \leq \max\{a, b\},$$

giving (1).

Let us consider now the function

$$h(t) = \left( \frac{L^2(x^t, y^t)}{I(x^t, y^t)} \right)^{\frac{1}{t}}, \ t \geq 1,$$

which has a derivative

$$h'(t) = \frac{h(t)}{t^2} \left( \log \frac{I^2}{L^2} - 1 + \frac{G^2}{L^2} \right),$$

where $I = I(x^t, y^t)$, etc. (we omit the simple computations). Now, in [10] (inequality (21)) the following has been proved:

$$L < Ie^{\frac{G-L}{L}},$$

or with equivalently

$$\log \frac{I}{L} > 1 - \frac{G}{L}. \tag{4}$$

By (4) we can write

$$\log \frac{I^2}{L^2} > 2 - \frac{2G}{L} > 1 - \left( \frac{G}{L} \right)^2,$$

by $\left( \frac{G}{L} - 1 \right)^2 > 0$. Thus $h'(t) > 0$, yielding $h(t) > h(1)$ for $t > 1$.

**Remark 1.** A refinement of (3) can be obtained in the following manner. Let

$$f(t) = \frac{I(x^t, y^t)}{L(x^t, y^t)}, \ t > 0, \ x \neq y.$$

Then

$$f'(t) = \frac{f(t)}{t} \left[ 1 - \frac{G^2(x^t, y^t)}{L^2(x^t, y^t)} \right] > 0 \quad \text{by} \quad L > G.$$

So we can write:

$$\frac{I(x^t, y^t)}{L(x^t, y^t)} > \frac{I(x, y)}{L(x, y)}, \ \text{for } t > 1,$$

which according to $(*)$ gives

$$I(x^t, y^t) > \frac{L(x^t, y^t) I(x, y)}{L(x, y)} > I^t(x, y),\ t > 1. \tag{5}$$

**Remark 2.** By using the function

$$s(t) = (L(x^t, y^t))^{\frac{1}{t}} - (I(x^{t/2}, y^{t/2}))^{\frac{1}{t}}$$

and applying the same method (using inequality (4)), the following can be obtained:

$$\frac{L(x^t, y^t)}{I(x^{t/2}, y^{t/2})} \leq \frac{L^t(x, y)}{I^t(x^{1/2}, y^{1/2})}, \tag{6}$$

for any $t \in (0, 1]$.

**Theorem 2.** *The mean $E$ is subhomogeneous and additively homogeneous.*

**Proof.** Let

$$g(t) = \frac{E(tx, ty)}{t},\ t > 1.$$

Since

$$g'(t) = \frac{(x-y)^2}{t^2(e^{tx} - e^{ty})^2} \left[ \left( \frac{e^{tx} - e^{ty}}{x - y} \right)^2 - t^2 e^{t(x+y)} \right] \tag{7}$$

(we omit the elementary computations), it is sufficient to prove $g'(t) > 0$ for $t > 0$. We then can derive

$$E(tx, ty) \geq tE(x, y) \text{ for any } t \geq 1,$$

i.e. the mean $E$ is subhomogeneous. Let $e^t = A > 1$. The classical Hadamard inequality (see e.g. [10], [26])

$$\frac{1}{x-y} \int_y^x F(t)dt > F\left(\frac{x+y}{2}\right)$$

for a convex function $F : [x, y] \to \mathbb{R}$, applied to the function $F(t) = A^t$, $A > 1$, gives

$$\frac{A^x - A^y}{x - y} > (\log A) \cdot A^{\frac{x+y}{2}},$$

giving

$$\frac{e^{tx}-e^{ty}}{x-y} > te^{\frac{t(x+y)}{2}}$$

thus $g'(t) > 0$, by (7). The additive homogeneity of $E$ is a consequence of the simple equality

$$E(x+t,y+t) = t + E(x,y).$$

**Theorem 3.** *The means $L$, $I$, $L^2/I$, $2I-A$, $3I-2A$ are additively superhomogeneous, while the mean $2A-I$ is additively subhomogeneous.*

**Proof.**

$$\frac{d}{dt}[L(x+t,y+t)-t] = \frac{L^2}{G^2} - 1,$$

$$\frac{d}{dt}[I(x+y,y+t)-t] = \frac{I}{L} - 1,$$

$$\frac{d}{dt}\left[\frac{L^2(x+y,y+t)}{I(x+y,y+t)} - t\right] = \frac{L}{I}\left(\frac{2L^2}{G^2} - 1\right) - 1,$$

$$\frac{d}{dt}[2I(x+y,y+t) - A(x+y,y+t) - t] = \frac{2I}{L} - 2,$$

$$\frac{d}{dt}[3I(x+y,y+t) - 2A(x+y,y+t) - t] = \frac{3I}{L} - 3,$$

$$\frac{d}{dt}[2A(x+y,y+t) - I(x+y,y+t) - t] = 1 - \frac{I}{L},$$

where in all cases $I = I(x+y,y+t)$, etc. From the known inequalities $G < L < I$, some of the stated properties are obvious. We note that $2I-A$ and $3I-2A$ are means, since

$$L < 2I - A < A \quad \text{by} \quad I > \frac{A+L}{2}$$

(see [10]) and $I < A$. Similarly,

$$I > \frac{2A+G}{3}$$

(see [11]), gives $G < 3I - 2A < A$. We have to prove only the inequality

$$\frac{I}{L}\left(\frac{2L^2}{G^2} - 1\right) > 1$$

(implying the additive superhomogeneity of $\frac{L^2}{I}$). Since $L^2 > IG$ (see [2]), we have $\frac{L}{I} > \frac{G}{L}$. Now

$$\frac{G}{L}\left(\frac{2L^2}{G^2} - 1\right) = \frac{2L}{G} - \frac{G}{L} > 1$$

since $1 + \frac{G}{L} < 2 < 2\frac{L}{G}$ by $G < L$.

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## 2.9 Monotonicity and convexity properties of means

### 1

Let $a, b > 0$ be real numbers. The arithmetic and geometric means of $a$ and $b$ are

$$A = A(a,b) = \frac{a+b}{2} \quad \text{and} \quad G = G(a,b) = \sqrt{ab}.$$

The logarithmic mean $L$ is defined by

$$L = L(a,b) = \frac{b-a}{\log b - \log a}, \ q \neq b; \quad L(a,a) = a,$$

while the identric mean is

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}, \ (a \neq b), \quad I(a,a) = a.$$

For history, results and connection with other means, or applications, see the papers given in the References of the survey paper [1]. The aim of this paper is to study certain properties of a new type of the identric, logarithmic or related means. These properties give monotonicity or convexity results for the above considered means.

### 2

Let $0 < a < b$ and fix the variable $b$. Then

$$\frac{d}{da}L(a,b) = \left(\frac{b}{a} - \log\frac{b}{a} - 1\right) / \left(\log\frac{a}{b}\right)^2. \tag{1}$$

(We omit the simple computations), so by the known inequality

$$\log x < x - 1 \quad (x > 0, \ x \neq 1),$$

with $x := \frac{b}{a}$ we have proved that:

**Proposition 1.** *The mean $L(a,b)$ is a strictly increasing function of $a$, when $b$ is fixed.*

**Consequence 1.** *The mean $L$ of two variables is a strictly increasing function with respect to each of variables.*

## 3

An analogous simple computation gives

$$\frac{d}{da}I(a,b) = I(a,b)\left[\frac{\log a - \log I(a,b)}{a-b}\right]. \tag{2}$$

Since $I$ is a mean, for $0 < a < b$ one has $a < I(a,b) < b$, so from (2) we obtain:

**Proposition 2.** *The mean $I(a,b)$ is a strictly increasing function of $a$, when $b$ is fixed.*

**Consequence 2.** *The identric mean $I$ of two variables is a strictly increasing function with respect to each of its variables.*

## 4

We now calculate $\frac{d^2}{da^2}L(a,b)$ and $\frac{d^2}{da^2}I(a,b)$. From (1), (2) after certain elementary computations one can obtain:

$$\frac{d^2}{da^2}L(a,b) = \frac{2}{a^2(\log a - \log b)^2}\left[\frac{a-b}{\log a - \log b} - \frac{a+b}{2}\right] \tag{3}$$

$$\frac{d^2}{da^2}I(a,b) = I(a,b)\left[\frac{(\log a - \log I(a,b) - 1)^2 - \dfrac{b}{a}}{(a-b)^2}\right]. \tag{4}$$

By $L(a,b) < \frac{a+b}{2} = A(a,b)$, clearly $\frac{d^2}{da^2}L(a,b) > 0$. By application of $\log I(a,b)$ we get the numerator in (4) is

$$\frac{b^2}{L^2} - \frac{b}{a} = \frac{b(ab-L^2)}{aL^2} < 0 \quad \text{by} \quad G(a,b) = \sqrt{ab} < L(ab),$$

which is a known result. Thus, from the above remarks we can state:

**Proposition 3.** *The means $L$ and $I$ are strictly concave functions with respect to each variables.*

## 5

As we have seen in paragraph 1 and 3, one can write the equalities:

$$\frac{I'}{I} = \frac{\log a - \log I}{a - b} \tag{5}$$

and

$$\frac{L'}{L} = \frac{1}{a - b} - \frac{1}{\log a - \log b} \cdot \frac{1}{a}, \tag{6}$$

where $I'$ and $L'$ are derivatives with respect to the variable $a$ (and fixed $b$). Since

$$\log a - \log I = \log a - \frac{b \log b - a \log a}{b - a} + 1,$$

we get the identity:

$$\log a - \log I = -\frac{b}{L} + 1 \tag{7}$$

so that (5) and (6) can be rewritten as:

$$\frac{I'}{I} = \frac{1}{a - b} \left( -\frac{b}{L} + 1 \right) \tag{8}$$

and

$$\frac{L'}{L} = \frac{1}{a - b} \left( 1 - \frac{L}{a} \right). \tag{9}$$

Here $-\frac{b}{L} > -\frac{L}{a}$ (equivalent to $G < L$). Thus, via $a - b < 0$ we get

$$\frac{I'}{I} < \frac{L'}{L} \quad \text{for} \quad 0 < a < b.$$

**Proposition 4.** *The function $a \to \dfrac{I(a,b)}{L(a,b)}$ is a strictly decreasing function for $0 < a < b$.*

**Remark.** From (8) and (9) we can immediately see that, for $a > b$ the above function is strictly increasing.

## 6

From the definition of $L$ and $I$ we can deduce that (for $0 < a < b$)

$$L(a,I) = \frac{a-I}{\log a - \log I},$$

which by (5) yields

$$L(a,I) = \frac{a-I}{a-b} \cdot \frac{I'}{I} \tag{10}$$

where $I' = \frac{d}{da} I(a,b)$ and $I = I(a,b)$, etc. By $0 < \frac{a-I}{a-b} < 1$ a corollary of (10) is the interesting inequality

$$L(a,I) < \frac{I'}{I} \tag{11}$$

which holds true also for $a > b$. Similarly, from (9) and the analogous identity of (7) we obtain:

$$\log \frac{I}{b} > (a-b)\frac{L'}{L}. \tag{12}$$

Thus, from the definition of the logarithmic mean,

$$L(b,I) < \frac{I-B}{a-b} \cdot \frac{L'}{L}. \tag{13}$$

Clearly $0 < \frac{I-b}{a-b} < 1$, thus a consequence of (13) is the inequality

$$L(b,I) < \frac{L'}{L} \tag{14}$$

similar to (11).

## 7

We now study the convexity of $L$ and $I$, as functions of two arguments. We consider the Hessian matrix:

$$\nabla^2 L(a,b) = \begin{bmatrix} \frac{\partial^2 L}{\partial a^2} & \frac{\partial^2 L}{\partial a \partial b} \\ \frac{\partial^2 L}{\partial b \partial a} & \frac{\partial^2 L}{\partial b^2} \end{bmatrix},$$

where as we have seen (see (3))

$$\frac{\partial^2 L}{\partial a^2} = \frac{-2}{a^2(\log a - \log b)}\left(\frac{a+b}{2} - L(a,b)\right)$$

$$\frac{\partial L}{\partial a} = \left(\log a - \log b + \frac{b}{a} - 1\right)/(\log a - \log b)^2$$

$$\frac{\partial^2 L}{\partial b^2} = \frac{-2}{b^2(\log a - \log b)}\left(\frac{a+b}{2} - L(a,b)\right).$$

It is easy to deduce that

$$\frac{\partial^2 L}{\partial b \partial a} = \frac{-a}{b} \cdot \frac{\partial^2 L}{\partial a^2},$$

and since by Proposition 3 we have $\dfrac{\partial^2 L}{\partial a^2} < 0$, and by a simple computation $\det \nabla^2 L(a,b) = 0$, we can state that $L$ is a concave function of two arguments.

For the function $I$, by (2), (4) etc. we can see that $\det \nabla^2 L(a,b) = 0$.

We have proved

**Proposition 5.** *The functions $L$ and $I$ are concave functions, as functions of two arguments.*

**Corollary.** $L\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right) \geq \dfrac{L(a,b)+L(c,d)}{2}$,

$$I\left(\frac{a+c}{2}, \frac{b+d}{2}\right) \geq \frac{I(a,b)+I(c,d)}{2}$$

*for all $a, b, c, d > 0$.*

## 8

We now consider a function closely related to the means $L$ and $I$. Put

$$f(a) = \frac{a-b}{I(a,b)} \quad \text{and} \quad g(a) = \operatorname{arctg}\sqrt{\frac{a}{b}}.$$

It is easy to see that

$$g'(a) = \frac{b}{4AG},$$

where $A = A(a, b)$ etc. On the other hand, by (8) we get

$$f'(a) = \frac{b}{IL}.$$

Thus, for the function $h$, $h(a) = f(a) - 4g(a)$ we have

$$h'(a) = b\left(\frac{1}{IL} - \frac{1}{AG}\right) < 0$$

by Alzer's result $AG < IL$. Thus:

**Proposition 6.** *The function $h$ defined above is strictly decreasing.*

## 9

Monotonicity or convexity problems can be considered also for functions obtained by replacing the variables $a$ and $b$ with $x^t$ and $y^t$, where $x$ and $y$ are fixed (positive) real numbers, while $t$ is a real variable. In the same way, we are able to study similar problems with $a = x+t$, $b = y+t$. We introduce the functions

$$I(t) = \frac{1}{t}\log I(x^t, y^t),\ \ t \neq 0; \quad I(0) = G$$

and

$$L(t) = \frac{1}{t}\log L(x^t, y^t),\ \ t \neq 0; \quad L(0) = G.$$

By the definition of $I$ and $L$, it is a simple matter of calculus to deduce the following formulae:

$$\frac{d}{dt}\log L(x^t, y^t) = \frac{m\log m - n\log n - (m-n)}{t(m-n)} = \frac{1}{t}\log I(m, n) \qquad (15)$$

$$\frac{d}{dt}\log I(x^t, y^t) = \frac{(m-n)(m\log m - n\log n) - mn(\log m - \log n)^2}{t(m-n)^2}$$

where $m = x^t$, $n = y^t$. By using these relations, we get

$$\frac{d}{dt}L(t) = \frac{1}{t^2}\log\frac{1}{L}, \quad \frac{d}{dt}I(t) = \frac{1}{t^2}\left(1 - \frac{G^2}{L^2}\right),$$

where $G = G(x^t, y^t)$, etc. By $I > L$ and $G < L$ we get $L'(t) > 0$, $I'(t) > 0$ for all $t \neq 0$.

By extending the definition of $L$ and $I$ at $t = 0$, we have obtained:

**Proposition 7.** *The functions $t \to L(t)$ and $t \to I(t)$ are strictly increasing functions on $\mathbb{R}$.*

## 10

Closely related to the means $L$ and $I$ is the mean $S$ defined by

$$S = S(a, b) = (a^a b^b)^{\frac{1}{a+b}}.$$

By the identity

$$S(a, b) = \frac{I(a^2, b^2)}{I(a, b)},$$

we get

$$S(t) = \frac{\log S(x^t, y^t)}{t} = 2I(2t) - I(t),$$

so from (15) we can deduce

$$S'(t) = \frac{G^2}{t^2 L^2}\left(1 - \frac{G^2}{A^2}\right) \tag{16}$$

where $G = G(x^t, y^t)$, etc.

By extending the definition of $S$ to the whole real line by $S(0) = G$, we can state:

**Proposition 8.** *The function $t \to S(t)$ is strictly increasing function on $\mathbb{R}$.*

## 11

Let $f(t) = \dfrac{I(x^t, y^t)}{L(x^t, y^t)}$ for $t \neq 0,\ f(0) = 1$.

By logarithming and using relations (15), the following can be proved:

$$f'(t) = \frac{f(t)}{tL^2}(L^2 - G^2),\ t \neq 0 \tag{17}$$

where $L = L(x^t, y^t)$.

**Proposition 9.** *The function $t \to f(t)$ defined above is strictly increasing for $t > 0$ and strictly decreasing for $t < 0$ (thus $t = 0$ is the single minimum-point of this continuous function).*

## 12

Let

$$s(t) = \left(\frac{L^2(x^t, y^t)}{I(x^t, y^t)}\right)^{\frac{1}{t}},\ t \neq 0; \quad s(0) = e^G.$$

By $\log s(t) = 2L(t) - I(t)$ and from (15) we easily get:

$$\frac{s'(t)}{s(t)} = \frac{1}{t^2}\left(\log \frac{I^2}{L^2} - 1 + \frac{G^2}{L^2}\right) \tag{18}$$

where $L = L(x^t, y^t)$ for $t \neq 0$. By

$$\log \frac{I^2}{L^2} - 1 + \frac{G^2}{L^2} > 0$$

(see [1]), with the assumption $s(0) = e^G$, we obtain:

**Proposition 10.** *The function $t \to s(t)$ is strictly increasing on $\mathbb{R}$.*

**Corollary.** $\dfrac{L^2(x^t, y^t)}{I(x^t, y^t)} > e^{tG(x,y)}$ *for $t > 0$.*

## 13

In what follows we will consider the derivatives of forms

$$\frac{d}{dt}M(a + t, b + t),$$

which will be denoted simply by $M'$ (where $M$ is a mean). We then will be able to obtain other monotonicity and convexity properties. It is a simple exercise to see that

$$A' = 1, \quad G' = \frac{A}{G},$$

where $G = G(a+t, b+t)$, etc. Indeed,

$$F' = \frac{d}{dt}\sqrt{(a+t)(b+t)} = \frac{1}{2}\cdot\frac{1}{\sqrt{a+t}}\sqrt{b+t} + \frac{1}{2\sqrt{b+t}}\sqrt{a+t} = \frac{A}{G}.$$

Similarly one can show that

$$L' = \frac{L^2}{G^2}, \quad I' = \frac{I}{L}.$$

For the mean $S$ the following formula can be deduced:

$$S' = S\left(\frac{1}{A} - k\frac{1}{A^2 L}\right),$$

where $k = \left(\frac{a-b}{2}\right)^2 > 0$. Thus

$$(A-I)' = 1 - \frac{I}{L} < 0, \quad (G-I)' = \frac{A}{G} - \frac{I}{L} > 0$$

(since $A > I$, $L > G$, so $AL > GI$);

$$(G^2 - I^2)' = \frac{2(AL - I^2)}{L} < 0$$

by the known inequality

$$I > \frac{A+L}{2} > \sqrt{AL}.$$

From $AGL < A^2G < I^3$ (see [1]) and

$$(G^3 - I^3)' = 3\left(AG - \frac{I^3}{L}\right)$$

we can deduce $(G^3 - I^3)' < 0$.

These remarks give:

**Proposition 11.** *$A - I$, $G^2 - I^2$, $G^3 - I^3$ are strictly decreasing functions, while $G - I$ is strictly increasing on the real line.*

*(Here $A = A(a+t, b+t)$ etc., $t \in \mathbb{R}$).*

For an example of convexity, remark that

$$I'' = \left(\frac{I}{L}\right)' = \frac{I}{L^2}\left(1 - \frac{L^2}{G^2}\right),$$

$$L'' = \left(\frac{L^2}{G^2}\right) = \frac{2L^2}{G^4}(L - A)$$

(we omit the details), so we can state:

**Proposition 12.** *The functions $L$ and $I$ are strictly concave on the real line.*

**Corollaries. 1.** *For $t \geq 0$ one has:*

$$G(a+t, b+t) - I(a+t, b+t) \geq G(a,b) - I(a,b)$$

$$G^2(a+t, b+t) - I^2(a+t, b+t) \leq G^2(a,b) - I^2(a,b),$$

$$A(a+t, b+t) - I(a+t, b+t) \leq A(a,b) - I(a,b).$$

**2.** $\dfrac{I(a+t_1, b+t_1) + I(a+t_2, b+t_2)}{2} \leq I\left(a + \dfrac{t_1+t_2}{2}, b + \dfrac{t_1+t_2}{2}\right)$

*for all $t_1, t_2 > 0$, $a, b > 0$.*

## 14

Let

$$P = P(a,b) = \frac{a-b}{4\arctan\left(\sqrt{\dfrac{a}{b}}\right) - \pi} \text{ for } a \neq b, \quad P(a,a) = a.$$

This mean has been introduced by Seiffert [2]. It is not difficult to show that

$$P' = \frac{P^2}{AG}, \quad \left(\frac{1}{P}\right)' = \frac{-1}{AG}.$$

Since $\left(\frac{1}{I}\right) = \frac{-1}{IL}$, we get:

**Proposition 13.** $\frac{1}{I} - \frac{1}{P}$ *is a strictly increasing function of* $t$, *where* $P = P(a+t, b+t)$, *etc.*

Indeed, this follows from the known inequality $AG < LI$.

**Corollary.**

$$\frac{1}{I(a+t,b+t)} - \frac{1}{P(a+t,b+t)} > \frac{1}{I(a,b)} - \frac{1}{P(a,b)} \text{ for all } t > 0,\ a \neq b.$$

Finally, we prove:

**Proposition 14.** *The function* $\frac{P}{L}$ *is strictly increasing on* $\mathbb{R}$.

**Proof.** We have

$$\left(\frac{P}{I}\right)' = \frac{P}{I}\left(\frac{P}{AG} - \frac{1}{L}\right).$$

Now, it is known that $GA < LP$ ([3]) implying the desired result.

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## 2.10 Logarithmic convexity of the means $I_t$ and $L_t$

In paper [3] we have studied the subhomogeneity or logsubhomogeneity (as well as their additive analogue) of certain means, including the identric and logarithmic means. There appeared in a natural way the following functions:

$$f(t) = \frac{\log L(x^t, y^t)}{t} \quad \text{and} \quad g(t) = \frac{\log I(x^t, y^t)}{t},$$

where $x, y > 0$ while $t \neq 0$. The $t$-modification of a mean $M$ is defined by (see e.g. [4])

$$M_t(x, y) = (M(x^t, y^t))^{1/t}.$$

Therefore,

$$f(t) = \log L_t(x, y) \quad \text{and} \quad g(t) = \log I_t(x, y),$$

where $L$ and $I$ are well known logarithmic and identric means, defined by

$$L(a, b) = \frac{b - a}{\ln b - \ln a} \ (b > a > 0), \quad L(a, a) = a;$$

$$I(a, b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \ (b > a > 0), \quad I(a, a) = a.$$

In paper [3] we have proved that

$$f'(t) = \frac{1}{t^2} g(t) \tag{1}$$

and

$$g'(t) = \frac{1}{t^2} h(t) \tag{2}$$

where

$$h(t) = 1 - \frac{G^2}{L^2} \quad \text{and} \quad g(t) = \log \frac{I}{L},$$

where in what follows $G = G(x^t, y^t)$, etc.

Our aim is to prove the following result.

**Theorem.** *$L_t$ and $I_t$ are log-concave for $t > 0$ and log-convex for $t < 0$.*

**Proof 1.** First observe that as $G = \sqrt{x^t y^t}$, one has

$$G' = \frac{1}{2\sqrt{x^t y^t}}(x^t \ln x \cdot y^t + y^t \ln y \cdot x^t) = \frac{G \ln G}{t}.$$

Similarly, since

$$L(x^t, y^t) = L(x, y)\frac{x^t - y^t}{t(x - y)},$$

we easily get

$$L' = \frac{L \log I}{t}$$

(where we have used the fact that $\log I(a, b) = \dfrac{b \ln b - a \ln a}{b - a} - 1$). Now,

$$h'(t) = \frac{2 - G}{L}\left(\frac{G'L - L'G}{L^2}\right) = \frac{2G^2}{tL^2}\log\frac{I}{G},$$

after using the above established formulae for $G'$ and $L'$. By calculating

$$g''(t) = \frac{th'(t) - 2h(t)}{t^3},$$

after certain computations we get

$$g''(t) = \frac{1}{t^3}\left(\frac{2G^2}{L^2}\log\frac{I}{G} + \frac{2G^2}{L^2} - 2)\right).$$

Since $\log\dfrac{I}{G} = \dfrac{A - L}{L}$ (where $A = A(a, b) = \dfrac{a + b}{2}$ denotes the arithmetic mean; for such identities see e.g. [2]), we arrive at

$$g''(t) = \frac{2}{t^3}\left(\frac{G^2 A}{L^3} - 1\right). \tag{3}$$

Since by a result of Leach-Sholander [1], $G^2 A < L^3$, by (3) we get that $g''(t) < 0$ for $t > 0$ and $g''(t) > 0$ for $t < 0$.

**Proof 2.** First we calculate $I'$. Let $a = x^t$, $b = y^t$. By

$$\log I(a,b) = \frac{b\ln b - a\ln a}{b-a} - 1$$

one has

$$I'(a,b) = I(a,b)\left(\frac{b\ln b - a\ln a}{b-a}\right)'.$$

Here

$$\left(\frac{b\ln b - a\ln a}{b-a}\right)' = \frac{1}{t}\left[\frac{b\ln b - a\ln a}{b-a} - ab\left(\frac{\ln b - \ln a}{b-a}\right)^2\right],$$

after some elementary (but tedious) calculations, which we omit here. Therefore

$$I' = \frac{1}{t} I \left(\log I + 1 - \frac{G^2}{L^2}\right).$$

Now

$$f''(t) = \left(\frac{g(t)}{t^2}\right)' = \frac{1}{t^3}(g'(t)t - 2g(t)),$$

where

$$g'(t) = \frac{L}{I}\left(\frac{I}{L}\right)' = \frac{L}{I}\left(\frac{I'L - L'I}{L^2}\right) = \frac{1}{t}\left(1 - \frac{G^2}{L^2}\right)$$

(after replacing $L'$ and $I'$ and some computations). Therefore

$$g'(t)t - 2g(t) = 1 - \frac{G^2}{L^2} - \log\frac{I^2}{L^2}.$$

In our paper [3] it is proved that

$$\log\frac{I^2}{L^2} > 1 - \frac{G^2}{L^2}$$

(and this implies the logsubhomogenity of the mean $\frac{L^2}{I}$). Thus $f''(t) < 0$ for $t > 0$ and $f''(t) > 0$ for $t < 0$. This completes the proof of the theorem.

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## 2.11 On certain logarithmic inequalities

### 1. Introduction

In the very interesting problem book by K. Hardy and K.S. Williams [1] (see 3., page 1) one can find the following logarithmic inequality:

$$\frac{\ln x}{x^3 - 1} < \frac{1}{3} \cdot \frac{x+1}{x^3 + x}, \tag{1}$$

where $x > 0, x \neq 1$.

The proof of this surprisingly strong inequality is obtained in [1] by using a quite complicated study of auxiliary functions.

We wish to note in what follows, how inequality (1) is related to the famous logarithmic mean $L$, defined by

$$L(a,b) = \frac{a-b}{\ln a - \ln b}(a \neq b); L(a,a) = a, \tag{2}$$

where $a$ and $b$ are positive real numbers. We will show that, in terms of logarithmic mean, (1) is due in fact to J. Karamata [2]. For a survey of results on $L$ and connected means, see e.g. [4], [5], [6].

Let $A(a,b) = \dfrac{a+b}{2}$, $G(a,b) = \sqrt{ab}$ denote the classical arithmetic, resp. geometric mean of $a$ and $b$. It is well known that, the logarithmic mean separates the geometric and arithmetic mean:

$$G < L < A, \tag{3}$$

where $G = G(a,b)$, etc. and $a \neq b$. For the history of this inequality and new proofs, see [5], [15], [17], [18], [19].

As inequality (3) is important in many fields of mathematics, (see e.g. [9], [12], [15]), the following famous refinement of left side of (3), due to Leach and Sholander [3] should be mentioned

$$\sqrt[3]{G^2 \cdot A} < L \tag{4}$$

Now, let us introduce the following mean $K$ by

$$K(a,b) = \frac{a\sqrt[3]{b} + b\sqrt[3]{a}}{\sqrt[3]{b} + \sqrt[3]{a}} \tag{5}$$

Letting $x = \sqrt[3]{\frac{a}{b}}$ $(a \neq b)$, inequality (1) can be written, by using (2) and (5):

$$L(a,b) > K(a,b) \tag{6}$$

This inequality is due to Karamata [2].

We will show that inequality (6) refines (4). Also, we will give new proof and refinements to this inequality.

## 2. Main results

The first result shows that (6) is indeed a refinement of (4):

**Theorem 1.** *One has*

$$L > K > \sqrt[3]{G^2 A} \tag{7}$$

**Proof.** We have to prove the second inequality of (7); i.e.

$$\frac{a\sqrt[3]{b} + b\sqrt[3]{a}}{\sqrt[3]{b} + \sqrt[3]{a}} > \sqrt[3]{ab \cdot \left(\frac{a+b}{2}\right)} \tag{8}$$

Putting $a = u^3, b = v^3$, this inequality becomes

$$\frac{u^3 v + v^3 u}{u+v} > uv \cdot \sqrt[3]{\frac{u^3 + v^3}{2}},$$

or after elementary transformations:

$$2(u^2 + v^2)^3 > (u+v)^3 \cdot (u^3 + v^3) \tag{9}$$

This inequality, which is interesting in itself, can be proved by algebraic computations; here we present an analytic approach, used also in our paper [11]. By logarithmation, the inequality becomes

$$\ln 2 + 3\ln(u^2 + v^2) - 3\ln(u+v) - \ln(u^3 + v^3) = f(u) > 0 \tag{10}$$

Suppose $u > v$. Also, for simplicity one could take $v = 1$ (since (9) is homogeneous). Then one has

$$f'(u) = \frac{6u}{u^2+1} - \frac{3}{u+1} - \frac{3u^2}{u^3+1} = \frac{(u-1)^3}{(u^2+1)(u^3+1)} > 0,$$

after elementary computations, which we omit here. Thus

$$f(u) > f(1) = 0,$$

and the result follows.

**Remark 1.** Inequality $K > \sqrt[3]{G^2 A}$ has been discovered by the author in 2003 [10]. For the extensions of (9), see [10] and [[16].

**Theorem 2.** *Inequality $L > K$ is equivalent to inequality*

$$L > \frac{3AG}{2A+G} \tag{11}$$

**Proof.** By letting $a = u^3, b = v^3$ the inequality $L(a,b) > K(a,b)$ becomes the equivalent inequality

$$L(u^3, v^3) > K(u^3, v^3).$$

Now, remark that

$$L(u^3, v^3) = L(u,v) \cdot \frac{u^2+uv+v^2}{3} \quad \text{and} \quad K(u^3, v^3) = \frac{uv(u^2+v^2)}{u+v},$$

so we get the relation

$$L(u,v) > \frac{3uv(u^2+v^2)}{(u+v)(u^2+uv+v^2)} \tag{12}$$

Let now $u = \sqrt{p}, v = \sqrt{q}$ in (12), with $p \neq q$ positive real numbers. Remarking that

$$L(\sqrt{p}, \sqrt{q}) = \frac{2}{\sqrt{u}+\sqrt{v}} \cdot L(p,q),$$

after certain computations, (12) becomes

$$L(p,q) > \frac{3\sqrt{pq}(p+q)}{p+q+\sqrt{pq}} \tag{13}$$

As

$$\sqrt{pq} = G(p,q), \quad p+q = 2A(p,q),$$

inequality (13) may be written as

$$L > \frac{3AG}{2A+G}, \tag{14}$$

where $L = L(p,q)$ etc. Clearly, this inequality is independents of the variables $p$ and $q$, and could take $L = L(a,b), A = A(a,b), G = G(a,b)$ in inequality (14). This proves Theorem 2.

**Remark 2.** For inequalities related to (11), see also [11].

Now, the surprise is, that, though (6) is stronger than (4), inequality (4) implies inequality (6)!: One has

**Theorem 3.**

$$L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G} \tag{15}$$

**Proof.** The first inequality of (15) is the Leach-Sholander inequality (4).

Now, remark that $\sqrt[3]{G^2A} =$ geometric mean of $G$ and

$$A = \sqrt[3]{G \cdot G \cdot A},$$

which is greater than the harmonic mean of these three numbers:

$$\frac{3}{\frac{1}{G}+\frac{1}{G}+\frac{1}{A}} = \frac{3}{\frac{2}{G}+\frac{1}{A}} = \frac{3AG}{2A+G}.$$

Therefore, inequality (15) follows.

**Theorem 4.** *One has*

$$L > \sqrt[3]{\left(\frac{A+G}{2}\right)^2 \cdot G} > \frac{3G(A+G)}{A+5G} > \frac{3AG}{2A+G} \tag{16}$$

**Proof.** The first inequality of (16) is a refinement of (4), and is due to the author [7]. See also [13].

The second inequality of (16) follows by the same argument as the proof of Theorem 3: the geometric mean of the numbers $\frac{A+G}{2}, \frac{A+G}{2}, G$ is greater than their harmonic mean, which is

$$\frac{3}{\frac{2}{A+G}+\frac{2}{A+G}+\frac{1}{G}}=\frac{3G(A+G)}{5G+A}.$$

Finally, the last inequality is equivalent, after some computations with $A^2-2AG+G^2>0$, or $(A-G)^2>0$.

**Remark 3.** Connections of $L$ with other means are studied in papers [6], [8], [14].

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## 2.12 A note on the logarithmic mean

### 1. Introduction

The logarithmic mean $L(a, b)$ of two positive real numbers $a$ and $b$ is defined by

$$L = L(a, b) = \frac{b-a}{\ln b - \ln a} \text{ for } a \neq b, L(a, a) = a. \tag{1}$$

Let $A := A(a, b) = \dfrac{a+b}{2}$ and $G := G(a, b) = \sqrt{ab}$ denote the arithmetic, resp. geometric means of $a$ and $b$.

One of the basic inequalities connecting the above means is the following:

$$G < L < A, \text{ for } a \neq b. \tag{2}$$

Among the first discoveries of this inequality, we quote B. Ostle and H.L. Terwilliger [2] (right side of (2)) and B.C. Carlson [1] (left side of (2)). See also [4], [5] for other references.

Inequality (2) has been rediscovered and reproved many time (see e.g. [3], [4], [5], [6]).

The aim of this note is to offer a new proof of this inequality. The method is based on two simple algebraic inequalities and Riemann integration.

### 2. The proof

**Lemma.** *For all $t > 1$ one has*

$$\frac{4}{(t+1)^2} < \frac{1}{t} < \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}. \tag{3}$$

**Proof.** The left side of (3) holds true, being equivalent to $(t-1)^2 > 0$, while the right side, after reducing with $\sqrt{t}$ to $\dfrac{1}{2\sqrt{t}} < \dfrac{1}{2}$, or $t > 1$.

Now, let $b > a > 0$, and integrate both sides of inequalities (3) on $[1, b/a]$. As

$$\int_1^{b/a} \frac{2}{(t+1)^2} dt = \frac{b-a}{b+a} \text{ and } \int_1^{b/a} \frac{1}{2\sqrt{t}} dt = \sqrt{\frac{b}{a}} - 1,$$

$$\int_1^{b/a} \frac{1}{2t\sqrt{t}} dt = -\sqrt{\frac{a}{b}} + 1,$$

we get:

$$2\left(\frac{b-a}{b+a}\right) < \ln b - \ln a < \sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}} = \frac{b-a}{\sqrt{ab}}. \tag{4}$$

Relation (4) implies immediately (2) for $b > a$. Since $L(a,b) = L(b,a)$, (2) follows for all $a \neq b$.

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## 2.13 A basic logarithmic inequality, and the logarithmic mean

### 1. Introduction

Let $a, b > 0$. The logarithmic mean $L = L(a, b)$ of $a$ and $b$ is defined by

$$L = L(a, b) = \frac{b - a}{\ln b - \ln a} \text{ for } a \neq b \text{ and } L(a, a) = a. \tag{1}$$

Let $G = G(a, b) = \sqrt{ab}$ and $A = A(a, b) = \dfrac{a + b}{2}$ denote the classical geometric, resp. logarithmic means of $a$ and $b$.

One of the most important inequalities for the logarithmic mean (besides e.g. $a < L(a, b) < b$ for $a < b$) is the following:

$$G < L < A \text{ for } a \neq b \tag{2}$$

The left side of (2) was discovered by B.C. Carlson in 1966 ([1]) while the right side in 1957 by B. Ostle and H.L. Terwilliger [2].

We note that relation (2) has applications in many subject of pure or applied mathematics and physics including e.g. electrostatics, probability and statistics, etc. (see e.g. [3], [4]).

The following basic logarithmic inequality is well-known:

**Theorem 1.**

$$\ln x \leq x - 1 \textit{ for all } x > 0. \tag{3}$$

*There is equality only for $x = 1$.*

(3) may be proved e.g. by considering the auxiliary function

$$f(x) = x - \ln x - 1,$$

and it is easy to show that $x = 1$ is a global minimum to $f$, so

$$f(x) \geq f(1) = 0.$$

Another proof is based on the Taylor expansion of the exponential function, yielding $e^t = 1 + t + \frac{t^2}{2} \cdot e^{\theta}$, where $\theta \in (0, t)$. Put $t = x - 1$, and (3) follows.

The continuous arithmetic, geometric and harmonic means of positive, integrable function $f : [a, b] \to \mathbb{R}$ are defined by

$$A_f = \frac{1}{b-a} \int_a^b f(x)dx, \quad G_f = e^{\frac{1}{b-a} \int_a^b \ln f(x)dx}$$

and

$$H_f = \frac{b-a}{\displaystyle\int_a^b dx/f(x)},$$

where $a < b$ are real numbers.

By using (3) we will prove the following classical fact:

**Theorem 2.**

$$H_f \le G_f \le A_f \tag{4}$$

Then, by applying (4) for certain particular functions, we will deduce (2). In fact, (2) will be obtained in a stronger form. The main idea of this note is the use of very simple inequality (3) in the theory of means.

## 2. The proofs

**Proof of Theorem 2.** Put

$$x = \frac{(b-a)f(t)}{\displaystyle\int_a^b f(t)dt}$$

in (3), and integrate on $t \in [a, b]$ the obtained inequality. One gets

$$\int_a^b \ln f(t)dt - \left(\left(\frac{1}{b-a} \int_a^b f(t)dt\right)\right)(b-a) \le \frac{(b-a)\displaystyle\int_a^b f(t)dt}{\displaystyle\int_a^b f(t)dt} - (b-a) = 0.$$

This gives the right side of (4).

Apply now this inequality to $\frac{1}{f}$ in place of $f$. As

$$\ln \frac{1}{f(t)} = -\ln f(t),$$

we immediately obtain the left side of (4).

**Corollary 1.** *If $f$ is as above, then*

$$\left(\int_a^b f(t)dt\right)\left(\int_a^b \frac{1}{f(t)}dt\right) \geq (b-a)^2. \tag{5}$$

This follows by $H_f \leq A_f$ in (4).

**Remark 1.** Let $f$ be continuous in $[a,b]$. The above proof shows that there is equality e.g. in right side of (4) if

$$f(t) = \frac{1}{b-a}\int_a^b f(t)dt. \tag{6}$$

By the first mean value theorem of integrals, there exists $c \in [a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(t)dt = f(c).$$

Since by (6) one has $f(t) = f(c)$ for all $t \in [a,b]$, $f$ is a constant function.

When $f$ is integrable, as

$$\int_a^b \ln\left[(b-a)\frac{f(t)}{\displaystyle\int_a^b f(t)dt}\right]dt = 0,$$

as for $g(t) = \ln \dfrac{(b-a)f(t)}{\displaystyle\int_a^b f(t)dt} > 0$ one has

$$\int_a^b g(t)dt = 0,$$

it follows by a known result that $g(t) = 0$ almost everywhere (a.e.). Therefore

$$f(t) = \frac{1}{b-a}\int_a^b f(t)dt$$

a.e., thus $f$ is a constant a.e.

**Remark 2.** If $f$ is continuous, it follows in the same manner, that in the left side of (4) there is equality only for $f =$ constant. The same is true for inequality (5).

**Proof of (2).** Apply $G_f \leq A_f$ to $f(x) = \frac{1}{x}$. Remark that

$$\frac{1}{b-a}\int_a^b \ln x dx = \ln I(a,b),$$

where $a < I(a,b) < b$.

This mean is known in the literature as "identric mean" (see e.g. [3]). As $f(x) = \frac{1}{x}$ is not constant, we get by

$$A_f = \frac{1}{L(a,b)}, \quad G_f = \frac{1}{I(a,b)},$$

that

$$L < I \tag{7}$$

Applying the same inequality $G_f \leq A_f$ to $f(x) = x$ one obtains

$$I < A \tag{8}$$

**Remark 3.** (7) and (8) can be deduced at once by applying all relations of (4) to $f(x) = x$. Apply now (5) to $f(t) = e^t$. After elementary computations, we get

$$\frac{e^b - e^a}{b-a} > e^{\frac{a+b}{2}} \tag{9}$$

As $f(t) > 0$ for any $t \in \mathbb{R}$, inequality (9) holds true for any $a, b \in \mathbb{R}$, $b > a$. Replace now $b := \ln b$, $a := \ln a$, where now the new values of $a$ and $b$ are $> 0$. One gets from (9):

$$L > G \tag{10}$$

By taking into account of (7)-(10), we can write:

$$G < L < I < A, \tag{11}$$

i.e. (2) is proved (in improved form on the right side).

**Remark 4.** Inequality (4) (thus, relation (10)) follows also by $G_f \leq A_f$ applied to $f(t) = e^t$.

**Remark 5.** The right side of (2) follows also from (5) by the application $f(t) = t$. As

$$\int_a^b t dt = \frac{b^2 - a^2}{2} \text{ and } \int_a^b \frac{1}{t} dt = (\ln b - \ln a),$$

the relation follows.

**Remark 6.** Clearly, in the same manner as (4), the discrete inequality of means can be proved, by letting $x = \dfrac{nx_i}{x_1 + \ldots + x_n}$ $(x_1, \ldots, x_n > 0)$.

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## 2.14 On certain inequalities for means in two variables

### 1. Introduction

The logarithmic and identric means of two positive real numbers $a$ and $b$ with $a \neq b$ are defined by

$$L = L(a,b) = \frac{b-a}{\log b - a\log a} \quad \text{and} \quad I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)},$$

respectively. These means have been the subject of many intensive research, partly because they are related to many other important means and because these means have applications in physics, economics, meteorology, statistics, etc. For a survey of results, with an extended literature, see [3], [6]. For identities involving these, and other means, see e.g. [8], [10]. Particularly, the identity

$$I(a^2,b^2)/I(a,b) = (a^a \cdot b^b)^{1/(a+b)} = S = S(a,b)$$

leads to the weighted geometric mean of $a$ and $b$, denoted by $S(a,b)$ in [6], [8], [9].

In paper [12] there are proved the following two inequalities

$$G(a,b)\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < I(a,b) < A(a,b)\exp\left(-\frac{1}{6}\left(\frac{b-a}{b+a}\right)^2\right), \tag{1}$$

where $a \neq b$, $a, b > 0$.

In paper [3] by H. Alzer and S.-L. Qiu appears among many other relations, the following one:

$$G(a,b)\exp\left(\frac{1}{6}\left(\frac{b-a}{b+a}\right)^2\right) < L(a,b) < A(a,b)\exp\left(-\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right), \tag{2}$$

$a \neq b$, $a, b > 0$.

We note that, the right hand side inequality of (1) was first proved by the author in 1989 [5]. In that paper it was shown also the following inequality:

$$\frac{A^2(a,b)}{I(a^2,b^2)} < \exp\left(-\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right). \tag{3}$$

The aim of this note is to prove that the above inequalities are connected to each other by a chain of relations, and that, in fact, all are consequences of (3).

## 2. Main results

First write in another form all the inequalities. The left and right sides of (1) many be written respectively as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{I(a,b)}{G(a,b)}; \tag{4}$$

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{A^2(a,b)}{I^2(a,b)} \tag{5}$$

while the inequalities of (2) as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{L^2(a,b)}{G^2(a,b)}; \tag{6}$$

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{A(a,b)}{L(a,b)}. \tag{7}$$

Finally note that, (3) may be written as

$$\exp\left(\frac{1}{3}\left(\frac{b-a}{b+a}\right)^2\right) < \frac{I(a^2,b^2)}{A^2(a,b)} = \frac{I(a,b)S(a,b)}{A^2(a,b)}. \tag{8}$$

**Theorem 1.** *The following chain of implications holds true:*

$$(8) \Rightarrow (5) \Rightarrow (7) \Rightarrow (4) \Rightarrow (6).$$

**Proof.** (8) $\Rightarrow$ (5) means that $\frac{I \cdot S}{A^2} < \frac{A^2}{I^2}$, or $S < \frac{A^4}{I^3}$. This inequality is proved in [9] (see Theorem 1 there).

(5) $\Rightarrow$ (7) by $\frac{A^2}{I^2} < \frac{A}{L}$, i.e. $I^2 > A \cdot L$. For this inequality, see [7] (Relation (9)).

(7) $\Rightarrow$ (4) by $\frac{A}{L} < \frac{I}{G}$, i.e. $A \cdot G < L \cdot I$, see [1].

(4) $\Rightarrow$ (6) by $\frac{I}{G} < \frac{L^2}{G^2}$, i.e. $\sqrt{GI} < L$, see [2].

Therefore all implications are valid.

We note that inequality (8) was a consequence of an integral inequality due to the author [4], (discovered in 1982), to the effect that:

**Theorem 2.** *Let $f : [a,b] \to \mathbb{R}$ be a $2k$-times ($k \geq 1$) differentiable function such that $f^{(2k)}(x) > 0$. Then*

$$\int_a^b f(x)dx > \sum_{j=0}^{k-1} \frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!} f^{(2j)}\left(\frac{a+b}{2}\right). \tag{9}$$

For $k = 2$ we get that if $f$ is 4-times differentiable, then

$$\frac{1}{b-a}\int_a^b f(x)dx > f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right). \tag{10}$$

Clearly, (9) and (10) are extensions of the classical Hadamard inequality, which says that, if $f$ is convex on $[a,b]$ then

$$\frac{1}{b-a}\int_a^b f(x)dx > f\left(\frac{a+b}{2}\right). \tag{11}$$

Applying (10) for $f(x) = x \log x$, and using the identity

$$\int_a^b x \log x dx = \frac{1}{4}(b^2 - a^2) \log I(a^2, b^2) \tag{12}$$

(see [6]), we get (8). Applying (10) to $f(x) = -\log x$, we get (5), i.e. the right side of (1) (see [5]). For another proof, see [11].

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## 2.15 On a logarithmic inequality

### 1. Introduction

In the recent paper [1], the following logarithmic inequality has been proved (see Lemma 2.7 of [1]):

**Theorem 1.** *For any $k \geq 1$ and $t \in [t_0, 1)$, where $t_0 = \dfrac{e-1}{e+1}$ one has:*

$$\log\left(\frac{1+t^{1/k}}{1-t^{1/k}}\right) \leq k \log\left(\frac{1+t}{1-t}\right). \tag{1}$$

The proof of (1) given in [1] is very complicated, based on more subsequent Lemmas on various hyperbolic functions. We note that (1) has important applications in the study of quasiconformal mappings and related vector function inequalities [1].

The aim of this note is to offer a very simple proof of (1), and in fact to obtain a more general result.

### 2. The proof

Our method will be based on the study of monotonicity of a certain function, combined with a well-known result related to the logarithmic mean

$$L = L(x, y) = \frac{x-y}{\log x - \log y} \ (x \neq y),\ L(x, x) = x.$$

The following result is well-known (see e.g. [2]):

**Lemma.** *One has $L > G$ for any $x, y > 0$, $x \neq y$, where*

$$G = G(x, y) = \sqrt{xy}$$

*denotes the geometric mean of $x$ and $y$.*

Put now $t = \dfrac{1}{p}$, where $1 < p \leq \dfrac{e+1}{e-1}$ and $\dfrac{1}{k} = x$ in (1). Then the inequality becomes

$$f(x) = x \log\left(\frac{p^x+1}{p^x-1}\right) \leq f(1),\ \text{where } 0 < x \leq 1,$$

and $f(1)=\log\left(\dfrac{p+1}{p-1}\right)\ge 1$.

Now the following result will be proved:

**Theorem 2.** *Assuming the above conditions, the function $f(x)$ is strictly increasing on $(0,1]$.*

*Particularly, one has $f(x)\le f(1)$ for $0<x\le 1$.*

**Proof.** An easy computation gives

$$f'(x)=\log\left(\frac{p^x+1}{p^x-1}\right)-\frac{2xp^x\log p}{p^{2x}-1}=\log\left(\frac{a+1}{a-1}\right)-\frac{2a\log a}{a^2-1}=g(a),$$

where $a=p^x$. Since $0<x\le 1$, $a\le p$ and as $p\le\dfrac{e+1}{e-1}$, one has $a\le\dfrac{e+1}{e-1}$, i.e. $\log\left(\dfrac{a+1}{a-1}\right)\ge 1$. This implies

$$g(a)\ge 1-\frac{2a\log a}{a^2-1}=1-\frac{a\log a^2}{a^2-1}>0,$$

as this is equivalent with $L(a^2,1)<G(a^2,1)$ of the Lemma.

Since $f'(x)>0$, the function $f$ is strictly increasing, and the proof of Theorem 2 is finished.

**Remarks.** 1) Particularly, by letting $p_0=\dfrac{e+1}{e-1}$ we get $f(1)=1$, and the inequality

$$\log\left(\frac{p_0^x+1}{p_0^x-1}\right)\le\frac{1}{x}\tag{2}$$

follows. For $x=\dfrac{1}{k}$ and $p_0=\dfrac{1}{t_0}$, with the use of (2) an easier proof of Lemma 2.9 of [1] can be deduced.

2) Let $0<x\le y\le 1$. Then

$$x\log\left(\frac{p^x+1}{p^x-1}\right)\le y\log\left(\frac{p^y+1}{p^y-1}\right)\le\log\left(\frac{p+1}{p-1}\right).\tag{3}$$

This offers an extension of inequality (1) for $x=\dfrac{1}{k}$ and $p=\dfrac{1}{t}$.

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## 2.16 Series expansions related to the logarithmic mean

### 1. Introduction

In what follows, we let $x \in (-1, 1)$. The well-known series expansion for the logarithmic function

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{2.1}$$

(discovered for the first time by N. Mercator (1668), see e.g. [3]) applied to "$-x$" in place of $x$ gives

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \tag{2.2}$$

By subtracting equations (1) and (2) we get the Gregory series (J. Gregory (1668), see [3])

$$\log\frac{1+x}{1-x} = 2 \cdot \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \tag{2.3}$$

The Newton's binomial expansion (stated for the first time by I. Newton (1665), and considered later also by L. Euler (1748) states that for any rational $\alpha$ one has

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots, \tag{2.4}$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

denotes a generalized binomial coefficient.

Particularly, for $\alpha = -\dfrac{1}{2}$ we get

$$\binom{-1/2}{k} = (-1)^k \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k \cdot k!},$$

and for "$-x^2$" in place of $x$ in (4), we get the expansion

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots + \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k \cdot k!}x^{2k} + \cdots, \quad (2.5)$$

which we will need later.

The logarithmic mean of two positive real numbers $a$ and $b$ is defined by

$$L(a,b) = \frac{b-a}{\log b - \log a}(b \neq a); L(a,a) = a \quad (2.6)$$

This mean has many connections and applications in various domains of Mathematics (see e.g. [1]; [4]-[16]). Particularly, the following classical relations are true for $a \neq b$ :

$$G < L < A; \quad (2.7)$$

$$L < A_{1/3}, \quad (2.8)$$

where

$$A_r = A_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{1/r};$$

$$A = A_1(a,b) = \frac{a+b}{2},$$

$$G = A_0(a,b) = \lim_{r \to 0} A_r(a,b) = \sqrt{ab}$$

are the classical power mean, resp. arithmetic and geometric means of $a$ and $b$. Many applications to (7) and (8), as well as new proofs are known in the literature. New proofs to (7) and (8), based on integral inequalities have been obtained by the author in [6], [10]. For recent new proofs of (7), see [14], [15]. For the history of (7), see [2], [7], [13].

In what follows, we will show that the Gregory series (3) and Newton's series expansion (5) offer new proofs to (7), as well as to (8). Such a method with series, for the arithmetic-geometric mean of Gauss, appears in [9].

## 2. The proofs

Since the means $L$ and $A_r$ are homogeneous, it is easy to see that (7), resp. (8) are equivalent to

$$\sqrt{t} < \frac{t-1}{\log t} < \frac{t+1}{2} \tag{7'}$$

resp.

$$\frac{t-1}{\log t} < \left(\frac{\sqrt[3]{t}+1}{2}\right)^3, \tag{8'}$$

where $t = \dfrac{b}{a} > 1$ (say)

Now, to prove the right side of (7') remark that for $x = \dfrac{t-1}{t+1} \in (0,1)$ in (3) we have, by $t = \dfrac{1+x}{1-x}$ that

$$\log t = \log \frac{1+x}{1-x} > 2x = 2 \cdot \frac{t-1}{t+1}.$$

This gives immediately the right side of (7') for $t > 1$.

For the proof of (8') apply the same idea, by remarking that

$$\log t > 2 \cdot \left(x + \frac{x^3}{3}\right) = 2 \cdot \frac{t-1}{t+1} \cdot \frac{t^2+t+1}{(t+1)^2} \cdot \frac{4}{3}$$

As $(t-1)(t^2+t+1) = t^3 - 1$, we get the inequality

$$\frac{t^3-1}{3\log t} < \left(\frac{t+1}{2}\right)^3 \tag{8''}$$

Now, to get (8') from (8"), it is enough to replace $t$ with $\sqrt[3]{t}$, and the result follows.

For the proof of left side of (7') we will remark first that for the $k$th terms in the right side of (3) and (5) one has

$$\frac{x^{2k}}{2k+1} < \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k \cdot k!} \cdot x^{2k} \tag{2.9}$$

Since $0 < x$, it is sufficient to prove the inequality

$$1 \cdot 3 \cdot \ldots \cdot (2k-1)(2k+1) > 2^k \cdot k! \tag{2.10}$$

This follows immediately e.g. by mathematical induction. For $k = 1, 2$ it is true; and assuming it for $k$, we get

$$1 \cdot 3 \cdot \ldots \cdot (2k+1)(2k+3) > 2^k \cdot k!(2k+3) > 2^{k+1}(k+1)!,$$

where the last inequality holds by $2k + 3 > 2k + 2$.

Therefore, we get the inequality

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots < x\left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots\right), \tag{2.11}$$

and by letting $\dfrac{1+x}{1-x} = t > 1$ we obtain

$$\log t < 2 \cdot \left(\frac{t-1}{t+1}\right) \cdot \frac{t+1}{2\sqrt{t}},$$

so the left side of (7') follows, too.

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# 2.17 On some inequalities for the identric, logarithmic and related means

## 1. Introduction

Since last few decades, the inequalities involving the classical means such as arithmetic mean $A$, geometric mean $G$, identric mean $I$ and logarithmic mean $L$ and weighted geometric mean $S$ have been studied extensively by numerous authors, e.g. see [1], [2], [4], [7], [8], [15], [16], [17].

For two positive real numbers $a$ and $b$, we define

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab},$$

$$L = L(a,b) = \frac{a-b}{\log(a) - \log(y)}, \quad a \neq b,$$

$$I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}, \quad a \neq b,$$

$$S = S(a,b) = (a^a b^b)^{1/(a+b)}.$$

For the historical background of these means we refer the reader to [2], [4], [5], [12], [15], [16], [17]. Generalizations, or related means are studied in [3], [8], [7], [10], [12], [14], [18]. Connections of these means with trigonometric or hyperbolic inequalities are pointed out in [3], [13], [6], [14], [17].

Our main result reads as follows:

**Theorem 1.1.** *For all distinct positive real numbers $a$ and $b$, we have*

$$1 < \frac{I}{\sqrt{I(A^2, G^2)}} < \frac{2}{\sqrt{e}}. \tag{1.1}$$

*Both bounds are sharp.*

**Theorem 1.2.** *For all distinct positive real numbers $a$ and $b$, we have*

$$1 < \frac{2I^2}{A^2 + G^2} < c \tag{1.2}$$

*where* $c = 1.14\ldots$. *The bounds are best possible.*

**Remark 1.3. A.** The left side of (1.2) may be rewritten also as

$$I > Q(A, G), \tag{1.3}$$

where $Q(x, y) = \sqrt{(x^2 + y^2)/2}$ denotes the root square mean of $x$ and $y$. In 1995, Seiffert [25] proved the first inequality in (1.1) by using series representations, which is rather strong. Now we prove that, (1.3) is a refinement of the first inequality in (1.1). Indeed, by the known relation $I(x, y) < A(x, y) = (x + y)/2$, we can write

$$I(A^2, G^2) < (A^2 + G^2)/2 = Q(A, G)^2,$$

so one has:

$$I > Q(A, G) > \sqrt{I(A^2, G^2)}. \tag{1.4}$$

As we have $I(x^2, y^2) > I(x, y)^2$ (see Sándor [15]), hence (1.4) offers also a refinement of

$$I > I(A, G). \tag{1.5}$$

Other refinements of (1.5) have been provided in a paper by Neuman and Sándor [10]. Similar inequalities involving the logarithmic mean, as well as Sándor's means $X$ and $Y$, we quote [3], [13], [14]. In the second part of paper, similar results will be proved.

**B.** In 1991, Sándor [16] proved the inequality

$$I > (2A + G)/3. \tag{1.6}$$

It is easy to see that, the left side of (1.2) and (1.6) cannot be compared.

In 2001 Sándor and Trif [21] have proved the following inequality:

$$I^2 < (2A^2 + G^2)/3. \tag{1.7}$$

The left side of (1.2) offers a good companion to (1.7). We note that the inequality (1.7) and the right side of (1.2) cannot be compared.

In [25], Seiffert proved the following relation:

$$L(A^2, G^2) > L^2, \tag{1.8}$$

which was refined by Neuman and Sándor [10] (for another proof, see [8]) as follows:

$$L(A, G) > L. \tag{1.9}$$

We will prove with a new method the following refinement of (1.8) and a counterpart of (1.9):

**Theorem 1.4.** *We have*

$$L(A^2, G^2) = \frac{(A+G)}{2} L(A, G) > \frac{(A+G)}{2} L > L^2, \tag{1.10}$$

$$L(I, G) < L, \tag{1.11}$$

$$L < L(I, L) < L \cdot (I - L)/(L - G). \tag{1.12}$$

**Corollary 1.5.** *One has*

$$G \cdot I/L < \sqrt{I \cdot G} < L(I, G) < L, \tag{1.13}$$

$$(L(I, G))^2 < L \cdot L(I, G) < L(I^2, G^2) < L \cdot (I + G)/2. \tag{1.14}$$

**Remark 1.6. A.** Relation (1.13) improves the inequality

$$G \cdot I/L < L(I, G),$$

due to Neuman and Sándor [10]. Other refinements of the inequality

$$L < (I + G)/2 \tag{1.15}$$

are provided in [19].

**B.** Relation (1.12) is indeed a refinement of (1.15), as the weaker inequality can be written as $(I - L)/(L - G) > 1$, which is in fact (1.15).

The mean $S$ is strongly related to other classical means. For example, in 1993 Sándor [17] discovered the identity

$$S(a, b) = I(a^2, b^2)/I(a, b), \tag{1.16}$$

where $I$ is the identric mean. Inequalities for the mean $S$ may be found in [15], [17], [20].

The following result shows that $I$ and $S(A,G)$ cannot be compared, but this is not true in case of $I$ and $S(Q,G)$. Even a stronger result holds true.

**Theorem 1.7.** *None of the inequalities $I > S(A,G)$ or $I < S(A,G)$ holds true. On the other hand, one has*

$$S(Q,G) > A > I, \tag{1.17}$$

$$I(Q,G) < A. \tag{1.18}$$

**Remark 1.8.** By (1.17) and (1.18), one could ask if $I$ and $I(Q,G)$ may be compared to each other. It is not difficult to see that, this becomes equivalent to one of the inequalities

$$\frac{y\log y}{y-1} < (\text{or} >)\frac{x}{\tanh(x)}, \quad x > 0, \tag{1.19}$$

where $y = \sqrt{\cosh(2x)}$. By using the Mathematica Software [11], we can show that (1.19) with "$<$ " is not true for $x = 3/2$, while (1.19) with "$>$" is not true for $x = 2$.

## 2. Lemmas and proofs of the main results

The following lemma will be utilized in our proofs.

**Lemma 2.1.** *For $b > a > 0$ there exists an $x > 0$ such that*

$$\frac{A}{G} = \cosh(x),\ \frac{I}{G} = e^{x/\tanh(x)-1}. \tag{2.1}$$

**Proof.** For any $a > b > 0$, one can find an $x > 0$ such that $a = e^x \cdot G$ and $b = e^{-x} \cdot G$. Indeed, it is immediate that such an $x$ is (by considering $a/b = e^{2x}$),

$$x = (1/2)\log(a/b) > 0.$$

Now, as
$$A = G \cdot (e^x + e^{-x})/2 = G\cosh(x),$$
we get
$$A/G = \cosh(x).$$
Similarly, we get
$$I = G \cdot (1/e)\exp(x(e^x + e^{-x})/(e^x - e^{-x})),$$
which gives $I/G = e^{x/\tanh(x)-1}$.

**Proof of Theorem 1.1.** For $x > 0$, we have
$$I/G = e^{x/\tanh(x)-1} \quad \text{and} \quad A/G = \cosh(x)$$
by Lemma 2.1. Since
$$\log(I(a,b)) = \frac{a\log a - b\log b}{a-b} - 1,$$
we get
$$\log(\sqrt{I((A/G)^2, 1)}) = \frac{\cosh(x)^2\log(\cosh(x))}{\cosh(x)^2 - 1} - \frac{1}{2}.$$
By using this identity, and taking the logarithms in the second identity of (2.1), the inequality
$$0 < \log(I/G) - \log(\sqrt{I(A/G)^2, 1}) < \log 2 - 1/2$$
becomes
$$\frac{1}{2} < f(x) < \log 2, \tag{2.}$$
where
$$f(x) = \frac{x}{\tanh(x)} - \frac{\log(\cosh(x))}{\tanh(x)^2}.$$
A simple computation (which we omit here) for the derivative of $f(x)$ gives:
$$\sinh(x)^3 f'(x) = 2\cosh(x)\log(\cosh(x)) - x\sinh(x). \tag{2.3}$$

The following inequality appears in [6]:

$$\log(\cosh(x)) > \frac{x}{2}\tanh(x),\ \ x > 0, \tag{2.4}$$

which gives $f'(x) > 0$, so $f(x)$ is strictly increasing in $(0, \infty)$. As

$$\lim_{x\to 0} f(x) = 1/2 \quad \text{and} \quad \lim_{x\to\infty} f(x) = \log 2,$$

the double inequality (2.2) follows. So we have obtained a new proof of (1.1). □

We note that Seiffert's proof is based on certain infinite series representations. Also, our proof shows that the constants 1 and $2/\sqrt{e}$ in (1.1) are optimal.

**Lemma 2.2.** *Let*

$$f(x) = \frac{2x}{\tanh(x)} - \log\left(\frac{\cosh(x)^2+1}{2}\right),\ \ x > 0.$$

*Then*

$$2 < f(x) < f(1.606\ldots) = 2.1312\ldots. \tag{2.5}$$

**Proof.** One has $(\cosh(x)^2+1)/2f'(x) = g(x)$, where

$$g(x) = \sinh(x)\cosh(x)^3 x\cosh(x)^2 + \sinh(x)\cosh(x) - x$$

$$-\cosh(x)\sinh(x)^3 2\sinh(x)\cosh(x) - x\cosh(x)^2 x,$$

by remarking that

$$\sinh(x)\cosh(x)^3 - \cosh(x)\sinh(x)^3 = \sinh(x)\cosh(x).$$

Now, a simple computation gives

$$g'(x) = \sinh(x)\cdot(3\sinh(x) - 2x\cosh(x)) = 3\sinh(x)\cosh(x)\cdot k(x),$$

where $k(x) = \tanh(x) - 2x/3$. As it is well known that the function $\tanh(x)/x$ is strictly decreasing, the equation $\tanh(x)/x = 2/3$ can have at most a single solution. As $\tanh(1) = 0.7615\ldots > 2/3$ and $\tanh(3/2) = 0.9051\ldots < 1 = (2/3)\cdot(3/2)$, we find that the equation $k(x) = 0$ has a single solution $x_0$ in $(1, 3/2)$, and also that $k(x) > 0$ for $x$ in $(0, x_0)$ and $k(x) < 0$ in $(x_0, 3/2)$. This means that the function $g(x)$ is strictly increasing in the interval $(0, x_0)$ and strictly decreasing in $(x_0, \infty)$. As $g(1) = 0.24 > 0$, clearly $g(x_0) > 0$, while $g(2) = -3.01.. < 0$ implies that there exists a single zero $x_1$ of $g(x)$ in $(x_0, 2)$. In fact, as $g(3/2) = 0.21 > 0$, we get that $x_1$ is in $(3/2, 2)$.

From the above consideration we conclude that $g(x) > 0$ for $x \in (0, x_1)$ and $g(x) < 0$ for $x \in (x_1, \infty)$. Therefore, the point $x_1$ is a maximum point to the function $f(x)$. It is immediate that $\lim_{x\to 0} f(x) = 2$. On the other hand, we shall compute the limit of f(x) at $\infty$. Clearly $t = \cosh(x)$ tends to $\infty$ as $x$ tends to $\infty$. Since

$$\log(t^2+1) - \log(t^2) = \log((t^2+1)/t^2)$$

tends to $\log 1 = 0$, we have to compute the limit of

$$l(x) = 2x\cosh(x)/\sinh(x) - 2\log(\cosh(x)) + \log 2.$$

Here

$$2x\frac{\cosh(x)}{\sinh(x)} - 2\log(\cosh(x)) = 2\log\left(\frac{\exp(x\cosh(x)/\sinh(x))}{\cosh x}\right).$$

Now remark that

$$(x\cosh(x) - x\sinh(x))/\sinh(x)$$

tends to zero, as

$$x\cosh(x) - x\sinh(x) = x\exp(-x).$$

As $\exp(x)/\cosh x$ tends to 2, by the above remarks we get that the the limit of $l(x)$ is $2\log 2 + \log 2 = 3\log 2 > 2$. Therefore, the left side of inequality (2.5) is proved. The right side follows by the fact that $f(x) < f(x_1)$. By Mathematica Software® [11], we can find $x_1 = 1.606\ldots$ and $f(x_1) = 2.1312\ldots$.

**Proof of Theorem 1.2.** By Lemma 2.1, one has

$$(I/G)^2 = \exp(2(x/\tanh(x) - 1)),$$

while $(A/G)^2 = \cosh(x)^2$, $x > 0$. It is immediate that, the left side of (2.5) implies the left side of (1.2). Now, by the right side of (2.5) one has

$$I^2 < \exp(c_1)(A^2 + G^2)/2,$$

where $c_1 = f(x_1) - 2 = 0.13\cdots$. Since $\exp(0.13\cdots) = 1.14$, we get also the right side of (1.2). □

**Proof of Theorem 1.4.** The first relation of (1.10) follows from the identity

$$L(x^2, y^2) = ((x + y)/2) \cdot L(x, y),$$

which is a consequence of the definition of logarithmic mean, by letting $x = A, y = G$. The second inequality of (1.10) follows by (1.9), while the third one is a consequence of the known inequality

$$L < (A + G)/2. \tag{2.6}$$

A simple proof of (2.6) can be found in [12]. For (1.11), by the definition of logarithmic mean, one has

$$L(I, G) = (I - G)/\log(I/G),$$

and on base of the known identity

$$\log(I/G) = A/L - 1$$

(see [15], [22]), we get

$$L(I, G) = ((I - G)/(A - L))L < L,$$

since the inequality $(I-G)/(A-L)<1$ can be rewritten as

$$I+L<A+G$$

due to Alzer (see [15]).

The first inequality of (1.12) follows by the fact that $L$ is a mean (i.e. if $x<y$ then $x<L(x,y)<y$), and the well known relation $L<I$ (see [15]). For the proof of last relation of (1.12) we will use a known inequality of Sándor ([15]), namely:

$$\log(I/L)>1-G/L. \tag{2.7}$$

Write now that $L(I,L)=(I-L)/\log(I/L)$, and apply (2.7). Therefore, the proof of (1.12) is finished. □

**Proof of Corollary 1.5.** The first inequality of (1.13) follows by the well known relation $L>\sqrt{GI}$ (see [2]), while the second relation is a consequence of the classical relation $L(x,y)>G(x,y)$ (see e.g. [15]) applied to $x=I, y=G$. The last relation is inequality (1.10).

The first inequality of (1.14) follows by (1.10), while the second one by

$$L(I^2,G^2)=L(I,G)\cdot(I+G)/2$$

and inequality $L<(I+G)/2$. The last inequality follows in the same manner. □

**Proof of Theorem 1.7.** Since the mean $S$ is homogeneous, the relation $I>S(A,G)$ may be rewritten as $I/G>S(A/G,1)$, so by using logarithm and applying Lemma 2.1, this inequality may be rewritten as

$$\frac{x}{\tanh(x)}-1>\frac{\cosh(x)\log(\cosh(x))}{1+\cosh(x)}, \quad x>0. \tag{2.8}$$

By using Mathematica Software® [11], one can see that inequality (2.8) is not true for $x>2.284$. Similarly, the reverse inequality of (2.8) is not true, e.g. for $x<2.2$. These show that, $I$ and $S(A,G)$ cannot be compared to each other. In order to prove inequality (1.17), we will use the following result proved in [20]: The inequality

$$S > Q \tag{2.9}$$

holds true. By writing (2.9) as $S(a, b) > Q(a, b)$ for $a = Q$, $b = G$, and remarking that $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ and that $(Q^2 + G^2)/2 = A^2$, we get the first inequality of (1.17). The second inequality is well known (see [15] for history and references).

By using $I(a, b) < A(a, b) = (a + b)/2$ for $a = Q$ and $b = G$ we get

$$I(Q, G) < (Q + G)/2.$$

On the other hand by inequality

$$(a + b)/2 < \sqrt{(a^2 + b^2)/2} \quad \text{and} \quad (Q^2 + G^2)/2 = A^2,$$

inequality (1.18) follows as well. This completes the proof. □

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## 2.18 New refinements of two inequalities for means

### 1. Introduction

The logarithmic and identric means of two positive numbers $a$ and $b$ are defined by

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}\ (a \neq b);\ L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)}\ (a \neq b);\ I(a,a) = a,$$

respectively.

Let $A = A(a,b) := \frac{a+b}{2}$ and $G = G(a,b) := \sqrt{ab}$ denote the arithmetic and geometric means of $a$ and $b$, respectively. For these means many interesting inequalities have been proved. For a survey of results, see [1], [3], [7], [11], [12]. It may be surprising that the means of two arguments have applications in physics, economics, statistics, or even in meteorology. See e.g. [3], [6] and the references therein. For connections of such means with Ky Fan, or Huygens type inequalities; or with Seiffert and Gini type means,we quote papers [13] and [14]; as well as [5], [8], or [12].

In what follows we shall assume that $a \neq b$. In paper [2] H. Alzer proved that

$$\sqrt{GI} < L < \frac{G+I}{2} \tag{1}$$

and in [1] he proved that

$$AG < LI \text{ and } L + I < A + G. \tag{2}$$

In paper [8] the author proved that the first inequality of (2) is weaker than the left side of (1), while the second inequality of (2) is stronger than the right side of (1). In fact, these statements are consequences of

$$I > \sqrt[3]{A^2G} \tag{3}$$

and

$$I > \frac{2A+G}{3}. \tag{4}$$

Clearly, by the weighted arithmetic-geometric mean inequality, (4) implies (3), but one can obtain different methods of proof for these results (see [8]). In [7] J. Sándor has proved that

$$\ln\frac{I}{L} > 1 - \frac{G}{L} \tag{5}$$

and this was used in [9] to obtain the following refinement of right side of (1):

$$L < \frac{I+aG}{1+a} < \frac{I+G}{2}, \tag{6}$$

where $a = \sqrt{I}/\sqrt{L} > 1$.

In paper [11] the following refinements of left side of (1) has been also proved:

$$\sqrt{IG} < \frac{I-G}{A-L} \cdot L < L. \tag{7}$$

The aim of this paper is to offer certain new refinements of other type for inequalities (1).

## 2. Main results

The main result of this paper is contained in the following:

**Theorem.** *One has*

$$L < \sqrt{\frac{(A+G)(L+G)}{4}} < \frac{A+L+2G}{4} < \frac{I+G}{2} \tag{8}$$

*and*

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \sqrt{GI}. \tag{9}$$

**Proof.** First we note that the second inequality of (8) follows by $\sqrt{xy} < \frac{x+y}{2}$, applied to

$$x := \frac{A+G}{2} \quad \text{and} \quad y = \frac{L+G}{2},$$

while the last inequality can be written as

$$I > \frac{A+L}{2}. \tag{10}$$

This appears in [7], but we note that follows also by (4) and

$$\frac{2A+G}{3} > \frac{A+L}{2}, \tag{11}$$

which can be written equivalently as

$$L < \frac{2G+A}{3}, \tag{12}$$

due to B.C. Carlson [4] and G. Pólya and G. Szegö [6].

Thus we have to prove only the first inequality of (8).

For this purpose, we shall use inequality (5) combined with the identity

$$\ln \frac{I}{G} = \frac{A-L}{L}, \tag{13}$$

due to H.J. Seiffert [15]. See also [9] for this and related identities.

Since $\ln x > \frac{2(x-1)}{x+1}$ for $x > 1$ (equivalent in fact with the classical inequality $L(x,1) < A(x,1)$), by letting $x = \frac{L}{G}$, and by

$$\ln \frac{I}{L} = \ln \frac{I}{G} - \ln \frac{L}{G}, \;\; \ln \frac{L}{G} > 2 \cdot \frac{L-G}{L+G}$$

and (13) combined with (5) gives the following inequality:

$$2 \cdot \frac{L-G}{L+G} < \frac{A+G}{L} - 2, \tag{14}$$

which after some elementary computations gives the first inequality of (8).

**Remark.** The first and third term of (8) is exactly inequality (12). Therefore, the first two inequalities provide also a refinement of (12).

Now, for the proof of relation (9) remark first that the first inequality has been proved by the author in paper [10]. The second inequality will be reduced to an inequality involving hyperbolic functions. Put $a = e^x G$, $b = e^{-x} G$, where $x > 0$ (for this method see e.g. [1]). Then the inequality to be proved becomes equivalent to

$$\ln\left(\frac{\cosh x + 1}{2}\right) > \frac{3}{4}\left(\frac{x\cosh x - \sinh x}{\sinh x}\right). \tag{15}$$

Let us introduce the function

$$f(x) = 4\ln\left(\frac{\cosh x + 1}{2}\right) - 3x\coth x + 3, \ \ x > 0.$$

An immediate computation gives

$$(\cosh x + 1)\sinh^2 x \cdot f'(x)$$

$$= \sinh^3 x - 3\sinh x + 3x\cosh x + 3x - 3\sinh x\cosh x = g(x).$$

One has

$$g'(x) = 3\sinh x(\sinh x\cosh x + x - 2\sinh x).$$

Now, as it is well known that $\sinh x < x\cosh x$, we can remark that

$$\sinh x < \sqrt{x\sinh x\cosh x} \le \frac{x + \sinh x\cosh x}{2} \text{ by } \sqrt{uv} \le \frac{u+v}{2}.$$

This in turn implies $g'(x) > 0$, and as $g(x)$ can be defined for $x \ge 0$ and $g(0) = 0$, we get $g(x) \ge 0$, and $g(x) > 0$ for $x > 0$. Thus $f'(x) > 0$ for $x > 0$, so $f$ is strictly increasing and as $\lim\limits_{x\to 0} f(x) = 0$, inequality (15) follows.

This finishes the proof of the Theorem.

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## 2.19 On certain entropy inequalities

### 1. Introduction

**1.** Let $p, q$ be a positive real numbers such that $p+q=1$. The entropy of the probability vector $(p, q)$ introduced in [7] is

$$H(p,q) = -p\ln p - q\ln q.$$

In the note [1] a new proof of the following double inequality (see [7]) has been provided:

**Theorem 1.1.** *One has*

$$\ln p \cdot \ln q \leq H(p,q) \leq \frac{(\ln p \cdot \ln q)}{\ln 2}. \tag{1.1}$$

Our aim in what follows is twofold. First, by remaking a connection with the logarithmic mean, we will obtain improvements of (1.1), in fact, a new proof. Secondly, the connection of $H$ with a mean $S$ introduced and studied, for example in [2], [4], [6], will give new relations for the entropy $H(p, q)$.

### 2. Main results

**2.** Let $p, q$ be as above. First, we will prove the following relation:

**Theorem 2.1.**

$$\begin{aligned} \ln p \cdot \ln q &\leq (\sqrt{p} + \sqrt{q}) \ln p \cdot \ln q \\ &\leq H(p,q) \\ &\leq A(p,q) \ln p \cdot \ln q \\ &\leq \frac{(\ln p \cdot \ln q)}{\ln 2}, \end{aligned} \tag{2.1}$$

where

$$A(p,q) = \frac{2}{q-p} \int_p^q \frac{s-1}{\ln s} ds.$$

**Proof.** We note that, since $q-1=-p$ and $p-1=-q$, one can write

$$H(p,q)=(q-1)\ln p+(p-1)\ln q=(\ln p)(\ln q)\left[\frac{q-1}{\ln q}+\frac{p-1}{\ln p}\right].$$

Now, $\dfrac{q-1}{\ln q}$ is equal to $L(q,1)$, where $L(x,y)$ is the logarithmic mean of $x$ and $y$ $(x \neq y)$ defined by

$$L(x,y)=\frac{x-y}{\ln x-\ln y}.$$

For the mean $L$ there exists an extensive amount of literature. We shall use only the following relations

$$G<L<A, \tag{2.2}$$

where

$$G=G(x,y)=\sqrt{xy} \quad \text{and} \quad A=A(x,y)=\frac{x+y}{2}$$

respectively denote the geometric and arithmetic means of $x,y$ (see e.g. [2], [5]). By the left hand side of (2.2) one has

$$L(p,1)+L(q,1)>\sqrt{p}+\sqrt{q}.$$

Here $\sqrt{p}+\sqrt{q}>1$ since this is equivalent to

$$(\sqrt{p}+\sqrt{q})^2>1, \quad \text{i.e.} \quad p+q+2\sqrt{pq}=1+2\sqrt{pq}>1,$$

which is trivial.

Therefore, the left sides of (2.1) are proved. For the right side, let us introduce the following function:

$$f(p)=\frac{p-1}{\ln p},\ p\in(0,1).$$

An easy computation shows that

$$f'(p)=\frac{p\ln p-p+1}{p\ln^2 p}, \quad f''(p)=\frac{2(p-1)-(p+1)\ln p}{p^2\ln^3 p}.$$

By the right side of (2.2), one has

$$L(p,1) < \frac{p+1}{2}, \quad \text{i.e.} \quad 2(p-1) > (p+1)\ln p.$$

Since $\ln^3 p < 0$, we get that $f''(p) < 0$. Therefore, $f$ is a concave function on $(0,1)$. Now, by the Jensen-Hadamard inequality, one has

$$\frac{f(p)+f(q)}{2} \le \frac{1}{q-p}\int_p^q f(s)ds \le f\left(\frac{p+q}{2}\right),$$

so

$$L(p,1) + L(q,1) \le A(p,q) \le 2L\left(\frac{1}{2},1\right) = \frac{1}{\ln 2},$$

completing the proof of the right side inequality in (2.1). □

**3.** We now obtain an interesting connection of the entropy $H(p,q)$ with a mean $S$, defined by (see [2], [6])

$$S = S(a,b) = (a^a \cdot b^b)^{\frac{1}{a+b}}, \quad a,b > 0. \tag{2.3}$$

Let $p,q$ be as in the introduction. Then (2.3) implies the interesting relation

$$H(p,q) = \ln \frac{1}{S(p,q)}. \tag{2.4}$$

Since there exist many known results for the mean $S$, by equality (2.4), these give some information on the entropy $H$. For example, in [6], the following are proved:

$$\sqrt{\frac{a^2+b^2}{2}} \le S(a,b) \le \frac{A^2}{G}, \tag{2.5}$$

$$\frac{A^2-G^2}{A^2} \le \ln\frac{S}{G} \le \frac{A^2-G^2}{GA}, \tag{2.6}$$

$$S \le \frac{a\sqrt{2}-G}{\sqrt{2}-1}, \tag{2.7}$$

where $S = S(a,b)$, etc. By (2.4)-(2.7) the following results are immediate:

**Theorem 2.2.**

$$2\ln 2 + \frac{1}{2}\ln(pq) \le H(p,q) \le \frac{1}{2}\ln 2 - \frac{1}{2}\ln(1-2pq), \tag{2.8}$$

$$\frac{4pq-1}{2\sqrt{pq}} - \frac{1}{2}\ln(pq) \le H(p,q) \le 4pq - 1 - \frac{1}{2}\ln(pq), \tag{2.9}$$

$$H(p,q) \ge \ln\left(\frac{2-\sqrt{2}}{1-\sqrt{2pq}}\right). \tag{2.10}$$

**Proof.** Apply (2.5)-(2.7) and remark that

$$A = \frac{p+q}{2} = \frac{1}{2}, \quad G = \sqrt{pq}. \qquad \square$$

The following result shows a connection with the so-called identric mean $I$, defined by

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a \ne b.$$

In the paper [3], the following identity appears:

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}. \tag{2.11}$$

By (2.3), the entropy $H$ is connected to the mean $I$ by

$$H(p,q) = \ln I(p,q) - \ln I(p^2,q^2). \tag{2.12}$$

Since

$$\frac{4A^2 - G^2}{3I} \le I \le \frac{A^4}{I^3}$$

(see [6]), the following holds.

**Theorem 2.3.**

$$4\ln 2 + 3\ln I(p,q) \le H(p,q) \le \ln 3 + \ln I(p,q) - \ln(1-pq). \tag{2.13}$$

**4.** Finally, we shall deduce two series representations for $H$. In [6], the following representations are proved

$$\ln \frac{S}{A} = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \cdot z^{2k}, \tag{2.14}$$

$$\ln \frac{S}{G} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot z^{2k}, \tag{2.15}$$

where $z = \dfrac{b-a}{b+a}$. Now, let $a = p$, $b = q$ with $p + q = 1$. Then, by (2.3), one can deduce:

**Theorem 2.4.**

$$H(p,q) = \ln 2 - \sum_{k=1}^{\infty} \frac{(p-q)^{2k}}{2k(2k-1)} \tag{2.16}$$

*and*

$$H(p,q) = \frac{1}{2}\ln(pq) - \sum_{k=1}^{\infty} \frac{(p-q)^{2k}}{2k-1}. \tag{2.17}$$

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# Chapter 3

# Integral inequalities and means

"Mathematics is not a dead letter which can be stored in libraries, it is a living thinking."

(J. Leray)

"I love inequalities. So, if somebody shows me a new inequality, I say, "Oh, that's beautiful, let me think about it", and I may have some ideas connected with it."

(L. Nirenberg)

## 3.1 Some integral inequalities

The aim of this note is to prove some integral inequalities and to find interesting applications for the logarithmic and exponential functions. These relations have some known corollaries ([3], [4], [5], [8]).

**Theorem 1.** *Let* $f : [a, b] \to \mathbb{R}$ $(a < b)$ *be a differentiable function with increasing (strictly increasing) derivative on* $[a, b]$*. Then one has the* *following inequalities:*

$$\int_a^b f(t)dt \underset{(>)}{\geq} (b-a)f\left(\frac{a+b}{2}\right) \tag{1}$$

$$2\int_a^b f(t)dt \underset{(<)}{\leq} (b-a)f\left(\sqrt{ab}\right) + \left(\sqrt{b}-\sqrt{a}\right)\left(\sqrt{b}f(b)+\sqrt{a}f(a)\right) \tag{2}$$

*(Here $0 \leq a < b$).*

**Proof.** The Lagrange mean-value theorem implies:

$$f(y) - f(x) \underset{(>)}{\geq} (y-x)f'(x) \text{ for all } x, y \in [a,b].$$

Take $x = (a+b)/2$ and integrate the obtained inequality:

$$\int_a^b f(y)dy - (b-a)f\left(\frac{a+b}{2}\right) \underset{(>)}{\geq} f'\left(\frac{a+b}{2}\right)\int_a^b \left(y - \frac{a+b}{2}\right)dy = 0,$$

i.e. relation (1).

In order to prove (2) consider as above the inequality

$$f(y) - f(x) \underset{(<)}{\leq} (y-x)f'(y)$$

with $x = \sqrt{ab}$. Integrating by parts on $[a,b]$ we get

$$\int_a^b f(y)dy - (b-a)f\left(\sqrt{ab}\right) \underset{(<)}{\leq} \left(y - \sqrt{ab}\right)f(y)\Big|_a^b - \int_a^b f(y)dy$$

which easily implies (2).

**Remark.** Inequality (1) is called sometimes "Hadamard's inequality" and it is valid for convex functions $f$ as well with the same proof, but using $f'_+\left(\frac{a+b}{2}\right)$ instead of $f'\left(\frac{a+b}{2}\right)$ (see also [1]).

In applications is useful the following generalization of (1) (see [9]).

**Theorem 2.** *Let $f : [a,b] \to \mathbb{R}$ be a $2k$-times differentiable function, having continuous $2k$-th derivative on $[a,b]$ and satisfying*

$$f^{(2k)}(t) \underset{(>)}{\geq} 0 \textit{ for } t \in (a,b).$$

*Then one has the inequality:*

$$\int_a^b f(t)dt \underset{(>)}{\geq} \sum_{p=1}^{k} \frac{(b-a)^{2p-1}}{2^{2p-2}(2p-1)!} f^{(2p-2)}\left(\frac{a+b}{2}\right). \tag{3}$$

**Proof.** Apply Taylor's formula (with Lagrange remainder term) for $f$ around $\left(\frac{a+b}{2}\right)$ and integrate term by term this relation. Remarking that

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2m-1} dx = 0, \;\; m = 1, 2, 3, \ldots$$

we obtain

$$\begin{aligned}\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) &+ \frac{(b-a)}{2^2 \cdot 3!} f''\left(\frac{a+b}{2}\right) \\ &+ \ldots + \frac{(b-a)^{2k-1}}{2^{2k-2}(2k-1)!} f^{(2k-2)}\left(\frac{a+b}{2}\right) \\ &+ \int_a^b \frac{(x-(a+b)/2)^{2k}}{(2k)!} f^{(2k)}(\xi)dx.\end{aligned}$$

Taking into account $f^{(2k)}(\xi) \underset{(>)}{\geq} 0$, we get the desired inequality (3).

**Applications.** 1) Let $a > 0$, $b = a + 1$, $f_1(t) = \frac{1}{t}$ and $f_2(t) = -\ln t$ in (1). We can easily deduce the following double inequality:

$$\frac{2a+2}{2a+1} < \frac{e}{\left(1 + \frac{1}{a}\right)^a} < \sqrt{1 + \frac{1}{a}} \tag{4}$$

containing inequalities studied by E.R. Love [4] and G. Pólya- G. Szegö [7]. Using Bernoulli's inequality we have

$$\left(1 + \frac{1}{2a+1}\right)^{5/2} > 1 + \frac{5}{4a+2} \geq 1 + \frac{1}{a}, \text{ for } a \geq 2.$$

Hence we have:

$$\left(1 + \frac{1}{a}\right)^{a+2/5} < e < \left(1 + \frac{1}{a}\right)^{a+1/2}, \; a \geq 2. \tag{5}$$

2) By repeating the same argument in (3) for $k = 2$, $b = a+1$, $(a > 0)$,

$$f_1(t) = \frac{1}{t}, \quad f_2(t) = -\ln t,$$

we obtain

$$\frac{2a+2}{2a+1} e^{\frac{1}{6(2a+1)^2}} < \frac{e}{\left(1+\frac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \cdot e^{\frac{1}{3(2a+1)^2}}. \tag{6}$$

This inequality implies for $a > 0$ e.g. that

$$e^{\frac{1}{2a}\left(1-\frac{1}{a}\right)} < \frac{e}{\left(1+\frac{1}{a}\right)^a} < e^{\frac{1}{2a}\left(1-\frac{1}{2a}\right)} \tag{7}$$

and so

$$A_n = \left(\frac{1}{2} - n \ln e / \left(1+\frac{1}{n}\right)^n\right) = 0(1/n)$$

which can be compared with the more familiar $\lim\limits_{n\to\infty} A_n = 0$.

3) Apply (1), (2) for $f(t) = \dfrac{1}{t}$ to deduce

$$\sqrt{ab} < L(a,b) < \frac{a+b}{2}, \tag{8}$$

where

$$L(a,b) = \frac{b-a}{\ln b - \ln a}$$

denotes the logarithmic means (see [2], [3]). The right-hand side of this inequality is due to B. Ostle and H.L. Terwilliger [6]. The left-hand inequality was stated by B.C. Carlson [2]. (8) was rediscovered also by A. Lupaş [5].

4) Select $f(t) = -\ln t$ in (2). This application yields the following improvement of the right-hand side of (8):

$$L(a,b) < \left(\frac{a+b}{2} + \sqrt{ab}\right)/2. \tag{9}$$

5) An interesting remark is that one can use (8) (and also (9)) to obtain refinements of this inequality. Indeed, let us consider $a = \sqrt{x}$, $b = \sqrt{y}$ in (8). It follows that

$$\sqrt{xy} < \sqrt[4]{xy}\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right) < L(x,y) < \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{x+y}{2}. \tag{10}$$

With the same argument we can derive (on base of (9)):

$$L(x,y) < \frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 + \frac{1}{4}\left(\sqrt{z}+\sqrt{y}\right)\sqrt[4]{xy}. \tag{11}$$

6) In order to arrive to a better refinement, we can consider the relation (3) for $f(t) = 1/t$, $k = 2$ $(0 < a < b)$. It results

$$L(a,b) < \frac{3}{8} \cdot \frac{(a+b)^3}{a^2+ab+b^2}.$$

Letting $a = \sqrt[3]{x}$, $b = \sqrt[3]{y}$, this is just one of the Lin [3] and Rüthing [8] inequalities:

$$L(x,y) < \left(\frac{\sqrt[3]{x}+\sqrt[3]{y}}{2}\right)^3. \tag{12}$$

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## 3.2 Some simple integral inequalities

### 1. Introduction

Integral inequalities have a long history. For many remarkable results see e.g. the monographs [4], [1], [6], [7], etc. For more recent inequalities of the author, with applications, see [10], [11], [12], [13].

We will consider here some simple inequalities for monotonous functions. An application $f : I \to \mathbb{R}$ is called monotonous increasing on interval $I$, if $x < y \Rightarrow f(x) \leq f(y)$ for all $x, y \in I$. Clearly, this condition may be written also as

$$(x - y)(f(x) - f(y)) \geq 0 \text{ for all } x, y \in I. \tag{1}$$

The function $f$ is strictly increasing, if (1) holds with strict inequality for $x \neq y$. If the inequality, or strict inequality of (1) is reversed, then we speak of decreasing, or strictly decreasing functions on $I$.

Let $p : I \to \mathbb{R}$ be a given function. We will say that the function $f : I \to \mathbb{R}$ is $p$-increasing, if the relation

$$(p(x) - p(y))(f(x) - f(y)) \geq 0, \ \forall \ x, y \in I \tag{2}$$

holds true. The other similar notions can be introduced for the corresponding signs of inequalities of (2).

Clearly, when $p(x) = x$, we reobtain the classical notions of monotonicity. However, it should be noted that e.g. a $p$-increasing function $f$ is not necessarily increasing in the usual sense. Take e.g. $I = \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$, $p(x) = \sin x$, $f(x) = \cos x$. Then the function $f$ is strictly $p$-increasing on $I$, but clearly it is strictly decreasing.

If the function $f$ satisfies (1), then it is Riemann integrable, a well-known fact of real analysis (see e.g. [9]). This is not true for functions satisfying (2), as it is shown by the example

$$f(x) = \begin{cases} p(x), & \text{if} \quad x \in Q \\ 0, & \text{if} \quad x \notin Q \end{cases} \qquad \text{where} \quad p(t) = 0 \quad \text{for} \quad t \notin Q.$$

If $x, y \in Q$, then (2) becomes $(p(x) - p(y))^2 \geq 0$; while $x \in Q$, $y \notin Q$, then

$$p(x)p(x) \geq 0, \quad x \notin Q,\ y \in Q \Rightarrow (-p(y))(-p(y)) \geq 0.$$

For example, when $p(x) = 1$ for $x \in Q$, we obtain the well-known Dirichlet function, which is not integrable.

Therefore, when dealing with integral inequalities for $p$-increasing functions, we must suppose that $p$ and $f$ are integrable on $I$.

## 2. Main results

**Theorem 1.** *Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Then for any positive integer $n \geq 1$, one has*

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2n-1} f(x)dx \geq 0. \tag{3}$$

*Particularly, for $n = 1$ we get*

$$\int_a^b xf(x)dx \geq \frac{a+b}{2} \int_a^b f(x)dx. \tag{4}$$

*When $f$ is strictly decreasing, all inequalities are strict. The inequalities are reversed, when $f$ is decreasing, resp. strictly decreasing functions.*

**Proof.** We shall apply the following remark:

**Lemma.** *When $\varphi : [a, b] \to \mathbb{R}$ is an integrable, odd function, then*

$$\int_a^b \varphi\left(x - \frac{a+b}{2}\right) dx = 0. \tag{5}$$

**Proof.** Put $x - \dfrac{a+b}{2} = y$. Then

$$\int_a^b \varphi\left(x - \frac{a+b}{2}\right) = \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \varphi(y)dy = \int_{-u}^{u} \varphi(y)dy,$$

where $u = \dfrac{b-a}{2}$. Now, letting $y = -z$, and using $\varphi(-z) = -\varphi(z)$, one has

$$I = \int_{-u}^{u} \varphi(y)dy = -\int_{u}^{-u} \varphi(-z)dz = -\int_{-u}^{u} \varphi(z)dz;$$

so $2I = 0$, giving $I = 0$.

Particularly, when $\varphi(x) = x^{2n-1}$, by (5) we get

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2n-1} dx = 0. \tag{6}$$

Since $f$ is increasing, by letting $y = \dfrac{a+b}{2}$ in (1) we have

$$\left(x - \frac{a+b}{2}\right)\left(f(x) - f\left(\frac{a+b}{2}\right)\right) \geq 0,$$

which multiplied with $\left(x - \dfrac{a+b}{2}\right)^{2n-2} \geq 0$ gives

$$\left(x - \frac{a+b}{2}\right)^{2n-1}\left(f(x) - f\left(\frac{a+b}{2}\right)\right) \geq 0 \tag{7}$$

Relation (7) implies

$$\left(x - \frac{a+b}{2}\right)^{2n-1} f(x) \geq f\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)^{2n-1},$$

which by integration, and taking into account of (6) implies relation (3). When $n = 1$, (3) implies immediately (4).

When $f$ is strictly increasing, for $x \neq \dfrac{a+b}{2}$ one has strict inequality in (7), so by integration we get strict inequality.

**Remark.** The proof shows that in fact the following general result is true:

**Theorem 2.** *Let $\varphi : [a,b] \to \mathbb{R}$ be an integrable, odd function such that $\varphi(t) > 0$ for $t > 0$. Then*

$$\int_a^b \varphi\left(x - \frac{a+b}{2}\right) f(x)dx \geq 0 \tag{8}$$

*for any increasing function $f$. The inequality is strict, when $f$ is strictly increasing, etc.*

**Proof.** Write

$$\varphi\left(x-\frac{a+b}{2}\right)\left(f(x)-f\left(\frac{a+b}{2}\right)\right)$$

as

$$\frac{\varphi\left(x-\dfrac{a+b}{2}\right)}{x-\dfrac{a+b}{2}}\left(x-\frac{a+b}{2}\right)\left(f(x)-f\left(\frac{a+b}{2}\right)\right) \text{ for } x\neq\frac{a+b}{2}.$$

We first prove that $\dfrac{\varphi(t)}{t}>0$ for any $t\neq 0$.

When $t>0$, this is true by assumption, while when $t<0$, put $t=-p$, $p>0$. Then

$$\frac{\varphi(-p)}{-p}=\frac{-\varphi(p)}{-p}=\frac{\varphi(p)}{p}>0,$$

as required.

This means, that for all $x$ in $[a,b]$ one has

$$\varphi\left(x-\frac{a+b}{2}\right)\left(f(x)-f\left(\frac{a+b}{2}\right)\right)\geq 0 \tag{9}$$

and the procedure may be repeated.

**Theorem 3.** *Let $p,f:[a,b]\to\mathbb{R}$ be integrable functions, and suppose that $f$ is $p$-increasing. Then*

$$\int_a^b\left[p(x)-p\left(\frac{a+b}{2}\right)\right]f(x)dx\geq f\left(\frac{a+b}{2}\right)\int_a^b\left[p(x)-p\left(\frac{a+b}{2}\right)\right]dx \tag{10}$$

*When $f$ is strictly $p$-increasing, there is strict inequality in (10); etc.*

**Proof.** Write

$$\left[p(x)-p\left(\frac{a+b}{2}\right)\right]\left[f(x)-f\left(\frac{a+b}{2}\right)\right]\geq 0, \tag{11}$$

and integrate after multiplication. When $f$ is strictly $p$-increasing, (11) holds with strict inequality for all $x \neq \dfrac{a+b}{2}$; thus the integral of left side of (11) will be $> 0$.

**Remarks.** 1) It is well-known from textbooks of real analysis that (see e.g. [9]) if $F : [a,b] \to \mathbb{R}$ is integrable and nonnegative, then

$$\int_a^b F(x)dx \geq 0 \quad \text{and} \quad \int_a^b F(x)dx = 0$$

holds true if and only if $F(x) = 0$ a.e. in $x \in [a,b]$.

Thus we have also strict inequalities in (4), (8), (10), if instead of strict monotonicity we suppose that e.g. $f$ is increasing and $f(x) \neq f\left(\dfrac{a+b}{2}\right)$ for almost every $x \in [a,b]$.

2) When $p(x) = x$, (10) coincides with (4). When $f(x) = \cos x$, $p(x) = \sin x$, $[a,b] \subset \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$, however; we obtain a new type of result.

## 3. A refinement

Suppose now that $f$ is an increasing (decreasing) continuous function on $[a,b]$. In this case we are able to prove the following result connected to (4):

**Theorem 4.** *If $a < b < c$, and if $f$ is continuous and increasing (decreasing) on $[a,c]$, then*

$$\int_a^c xf(x)dx \underset{(\leq)}{\geq} \left(\frac{a+c}{2}\right)\int_a^c f(x)dx + \int_a^b xf(x)dx - \left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$
$$\underset{(\leq)}{\geq} \frac{a+c}{2}\int_a^c f(x)dx. \tag{12}$$

*When $f$ is strictly increasing, all inequalities are strict.*

**Remark.** Inequality (12) refines (4) on the interval $[a,c]$.

**Proof.** Put

$$F(t) = \int_a^t xf(x)dx - \left(\frac{a+t}{2}\right)\int_a^t f(x)dx.$$

Then (12) may be rewritten as $F(c) \geq F(b) \geq 0$. While the last of these inequalities is in fact (4), the first one requires that $F$ is an increasing function (as $c > b$).

Since $f$ is continuous, the integrals are differentiable, so we may use derivatives. One has

$$F'(t) = tf(t) - \frac{1}{2}\int_a^b f(x)dx - \frac{a+t}{2}f(t) = \left[f(t)(t-a) - \int_a^t f(x)dx\right]/2.$$

Since $f$ is increasing, we have

$$\int_a^t f(x)dx \leq \int_a^t f(t)dx = f(t)(t-a),$$

so we get $F'(t) \geq 0$, and the result follows. When $f$ is strictly increasing, clearly

$$\int_a^t f(x)dx < \int_a^t f(t)dx,$$

so $F'(t) > 0$. This finishes the proof of Theorem 4.

**Remark.** As $\displaystyle\int_a^c = \int_a^b + \int_b^c$, the first inequalities of (12) can be written also as

$$\int_b^c \left(x - \frac{a+c}{2}\right) f(x)dx \geq \frac{c-b}{2}\int_a^b f(x)dx,$$

i.e.

$$\frac{2}{c-b}\int_b^c \left(x - \frac{a+c}{2}\right) f(x)dx \geq \int_a^b f(x)dx \tag{13}$$

When $f$ is decreasing, (13) holds with reversed sign of inequality.

## 4. Applications

1) The simplest inequalities of all paper are clearly contained in Theorem 1. We will show that this elementary inequality has surprizing applications.

a) Put $f(x) = \dfrac{1}{x}$, $(0 < a < b)$, which is strictly decreasing. As

$$\int_a^b f(x)dx = \log b - \log a,$$

we get with reversed sign in (4) that

$$L = \frac{b-a}{\log b - \log a} < \frac{a+b}{2} \quad (a, b > 0,\ a \neq b) \tag{14}$$

Here

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \quad \text{and} \quad A = A(a,b) = \frac{a+b}{2}$$

represent the famous logarithmic, resp. arithmetic means of $a$ and $b$.

b) Letting $f(x) = \dfrac{1}{x^2}$, we get

$$\frac{b-a}{\log b - \log a} > \frac{2ab}{a+b} \tag{15}$$

Put now $\sqrt{b}$ and $\sqrt{a}$ in place of $a, b$ in (15). Since

$$(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) = b - a,$$

we get

$$L > \sqrt{ab}, \tag{16}$$

which is another important inequality, with $\sqrt{ab} = G(a,b) = G$ denoting the geometric mean. The inequalities $G < L < A$ are frequently used in many fields of science (even applied ones, as electrostatics [8], heat conductors and chemistry [15], statistics and probability [5], etc.).

c) Another noteworthy mean, related to the above means is the so-called identric mean $I = I(a,b)$ defined by

$$I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad \text{for} \quad a \neq b.$$

It is easy to see that

$$\frac{1}{b-a}\int_a^b \log x dx = \log I(a,b) \tag{17}$$

and

$$\frac{1}{b-a}\int_a^b x\log x dx = \frac{b+a}{4}\log I(a^2,b^2) \tag{18}$$

(see [11], [12] for details). Since $f(x) = \log x$ is strictly increasing, we get from (4) the inequality

$$I(a^2,b^2) > (I(a,b))^2, \tag{19}$$

first discovered by the author in [11]. See also [12].

d) Let $f(x) = e^x$ in (4). Put

$$E = E(a,b) = \frac{be^b - ae^a}{e^b - e^a} - 1$$

an exponential mean introduced in [16]. From (4) we get (Toader's inequality):

$$E > A \tag{20}$$

Applying (4) with reversed sign to $f(x) = e^{-x}$, we get

$$\overline{E} < A \tag{21}$$

where

$$\overline{E} = \overline{E}(a,b) = \frac{ae^b - be^a}{e^b - e^a} + 1,$$

is a "complementary" exponential mean to $E$ (see [14]).

e) Let $f(x) = x^k$, where $k > 0$. Since $f$ is strictly increasing, we get from (4):

$$\frac{b^{k+2} - a^{k+2}}{b^{k+1} - a^{k+1}} > \frac{a+b}{2}\cdot\frac{k+2}{k+1} \tag{22}$$

For $k = 1$ this implies

$$\left(\frac{a+b}{2}\right)^2 < \frac{a^2+ab+b^2}{3}, \tag{23}$$

while for $k = 2$ that

$$\frac{a^2+ab+b^2}{3} < \frac{a^2+b^2}{2} \tag{24}$$

Note that (23) and (24) give a refinement of the frequently used elementary relation $\left(\frac{a+b}{2}\right)^2 < \frac{a^2+b^2}{2}$:

$$\left(\frac{a+b}{2}\right)^2 < \frac{a^2+ab+b^2}{3} < \frac{a^2+b^2}{2} \tag{25}$$

2) f) Apply now the first part of (12) to $f(x) = \frac{1}{x}$.

For $a < b < c$ one gets

$$2(c-b) < \frac{b^2-a^2}{L(a,b)} - \frac{c^2-a^2}{L(c,a)} \tag{26}$$

g) For $f(x) = \log x$, by taking into account of (17)-(18) we can deduce from (12) that:

$$1 < \left(\frac{\sqrt{I(a^2,c^2)}}{I(a,c)}\right)^{c^2-a^2} < \left(\frac{\sqrt{I(a^2,b^2)}}{I(a,b)}\right)^{b^2-a^2} \quad \text{for} \quad a<b<c, \tag{27}$$

which is a refinement of inequality (19).

By using the mean $S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}$, i.e. $S(a,b) = (a^a \cdot b^b)^{1/(a+b)}$ (see e.g. [11], [12], [14]) we get from (27)

$$1 > \frac{I(a,b)}{S(a,b)} > \left(\frac{I(a,c)}{S(a,c)}\right)^{\frac{c^2-a^2}{b^2-a^2}} > \frac{I(a,c)}{S(a,c)}, \quad \text{as} \quad \frac{c^2-a^2}{b^2-a^2} > 1. \tag{28}$$

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## 3.3 Generalization of the Hadamard integral inequalities

The famous Hadamard (or Hermite, or Hermite-Hadamard, or Jensen-Hermite-Hadamard) inequalities for integrals states that if $f : [a,b] \to \mathbb{R}$ is convex and continuous, then for all $x, y \in [a,b]$ one has

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2},\ x \ne y \tag{1}$$

Let $a_1 < a_2 < \ldots < a_n$ be elements of $I = [a,b]$. Applying (1) to $x = a_i$, $y = a_{i+1}$ one gets

$$f\left(\frac{a_i+a_{i+1}}{2}\right) \le \frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} f(t)dt \le \frac{f(a_i)+f(a_{i+1})}{2} \tag{2}$$

Applying (2) for $i = 1, 2, \ldots, n$, after term-by-term additions we get that

$$f\left(\frac{a_1+a_2}{2}\right) + f\left(\frac{a_2+a_3}{2}\right) + \ldots + f\left(\frac{a_n+a_{n+1}}{2}\right)$$

$$\le \sum_{k=1}^{n} \frac{1}{a_{k+1}-a_k}\int_{a_k}^{a_{k+1}} f(t)dt$$

$$\le \frac{f(a_1)}{2} + f(a_2) + \ldots + f(a_n) + \frac{f(a_{n+1})}{2} \tag{3}$$

But $f$ being convex,

$$f\left(\frac{a_1+a_2}{2}\right) + \ldots + f\left(\frac{a_n+a_{n+1}}{2}\right) \ge nf\left(\frac{\dfrac{a_1+a_2}{2} + \ldots + \dfrac{a_n+a_{n+1}}{2}}{n}\right)$$

$$= nf\left(\frac{a_1}{2n} + \frac{a_2+\ldots+a_n}{n} + \frac{a_{n+1}}{2n}\right).$$

So by (3) one gets:

**Theorem 1.** *For $f$ and $(a_i)$ satisfying the stated conditions, one has:*

$$f\left(\frac{a_1}{2n}+\frac{a_2+\ldots+a_n}{n}+\frac{a_{n+1}}{2n}\right)\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_{k+1}-a_k}\int_{a_k}^{a_{k+1}}f(t)dt$$

$$\le\frac{1}{n}\left[\frac{f(a_1)}{2}+f(a_2)+\ldots+f(a_n)+\frac{f(a_{n+1})}{2}\right]. \qquad (4)$$

Clearly, inequality (4) is a generalization of the Hadamard inequalities, as for $n=1$ we get (1) for $x=a_1$, $y=a_2$. Apply now inequalities (2) for $i=1,3,\ldots,2n-1$. By the same procedure, as above one obtains:

**Theorem 2.** *If $f$ and $(a_i)$ satisfy the stated conditions, then*

$$f\left(\frac{1}{2n}(a_1+a_2+\ldots+a_n)\right)\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_{2k}-a_{2k-1}}\int_{a_{2k-1}}^{a_{2k}}f(t)dt$$

$$\le\frac{1}{2n}[f(a_1)+\ldots+f(a_{2n})]. \qquad (5)$$

Inequality (5) has been first discovered by I.B. Lacković ([1]).

Let now $i=2,4,\ldots,2n$ in relation (2). After applying the same procedure, we can state the following:

**Theorem 3.** *If $f$ and $(a_i)$ are satisfying the stated conditions, then*

$$f\left(\frac{a_2+a_3+\ldots+a_{2n+1}}{2n}\right)\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_{2k+1}-a_{2k}}\int_{a_{2k}}^{a_{2k+1}}f(t)dt$$

$$\le\frac{f(a_2)+f(a_3)+\ldots+f(a_{2n+1})}{2n} \qquad (6)$$

Clearly, inequalities (4), (5), (6) can be applied to many particular $f$ and $(a_i)$. We invite the interested reader to perform such applications.

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## 3.4 Applications of the Cauchy-Bouniakowsky inequality in the theory of means

### 1. Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions. The classical inequality of Cauchy-Bouniakowsky states that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right). \tag{1}$$

One has equality in (1) iff there exists a real constant $k \in \mathbb{R}$ such that $f(x) = kg(x)$ almost everywhere in $x \in [a, b]$. When $f$ and $g$ are continuous, equality occurs when the above equality holds true for all $x \in [a, b]$ (see e.g. [2]).

Let $x, y > 0$ be positive real numbers. Let us denote by

$$A := A(x, y) = \frac{x+y}{2} \text{ and } G := G(x, y) = \sqrt{xy}$$

the classical arithmetic resp. geometric means of $x$ and $y$.

The logarithmic and identric means $L$ and $I$ are defined by

$$L := L(x, y) = \frac{x-y}{\ln x - \ln y}\ (x \neq y),\ L(x, x) = x \tag{2}$$

and

$$I := I(x, y) = \frac{1}{e}(y^y/x^x)^{1/(y-x)}\ (x \neq y),\ I(x, x) = x, \tag{3}$$

respectively (see e.g. [4], [5], [11]).

One of the most important inequalities satisfied by the mean $L$ is:

$$G < L < A \text{ for } x \neq y \tag{4}$$

Though the left side inequality of (4) is attributed to B.C. Carlson, while the right side to B. Ostle and H.L. Terwilliger (see [5] for references), the author has discovered recently ([13]) that (4) was proved in fact by Bouniakowsky in his paper [1] from 1859. In the proof, inequality (1) was used for certain particular continuous functions. The author has obtained more direct and simplified proofs of (4).

The aim of this paper is to obtain other applications of inequality (1) in the theory of means. Other means, besides $L$ and $I$ will be defined, when necessary.

Though there exist many integral inequalities with applications in the theory of means (some of them may be found e.g. in [5]) we will restrict here our interest **only** to the inequality (1) (in honour of V. Bouniakowsky).

## 2. Applications

1) Let $g(x) = 1$, $x \in [a, b]$ in (1). Then one obtains

$$\left(\int_a^b f(x)dx\right)^2 \leq (b-a)\int_a^b f^2(x)dx, \text{ with } a < b, \tag{5}$$

where equality occurs in case of continuous $f$, when $f$ is constant.

a) For a new proof of (4), apply (5) for $f(x) = \dfrac{1}{x}$. One obtains

$$(\ln b - \ln a)^2 < (b-a)\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{(b-a)^2}{ab},$$

where the inequality is strict, as the function is not constant. The left side of (4) follows:

$$G(a, b) < L(a, b).$$

b) Apply now (1) for $f(x) = e^x$, implying:

$$(e^b - e^a)^2 < \frac{b-a}{2}(e^b - e^a)(e^b + e^a),$$

so

$$\frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}. \tag{6}$$

Replace $a = \ln x$, $b = \ln y$ in (6), obtaining

$$L(x, y) < A(x, y),$$

i.e. the right side of (4).

c) Apply now (5) for $f(x) = \dfrac{1}{\sqrt{x}}$. One obtains

$$4(\sqrt{b} - \sqrt{a})^2 < (b - a)(\ln b - \ln a),$$

or

$$\frac{b-a}{\ln b - \ln a} < \left[\frac{b-a}{2(\sqrt{b} - \sqrt{a})}\right]^2 = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 = A_{1/2},$$

where $A_r = \left(\dfrac{a^r + b^r}{2}\right)^{1/r}$ is the Hölder mean of $a$ and $b$.

As $A_r$ is a strictly increasing function of $r$, we have obtained the following refinement of right side of (4):

$$L < A_{1/2} < A \tag{7}$$

In fact $A_{1/2} = \dfrac{A + G}{2}$, and inequality (7), with another method, has been deduced in [4], too.

d) Let now $f(x) = x^r$, where $r \neq -1$ and $-1/2$ (these cases have been applied in a), resp. c)). Then one obtains the inequality:

$$\left[\frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right]^2 < \frac{b^{2r+1} - a^{2r+1}}{(b-a)(2r+1)} \tag{8}$$

By denoting by $L_r = L_r(a, b)$ the usual $r$**-th logarithmic mean**

$$L_r(a, b) = \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r} \quad \text{for } r \neq -1,\ r \neq 0$$

(and $L_{-1} = \lim\limits_{r \to -1} L_r(a, b) = L$, $L_0 = \lim\limits_{r \to 0} L_r(a, b) = I$), relation (8) can be rewritten as $L_r^{2r} < L_{2r}^{2r}$, or

$$L_r^r < L_{2r}^r \tag{9}$$

When $r > 0$, particularly (9) contains the inequality $L_r < L_{2r}$.

e) Let $f(x) = \ln x$ in (5). It is well-known that (see e.g. [5])

$$\frac{1}{b-a}\int_a^b \ln x dx = \ln I(a,b) \tag{10}$$

On the other hand, by partial integration we can deduce

$$\frac{1}{b-a}\int_a^b \ln^2 x dx = \frac{b\ln^2 b - a\ln^2 a}{b-a} - 2\ln I(a,b), \tag{11}$$

where we have used (10). Therefore, by (5) we get:

$$\ln^2 I < \frac{b\ln^2 b - a\ln^2 a}{b-a} - 2\ln I, \tag{12}$$

which seems to be new. Remark that (12) may be rewritten as

$$(\ln I + 1)^2 < \frac{b\ln^2 b - a\ln^2 a}{b-a} + 1 \tag{13}$$

Now we shall prove that the expression $K(a,b)$ given by

$$\ln K(a,b) + 1 = \sqrt{\frac{b\ln^2 b - a\ln^2 a}{b-a} + 1} \tag{14}$$

defines a mean. Indeed, by the mean value theorem of Lagrange one has for the function $g(x) = x\ln^2 x$:

$$\frac{g(b)-g(a)}{b-a} = \ln^2 \xi + 2\ln \xi, \text{ with } \xi \in (a,b).$$

Therefore,

$$\sqrt{\frac{g(b)-g(a)}{b-a} + 1} = \ln \xi + 1$$

which lies between $\ln a + 1$ and $\ln b + 1$.

Thus $\ln a + 1 < \ln K + 1 < \ln b + 1$, implying

$$a < K < b \text{ for } a < b. \tag{15}$$

Since $K(a,a) = a$, this means that $K$ is indeed a mean. By (13) and (14) we get the inequality

$$I < K, \tag{16}$$

where

$$K := K(a,b) = \frac{1}{e} \cdot \exp\left( \sqrt{\frac{b \ln^2 b - a \ln^2 a}{b-a}} + 1 \right).$$

f) Let $f(x) = \dfrac{1}{\sqrt{x(a+b-x)}}$ in (5). Then we get

$$\left[ \frac{1}{b-a} \int_a^b \frac{1}{\sqrt{x(a+b-x)}} dx \right]^2 < \frac{1}{b-a} \int_a^b \left( \frac{1}{x} + \frac{1}{a+b-x} \right) \frac{1}{a+b} dx, \tag{17}$$

where we have used the remark that

$$\frac{1}{x(a+b-x)} = \left( \frac{1}{x} + \frac{1}{a+b-x} \right) \frac{1}{a+b}.$$

Remark also that

$$\frac{1}{b-a} \int_a^b \frac{1}{x} dx = \frac{1}{b-a} \int_a^b \frac{1}{a+b-x} dx = \frac{1}{L(a,b)}. \tag{18}$$

On the other hand one has

$$\frac{1}{b-a} \int_a^b \frac{1}{\sqrt{x(a+b-x)}} = \frac{1}{P(a,b)}, \tag{19}$$

where $P = P(a,b)$ is the Seiffert mean, defined by (see e.g. [10], [14], [15], [16])

$$P(a,b) = \frac{a-b}{2 \arcsin \dfrac{a-b}{a+b}} \text{ for } a \neq b, \ P(a,a) = a \tag{20}$$

For the integral representation (19) of the mean $P$ defined by (20), see e.g. [14]. Now, by (17), (18) and (19) we get the inequality

$$P^2 > L \cdot A, \tag{21}$$

discovered by more complicated arguments in [3].

Particularly, by the right side of (4), from (21) we get

$$P > L \tag{22}$$

Clearly, by $\sqrt{a(a+b-x)} < \dfrac{x+(a+b-x}{2} = A$ from (19) we get

$$A > P, \tag{23}$$

therefore (22) and (23) improve the right side of (4).

**Remark 1.** For improvements of (21) with stronger arguments, see [12].

As (21) is equivalent, with the following inequality (see [3]):

$$L^2 > P \cdot G \tag{21$'$}$$

inequality (21′) here follows by the proved inequality (21).

By (21) and (21′) one can deduce also

$$P^2 \cdot L^2 > (LA) \cdot (PG),$$

which implies the inequality

$$P \cdot L > A \cdot G, \tag{21$''$}$$

one of the main results in [16] (and proved by more difficult means).

2) Applying (1) for $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = \dfrac{1}{\sqrt{a+b-x}}$, and using (19) we can deduce again relation (22). We note that by the left side of (4) and (22) we get

$$P > G, \tag{24}$$

but this follows also from the observation that for any $t \in [a,b]$ one has $t(a+b-t) \geq ab$, or equivalently $(t-a)(b-t) \geq 0$. Now, using this fact, and the integral representation (19), we get (24).

b) Let $f(x) = \sqrt{\frac{\ln x}{x}}$, $g(x) = \sqrt{x \ln x}$ in (1), where $x > 1$. As

$$\frac{1}{b-a}\int_a^b \ln x dx = \ln I(a,b), \quad \frac{1}{b-a}\int_a^b \frac{\ln x}{x} dx = \frac{1}{2(b-a)}(\ln^2 b - \ln^2 a)$$

and

$$\frac{1}{b-a}\int_a^b x \ln x = \frac{A}{2} \ln I(a^2, b^2)$$

(see e.g. [7]), we get:

$$\ln^2 I < \frac{\ln b - lna}{b-a} \cdot \ln G \cdot \frac{A}{2} \cdot lnI(a^2, b^2) = \frac{A}{2L} \cdot \ln G \cdot \ln I(a^2, b^2).$$

Let $S = S(a,b)$ be the mean defined by

$$S = (a^a \cdot b^b)^{1/(a+b)} \tag{25}$$

Then it is known (see [5], [7]) that

$$S(a,b) = \frac{I(a^2, b^2)}{I(a,b)} \tag{26}$$

By using (26), from the above relations we get the inequality

$$\ln^2 I < \frac{A}{2L} \cdot \ln G \cdot \ln(S \cdot I), \tag{27}$$

which seems to be new.

**Remark 2.** As in the definitions of $f$ and $g$ we must suppose $x > 1$, clearly (27) holds true for $b > a > 1$, where $I = I(a,b)$, etc.

c) An exponential mean $E = E(a,b)$ is defined and studied e.g. in [6], [9] by

$$E = E(a,b) = \frac{be^b - ae^a}{e^b - e^a} - 1 \tag{28}$$

Apply now inequality (1) for $f(x) = \sqrt{e^x}$ and $g(x) = x\sqrt{e^x}$.

Remark that

$$\int_a^b xe^x dx = be^b - ae^a - (e^b - e^a),$$

$$\int_a^b x^2 e^x dx = b^2 e^b - a^2 e^a - 2\int_a^b x e^x dx$$

and that these imply

$$\int_a^b x e^x dx = (e^b - e^a)E, \quad \int_a^b x^2 e^x dx = b^2 e^b - a^2 e^a - 2(e^b - e^a)E,$$

so we get:

$$(e^b - e^a)^2 E^2 < (e^b - e^a)[b^2 e^b - a^2 e^a - 2(e^b - e^a)E] \tag{29}$$

or

$$(e^b - e^a)(E^2 + 2E) < b^2 e^b - a^2 e^a,$$

so

$$(E+1)^2 < \frac{b^2 e^b - a^2 e^a}{e^b - e^a} + 1 \tag{30}$$

Define a new exponential mean $F$ by

$$F = F(a,b) = \sqrt{\frac{b^2 e^b - a^2 e^a}{e^b - e^a} + 1} - 1 \tag{31}$$

By (30) we get

$$E < F \tag{32}$$

d) Let $f(x) = \dfrac{1}{\sqrt{x \ln x}}$ and $g(x) = \sqrt{\dfrac{\ln x}{x}}$ in (1). As

$$\int_a^b \frac{1}{x \ln x} dx = \ln(\ln b) - \ln(\ln a)$$

and

$$\int_a^b \frac{\ln x}{x} dx = \frac{1}{2}(\ln^2 b - \ln^2 a),$$

we get

$$(\ln b - \ln a)^2 < [\ln(\ln b) - \ln(\ln a)] \cdot \frac{1}{2}(\ln^2 b - \ln^2 a),$$

or

$$\ln b - \ln a < [\ln(\ln b) - \ln(\ln a)] \cdot \frac{1}{2}(\ln b + \ln a).$$

By letting $\ln b = y$, $\ln a = x$, this gives a new proof of right side of (4).

Applying (1) for $f(x) = \dfrac{1}{\sqrt{x} \cdot \ln x}$, $g(x) = \dfrac{1}{\sqrt{x}}$, as

$$\int_a^b \frac{1}{x \ln^2 x} dx = \frac{1}{\ln x} - \frac{1}{\ln b},$$

we get

$$[\ln(\ln b) - \ln(\ln a)]^2 < (\ln b - \ln a) \cdot \frac{\ln b - \ln a}{\ln b \cdot \ln a},$$

which by notation $\ln b = y$, $\ln a = x$, gives a new proof of left side of (4).

e) Let $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} \cdot \ln x$ in (1). As

$$\int x \ln^2 x dx = \frac{x^2 \ln^2 x}{2} - \int x \ln x dx,$$

by the formula used in b) we get:

$$(b-a)^2 \cdot \frac{A^2}{4} \cdot \ln^2(S \cdot I) < \frac{b^2 - a^2}{2} \left[ \frac{b^2 \ln^2 b - a^2 \ln^2 a}{2} - (b-a) \cdot \frac{A}{2} \cdot \ln(I \cdot S) \right] \tag{33}$$

As $\ln I = \dfrac{b \ln b - a \ln a}{b - a} - 1$ and $\ln S = a \ln a + b \ln b 2A$, after certain transformations, we get from (33):

$$\ln^2(S \cdot I) + 2 \ln(S \cdot I) < 4(1 + \ln I) \cdot \ln S \tag{34}$$

Put $u = \ln I$, $v = \ln S$ in (34). It is easy to see that

$$v = \ln S > \ln I = u \tag{35}$$

becomes equivalent, after elementary transformations to

$$ab(\ln b^2 - \ln a^2) < b^2 - a^2, \text{ or } L(a^2, b^2) > G(a^2, b^2),$$

which is the left side of (4).

Now, (34) can be written as

$$(v+u)^2 + 2(v+u) < 4v(1+u), \text{ or } v^2 + u^2 + 2u < 2v + 2vu,$$

or

$$(v-u)^2 < 2(v-u) \tag{36}$$

as, by (35), $v - u > 0$, we get from (36) that $v - u < 2$, i.e.

$$S < e^2 \cdot I \tag{37}$$

Therefore, inequality (37) is a consequence of the Cauchy-Bouniakowsky inequality.

3) We have shown by more applications of the inequality (1) that holds true relation (4). Now, this implies the logarithmic inequality

$$\ln x \le x - 1, \tag{38}$$

with equality only for $x = 1$. Indeed, let $x > 1$. Then by $L(x,1) > G(x,1)$ one has $\dfrac{x-1}{\ln x} > \sqrt{x}$, so $\ln x < \dfrac{x-1}{\sqrt{x}} < x - 1$. If $0 < x < 1$, then apply $L(1,x) < A(1,x)$, i.e. $\dfrac{1-x}{-\ln x} < \dfrac{x+1}{2}$, where $\dfrac{x+1}{2} < 1$. Thus $\dfrac{1-x}{-\ln x} < 1$, so $1 - x < -\ln x$ or $\ln x < x - 1$. There is equality in (38) only for $x = 1$.

Let $A_p(x) = p_1x_1 + \ldots + p_rx_r$, $G_p(x) = x_1^{p_1} \ldots x_r^{p_r}$ and

$$H_p(x) = \frac{1}{\dfrac{p_1}{x_1} + \ldots + \dfrac{p_r}{x_r}}$$

denote the weighted arithmetic, geometric resp. harmonic means of the positive real numbers $x_1, \ldots, x_r > 0$, where the positive weights $p_i$ ($i = 1, 2, \ldots, r$) satisfy $p_1 + \ldots + p_r = 1$.

Apply now inequality (38) for $x = \dfrac{x_i}{A_p(x)}$, and multiply both sides with $p_i$:

$$p_i \ln \frac{x_i}{A_p(x)} \le \frac{p_i x_i}{A_p(x)} - p_i \tag{39}$$

After summation in (39) we get

$$\ln \frac{x_1^{p_1} \dots x_r^{p_r}}{A_p(x)^{p_1+\dots+p_r}} \le \frac{p_1 x_1 + \dots + p_r x_r}{A_p(x)} - (p_1 + \dots + p_r).$$

As $p_1 + \dots + p_r = 1$, we get the weighted arithmetic-geometric inequality. This in turn gives also the weighted harmonic-geometric inequality:

$$H_p(x) \le G_p(x) \le A_p(x) \tag{40}$$

The left side of (40) follows by applying

$$G_p\left(\frac{1}{x}\right) \le A_p\left(\frac{1}{x}\right),$$

where $\frac{1}{x} = \left(\frac{1}{x_1}, \dots, \frac{1}{x_r}\right)$.

There is equality in both sides only if $\frac{x_i}{A_p(x)} = 1$ for all $i = 1, \dots, r$, which means that $x_1 = \dots = x_r$.

The continuous analogue of inequality (40) can be proved in the same manner. Let $f, p : [a, b] \to \mathbb{R}$ be two positive Riemann-integrable functions.

Suppose that $\int_a^b p(x)dx = 1$ and define

$$A_{p,f} = \int_a^b p(x)f(x)dx, \; G_{p,f} = e^{\int_a^b p(x)\ln f(x)dx}, \; H_{p,f} = \frac{1}{\displaystyle\int_a^b \frac{p(x)}{f(x)}dx} \tag{41}$$

Then one has

$$H_{p,f} \le G_{p,f} \le A_{p,f}. \tag{42}$$

Particularly, when $p(x) = \frac{1}{b-a}$, we get

$$A_f = \frac{1}{b-a}\int_a^b f(x)dx, \quad G_f = e^{\frac{1}{b-a}\int_a^b \ln f(x)}, \quad H_f = \frac{b-a}{\displaystyle\int_a^b dx/f(x)} \tag{43}$$

so

$$H_f \leq G_f \leq A_f \tag{44}$$

There is equality in both sides of (41) (or (44)) only if $f$ is a constant almost everywhere. If $f$ is continuous, the equality occurs only when $f$ is a constant function.

For the proof of (42) apply the same method, as in the proof of (40), but in place of summation, use integration.

Therefore, let $x = \dfrac{x_i}{A_{p,f}}$ in (38), and multiply both sides with $p(x) > 0$:

$$\ln \frac{f(x)^{p(x)}}{A_{p,f}^{p(x)}} \leq \frac{p(x)f(x)}{A_{p,f}} - p(x). \tag{45}$$

By integration in (45) we get the left side of (42). Then apply this inequality to $\dfrac{1}{f}$ in place of $f$ in order to deduce the left side of (42).

There are many applications to the discrete form (40), or continuous form (42) of the arithmetic-geometric-harmonic inequalities.

We will be mainly interest in the means studied before.

a) For the means $A, G, L, I$ and $S$, the following identities are easy to prove (see also [7], [8]):

$$\ln \frac{I}{G} = \frac{A}{L} - 1 \tag{46}$$

$$\ln \frac{S}{I} = 1 - \frac{G^2}{AL} \tag{47}$$

As

$$A \cdot L = A(a,b) \cdot L(a,b) = L(a^2, b^2) \quad \text{and} \quad G^2(a,b) = G(a^2, b^2),$$

by replacing $a$ with $\sqrt{a}$ and $b$ with $\sqrt{b}$ in (47), one obtains

$$\ln \frac{S}{I}(\sqrt{a}, \sqrt{b}) = 1 - \frac{G}{L} \tag{47'}$$

In base of identities (46) and (47′) one can state the following:

$$L < A \Leftrightarrow I > G \tag{48}$$

$$G < L \Leftrightarrow S > I \tag{49}$$

Therefore, inequalities (4) are equivalent to the following:

$$G < I < S \tag{4'}$$

Applying (44) to $f(x) = x$ we get

$$L < I < A \tag{50}$$

On the other hand, applying the left side of (40) for $r = 2$ and

$$p_1 = \frac{a}{a+b},\ p_2 = \frac{b}{a+b},\ x_1 = a,\ x_2 = b,$$

one has

$$\frac{1}{\frac{a}{a+b} \cdot \frac{1}{a} + \frac{b}{a+b} \cdot \frac{1}{b}} < a^{a/(a+b)} \cdot b^{b/(a+b)},$$

which gives

$$A < S, \tag{51}$$

see e.g. [8].

In fact, relations (4), (4′), (50) and (51) may be rewritten as

$$G < L < I < A < S \tag{52}$$

b) By (22) and (23) $P$ lies between $L$ and $A$, but we can strengthen this fact by applying the right side of (44) to

$$f(x) = \frac{1}{\sqrt{x(a+b-x)}}.$$

As $\int_a^b \ln(a+b-x)dx = \int_a^b \ln x dx = \ln I$, by (19) we get

$$P < I \tag{53}$$

Therefore, (52) may by completed as

$$G < L < P < I < A < S \tag{54}$$

**Remark.** Inequalities $L < P < I$ have been obtained for the first time by H.-J. Seiffert [15], by using more complicated arguments.

Let us apply now the left side of (44) to the same function $f$ as above. Let us introduce the new mean

$$J = J(a,b) = \frac{1}{b-a}\int_a^b \sqrt{x(a+b-x)}dx \tag{55}$$

As $G < \sqrt{x(a+b-x)} < A$ (see 1f) and 2a)), we get also

$$G < J < A \tag{56}$$

By the left side of (44) however, the left side of (56) may be improved to

$$I < J \tag{57}$$

Therefore the chain (54) may by rewritten as

$$G < L < P < I < J < A < S \tag{54$'$}$$

**Remark 3.** By inequalities (21) and (21′) one can strengthen the first two inequalities:

$$G < \sqrt{P \cdot G} < L < \sqrt{L \cdot A} < P$$

c) Let us introduce another new mean $R$ by

$$R = R(a,b) = 1/\left(\frac{1}{b-a}\cdot\int_a^b \frac{1}{\sqrt[4]{x(a+b-x)}}dx\right)^2 \tag{58}$$

As $\sqrt{G} < \sqrt[4]{x(a+b-x)} < \sqrt{A}$, clearly

$$G < R < A, \tag{59}$$

too. By inequality (5) applied to $f(x) = \dfrac{1}{\sqrt[4]{x(a+b-x)}}$ we get, using (19), that

$$P < R \tag{60}$$

Applying, as in b) the right side of (44) to this function, we get

$$R < I \tag{61}$$

Therefore, a completion of (54′) is valid:

$$G < L < P < R < I < J < A < S \tag{54''}$$

with two new means $J$, resp. $R$ defined by (55), resp. (58).

d) Apply now (41) for $p(x) = \dfrac{2x}{b^2 - a^2}$ and $f(x) = \dfrac{1}{x}$. As

$$A_{p,f} = \int_a^b p(x)f(x)dx = \frac{2}{b^2 - a^2}(b - a) = \frac{1}{A},$$

$$G_{p,f} = e^{\int_a^b p(x)\ln f(x)dx} = e^{-\frac{2}{b^2 a^2}\int_a^b x\ln x dx} = \frac{1}{\sqrt{I \cdot S}},$$

as

$$\int_a^b x\ln x dx = (b - a) \cdot \frac{A}{2}\ln(I \cdot S).$$

On the other hand,

$$H_{p,f} = \frac{1}{\displaystyle\int_a^b \frac{p(x)}{f(x)}dx} = \frac{b^2 - a^2}{2} \cdot \frac{3}{b^2 + ab + a^2} = \frac{A}{He(a^2, b^2)},$$

where

$$He(x, y) = \frac{x + \sqrt{xy} + y}{3}$$

denotes the **Heronian mean** of $x$ and $y$. One obtains the double inequality:

$$A^2 < I \cdot S < \left(\frac{He(a^2, b^2)}{A}\right)^2 \tag{59}$$

The left side of (59) has been proved also in [8], while the right side seems to be new.

For an extension of (59) repeat all above computations with

$$p(x) = \frac{x^{n-1}n}{b^n - a^n}.$$

Since by partial integration we get

$$\int_a^b x^{n-1} \ln x dx = \frac{(b^n - a^n) \ln I(a^n, b^n)}{n^2} \tag{60}$$

from (42) we get

$$\frac{b^n - a^n}{n(b^{n-1} - a^{n-1})} < \sqrt[n]{I(a^n, b^n)} < \frac{n}{n+1} \cdot \frac{b^{n+1} - a^{n+1}}{b^n - a^n} \tag{61}$$

This new inequality extends (59), as for $n = 2$, by $I(a^2, b^2) = I \cdot S$, one reobtains (59). Here $n$ is a positive integer, but as the proof shows, it holds true by replacing $n$ with any $r > 1$, i.e.

$$\frac{b^r - a^r}{r(b^{r-1} - a^{r-1})} < (I(a^r, b^r))^{1/r} < \frac{r}{r+1} \cdot \frac{b^{r+1} - a^{r+1}}{b^r - a^r}, \ r > 1. \tag{62}$$

By putting $a^n = x$, $b^n = y$ in (61), this inequality appears as

$$\frac{y - x}{n[y^{(n-1)/n} - x^{(n-1)/n}]} < \sqrt[n]{I(x, y)} < \frac{n}{n+1} \cdot \frac{y^{(n+1)/n} - x^{(n+1)/n}}{y - x}. \tag{61'}$$

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## 3.5 On some exponential means

### 1. Introduction

A mean of two positive real numbers is defined in [3] as a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the property:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\},\ \forall\ x, y \in \mathbb{R}_+.$$

Of course, it follows that $M(x, x) = x$.
Two means $M$ and $M'$ are sometimes comparable. We write $M < M'$ if

$$M(x, y) < M'(x, y) \text{ for } x \neq y.$$

The most common example of mean is the power mean $A_p$ defined by:

$$\begin{aligned} &A_p(x, y) = ((x^p + y^p)/2)^{1/p},\ \text{for } p \neq 0 \\ &A_0(x, y) = G(x, y) = (xy)^{1/2} \text{ (the geometric mean).} \end{aligned}$$

We have:

$$\begin{aligned} &A_1(x, y) = A(x, y) \text{ (the arithmetic mean)} \\ &A_{-1} = H(x, y) \text{ (the harmonic mean)} \end{aligned}$$

and, as limit cases:

$$\begin{aligned} &A_{-\infty}(x, y) = \min\{x, y\} \\ &A_{+\infty}(x, y) = \max\{x, y\}. \end{aligned}$$

It is proved in [3] that:

$$A_p < A_q \text{ for } p < q \tag{1}$$

so that for a given mean $M$ one looks for his place between two power means:

$$A_p < M < A_q$$

but $p$ or $q$ (or both) can be infinite.

As it is shown in [7], a general method of construction of means is offered by the mean-value theorem for integrals. If $f$ is a monotone and continuous function on $\mathbb{R}_+$ and $g$ is a positive continuous function on $\mathbb{R}_+$ which is not identical zero on any interval, then for any $x, y \in \mathbb{R}_+$ there is an unique $z \in \mathbb{R}_+$ such that:

$$f(z)\int_x^y g(t)dt = \int_x^y f(t)g(t)dt.$$

So, one can define a mean $V_{f,g}$ by:

$$V_{f,g}(x,y) = f^{-1}\left(\frac{\displaystyle\int_x^y f(t)g(t)dt}{\displaystyle\int_x^y g(t)dt}\right).$$

For example, taking $f(t) = t$ and $g(t) = e^t$ we get the mean:

$$E(x,y) = \frac{xe^x - ye^y}{e^x - e^y} - 1$$

which was studied in [8]. As it was proved there:

$$A_1 = A < E < A_\infty \tag{2}$$

but, while the upper bound is strong ($A_p$ is not comparable with $E$ for $p > 5/3$), it is conjectured that the lower bound can be lifted up to $A_{5/3}$.

In this paper we want to indicate some relations of the mean $E$ with other means.

## 2. The identric and the logarithmic mean

For $x \neq y$, the identric mean $I$ is defined by:

$$I(x,y) = e^{-1}(y^y/x^x)^{1/(y-x)}$$

and the logarithmic mean $L$ by:

$$L(x,y) = \frac{x-y}{\log x - \log y}.$$

Improving some other results, T.P. Lin has proved in [4] that:

$$A_0 = G < L < A_{1/3}$$

and the indices 0 and 1/3 are sharp, that is $L$ and $A_p$ are not comparable for $0 < p < 1/3$. Also, in [5] A.O. Pittenger proved that:

$$A_{2/3} < I < A_{\log 2} \tag{3}$$

and again the indices are sharp. Of course, it follows that:

$$L < I. \tag{4}$$

We remind also two results of H. Alzer:

$$A \cdot G < L \cdot I \tag{5}$$

proved in [1] and:

$$G \cdot I < L^2 \tag{6}$$

proved in [2].

In what follows we shall use also a result of J. Sándor from [6]:

$$I^2(x,y) < A^2(x,y) < I(x^2,y^2) \text{ for } x \neq y. \tag{7}$$

Finally we remind that compounding a mean $M$ with a bijection $f$ we can construct a new mean $M'$:

$$M'(x,y) = f^{-1}(M(f(x), f(y))).$$

We shall use two means obtained on this way:

$$F(x,y) = \log(L(e^x, e^y))$$

and

$$B_p(x,y) = \log(A_p(e^x, e^y)).$$

We denote also $B_1 = B$.

## 3. Main results

We start with the remark that:

$$\log(I(x,y)) = \frac{y\log y - x\log x}{y-x} - 1$$

hence:

$$E(x,y) = \log(I(e^x, e^y)).$$

Using this relation, (4) becomes:

$$E > F. \tag{8}$$

The inequality (8) improves the first part of the inequality (2) because $G < L$ implies, by logarithmation $A < F$. That is

$$E > F > A.$$

Also putting in (3) $x = e^u$, $y = e^v$ and logarithming, we get

$$B_{2/3} < E < B_{\log 2}. \tag{9}$$

From (5) we have

$$E > A + B - F \tag{10}$$

which is another refinement of the first inequality from (2), because, by Hadamard's inequality for the convex function $f(t) = e^t$, we get (see [6]):

$$A < F < B.$$

On the same way, (6) gives:

$$E < 2 \cdot F - A. \tag{11}$$

Hence, from (10) and (11) we have:

$$A + B - F < E < 2 \cdot F - A.$$

Another relation for $E$ we can obtain from (7). Putting $x = e^u$, $y = e^v$ and logarithming, we get

$$E(u, v) < B(u, v) < \frac{E(2u, 2v)}{2}$$

that is

$$E(a/2, b/2) < \frac{E(a, b)}{2}. \tag{12}$$

## 4. Homogeneity properties

This last relation suggest the study of a property of subhomogeneity. Most of the used means are homogeneous (of order one):

$$M(tx, ty) = t \cdot M(x, y),\ t > 0.$$

There are also some log-homogeneous (logarithmic-homogeneous) means:

$$M(x^t, y^t) = M^t(x, y),\ t > 0.$$

For example $G$. But $I$ and $E$ haven't these properties.

The relation (12) suggest the following definitions: for a given $t > 0$, the mean $M$ is called $t-$subhomogeneous ($t - \log -$subhomogeneous) if:

$$M(tx, ty) \leq tM(x, y) \quad (\text{respectively } M(x^t, y^t) \leq M^t(x, y)).$$

If the inequalities are reversed, the mean is called $t-$superhomogeneous respectively $t - \log -$superhomogenous. Of course, if $M$ is $t-$subhomogeneous then it is $1/t-$superhomogeneous.

From (1) we deduce that $A_p$ is $t - \log -$subhomogeneous for $t < 1$ and $B_p$ is $t-$subhomogeneous for $t < 1$.

Applying the first inequality of (9) to $x = 3u/2$, $y = 3v/2$, we get:

$$B(u, v) < (2/3) \cdot E(3u/2, 3v/2).$$

From the second inequality of (9) with $x = u/\log 2$, $y = v/\log 2$ follows:

$$(\log 2) \cdot E(u/\log 2, v/\log 2) < B(u, v)$$

thus:

$$(\log 2) \cdot E(u/\log 2, v/\log 2) < (2/3) \cdot E(3u/2, 3v/2)$$

hence $E$ is $t-$subhomogeneous for $t = 2/\log 8$. In fact is valid the following property.

**4.1. Theorem.** *The mean $E$ is $t-$subhomogeneous and the mean $I$ is $t - \log -$subhomogeneous for $t \geq 2/\log 8 = 0.961\ldots$*

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## 3.6 A property of an exponential mean

Let $x, y > 0$. The following exponential mean $E = E(x, y)$, has been introduced by the second author in [2]. Their properties and/or connections to other means are studied in the papers [3] and [1].

$$E(x,y) = \frac{xe^x - ye^y}{e^x - e^y} - 1 \ (x \neq y), \quad E(x,x) = x. \tag{1}$$

**Theorem.** *For all* $0 < a < b$, $0 < x < y$ *one has*

$$\min\{x - a, y - b\} \leq E(x,y) - E(a,b) \leq \max\{x - a, y - b\}. \tag{2}$$

**Proof.** Put

$$g(x) = \frac{x}{e^x - 1}, \quad h(x) = \frac{x}{1 - e^x}, \quad f(x,y) = \frac{xe^{-x} - ye^{-y}}{e^{-x} - e^{-y}}.$$

Then it is immediate that

$$f(x,y) = x - g(y - x) = y - h(y - x),$$

and

$$f(x,y) - f(a,b) = x - a + g(b - a) - g(y - x) = y - b + h(b - a) - h(y - x).$$

It can be seen immediately that $h$ is increasing and $g$ is decreasing on $(0, +\infty)$. This implies by the above identity, that for $0 < a < b$ and $0 < x < y$ we have

$$\min\{x - a, y - b\} \leq f(x,y) - f(a,b) \leq \max\{x - a, y - b\}. \tag{3}$$

Now, it is easy to see that

$$f(x,y) = \overline{E} - 1, \tag{4}$$

where $\overline{E}$ is the complementary mean to $E$ (see [3]), i.e.

$$\overline{E}(x,y) = 2A(x,y) - E(x,y), \quad \text{where} \quad A(x,y) = \frac{x+y}{2}.$$

Since $\min\{u, v\} - (u + v) = -\max\{u, v\}$, from (3) and (4), we can deduce (2), i.e. the Theorem is proved.

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## 3.7 Some new inequalities for means and convex functions

### 1

In what follows, for $a, b > 0$ let us denote

$$A = A(a, b) = \frac{a+b}{2}, \quad G = G(a, b) = \sqrt{ab},$$

$$W = W(a, b) = \frac{a^2+b^2}{a+b}, \quad H = H(a, b) = \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}.$$

If $f : [a, b] \to \mathbb{R}$ is increasing (decreasing) function, then the following property is immediate:

**Proposition 1.**

$$\frac{af(b) + bf(a)}{a+b} \leq \frac{f(a) + f(b)}{2} \leq \frac{af(a) + bf(b)}{a+b} \tag{1}$$

*All inequalities in (1) are reversed, when $f$ is decreasing.*

**Proof.** After simple computations, each parts of (1) become equivalent to

$$(f(b) - f(a))(b - a) \geq 0 \quad (\text{or} \quad (f(b) - f(a))(b - a) \leq 0).$$

For $f(x) = x$, relations (1) imply the classical inequality

$$H \leq A \leq W.$$

A more interesting example arises, when $f(x) = \ln x$. Then we get

$$(a^b b^a)^{1/(a+b)} \leq G \leq (a^a b^b)^{1/(a+b)}. \tag{2}$$

For the involved means in the extremal sides of (2), see e.g. [1]-[3].

If $f$ is convex, the following can be proved:

**Proposition 2.** *Let $f$ be convex on $[a,b]$. Then*

$$f(W) \le \frac{af(a)+f(b)}{a+b} \tag{3}$$

$$f(H) \le \frac{af(b)+bf(a)}{a+b} \tag{4}$$

$$\frac{af(b)+bf(a)}{a+b} + f(W) \le f(a)+f(b). \tag{5}$$

**Proof.**

$$\begin{aligned} f(W) &= f\left(\frac{a^2+b^2}{a+b}\right) = f\left(a \cdot \frac{a}{a+b} + b \cdot \frac{b}{a+b}\right) \\ &\le \frac{a}{a+b} \cdot f(a) + \frac{b}{a+b} \cdot f(b) = \frac{af(a)+bf(b)}{a+b}, \end{aligned}$$

by the convexity of $f$ (i.e. $f(a\lambda + b\mu) \le \lambda f(a) + \mu f(b)$ for $\lambda, \mu > 0$, $\lambda + \mu = 1$). This proved (3).

Now,

$$\begin{aligned} f(H) &= f\left(\frac{2ab}{a+b}\right) = f\left(\frac{a}{a+b} \cdot b + \frac{b}{a+b} \cdot a\right) \\ &\le \frac{a}{a+b} \cdot f(b) + \frac{b}{a+b} \cdot f(a) = \frac{af(b)+bf(a)}{a+b}, \end{aligned}$$

yielding (4).

Relation (5) follows by (3), since

$$\frac{af(b)+bf(a)}{a+b} + \frac{af(a)+bf(b)}{a+b} = f(a)+f(b).$$

## 2

By taking into account Propositions 1 and 2, one can ask the question of validity of relations of type

$$\frac{af(b)+bf(a)}{a+b} \le f(W) \le \frac{af(a)+bf(b)}{a+b},$$

or

$$f(H) \leq \frac{af(b)+bf(a)}{a+b} \leq f(A), \text{ etc.}$$

We will prove the following results:

**Theorem 1.** ([4]) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable, convex and increasing function. Suppose that the function*

$$g(x) = \frac{f'(x)}{x}, \; x \in [a,b]$$

*is decreasing. Then one has*

$$f(H) \leq \frac{af(b)+bf(a)}{a+b} \leq f(A). \tag{6}$$

**Proof.** The left side of (6) is exactly relation (4). Let us write the right-hand side of (6) in the form

$$a[f(b)-f(A)] \leq b[f(A)-f(a)]. \tag{$*$}$$

By

$$b - A = \frac{b-a}{2} = A - a,$$

and by the Lagrange mean value theorem one has

$$f(b)-f(A) = \frac{b-a}{2}f'(\xi_2), \quad f(A)-f(a) = \frac{b-a}{2}f'(\xi_1),$$

where $\xi_1 \in (a,A)$, $\xi_2 \in (A,b)$. Thus $a < \xi_1 < \xi_2 < b$. By $f'(x) \geq 0$ and $f'$ being increasing we get by the monotonicity of $g$:

$$\frac{f'(b)}{b} \leq \frac{f'(a)}{a},$$

so

$$af'(\xi_2) \leq af'(b) \leq bf'(a) \leq bf'(\xi_1).$$

This implies relation $(*)$, i.e. the proof of Theorem 1 is completed.

The following theorem has a similar proof.

**Theorem 2.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable, convex and increasing function. Suppose that the function*

$$h(x) = \frac{f'(x)}{\sqrt{x}}$$

*is decreasing on $[a,b]$. Then*

$$f(H) \leq \frac{af(b) + bf(a)}{a+b} \leq f(G). \tag{7}$$

*For $f(x) = x$, (7) gives the classical inequality $H \leq G$.*

**Theorem 3.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable, convex and increasing function. Suppose that the function*

$$g(x) = \frac{f'(x)}{x}$$

*is decreasing on $[a,b]$. Then*

$$f(A) \leq f(W) \leq \frac{f(a) + f(b)}{2}. \tag{8}$$

**Proof.** The left side of (8) is trivial by $A \leq W$ and the monotonicity of $f$. The proof of right side is very similar to the proof of right side of (6). Indeed,

$$W - a = \frac{b(b-a)}{a+b}, \quad b - W = \frac{a(b-a)}{a+b}.$$

By Lagrange's mean value theorem one has

$$f(W) - f(a) = \frac{b(b-a)}{a+b} f'(\eta_1), \quad f(b) - f(W) = \frac{a(b-a)}{a+b} f'(\eta_2),$$

where $\eta_1 \in (a, W)$, $\eta_2 \in (W, b)$. Now, we can write that

$$af'(\eta_1) \leq af'(b) \leq bf'(a) \leq bf'(\eta_2),$$

so $f(W) - f(a) \leq f(b) - f(W)$, and (8) follows.

## 3

Finally, we shall prove an integral inequality, which improves on certain known results.

**Theorem 4.** ([4]) *If* $f : [a,b] \to \mathbb{R}$ *is convex and differentiable, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[\frac{af(b)+bf(a)}{a+b} + f(W)\right] \le \frac{f(a)+f(b)}{2}. \qquad (9)$$

**Proof.** Since $f$ is convex and differentiable, we can write that

$$f(x) - f(y) \le (x-y)f'(x) \text{ for all } x, y \in [a,b]. \qquad (**)$$

Apply now $(**)$ for $y = W$ and integrate the relation on $x \in [a,b]$:

$$\int_a^b f(x)dx \le (b-a)f(W) + \int_a^b (x-W)f'(x)dx.$$

Here

$$\int_a^b (x-W)f'(x)dx = \int_a^b xf'(x)dx - W[f(b)-f(a)]$$
$$= bf(b) - af(a) - \int_a^b f(x)dx - W[f(b)-f(a)],$$

by partial integration. Thus

$$2\int_a^b f(x)dx \le (b-a)\left[\frac{af(b)+bf(a)}{a+b}\right] + (b-a)f(W),$$

and the left side of (9) follows. The right hand side inequality of (9) is a consequence of relation (5).

**Remarks.** 1) Relation (9) improves the Hadamard inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

2) If the conditions of Theorem 1 are satisfied, the following chain of inequalities holds true:

$$f(H) \le \frac{af(b)+bf(a)}{a+b} \le f(A) \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \frac{1}{2}\left[\frac{af(b)+bf(a)}{a+b} + f(W)\right] \le \frac{f(a)+f(b)}{2}. \qquad (10)$$

3) The methods of this paper show that the more general means

$$W_k = \frac{a^k + b^k}{a^{k-1} + b^{k-1}}$$

may be introduced.

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## 3.8 Inequalities for general integral means

### 1. Introduction

A mean (of two positive real numbers on the interval $J$) is defined as a function $M : J^2 \to J$, which has the property

$$\min(a,b) \le M(a,b) \le \max(a,b),\ \forall\ a,b \in J.$$

Of course, each mean $M$ is reflexive, i.e.

$$M(a,a) = a,\ \forall\ a \in J$$

which will be used also as the definition of $M(a,a)$ if it is necessary. The mean is said to be symmetric if

$$M(a,b) = M(b,a),\ \forall\ a,b \in J.$$

Given two means $M$ and $N$, we write $M < N$ (on $J$) if

$$M(a,b) < N(a,b),\ \forall\ a,b \in J,\ a \neq b.$$

Among the most known examples of means are the arithmetic mean $A$, the geometric mean $G$, the harmonic mean $H$, and the logarithmic mean $L$, defined respectively by

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{a \cdot b},$$

$$H(a,b) = \frac{2ab}{a+b}, \quad L(a,b) = \frac{b-a}{\ln b - \ln a}, \quad a,b > 0,$$

and satisfying the relation $H < G < L < A$.

We deal with the following weighted integral mean. Let $f : J \to \mathbb{R}$ be a strictly monotone function and $p : J \to \mathbb{R}_+$ be a positive function. Then $M(f,p)$ defined by

$$M(f,p)(a,b) = f^{-1}\left(\frac{\displaystyle\int_a^b f(x)\cdot p(x)dx}{\displaystyle\int_a^b p(x)dx}\right),\ a,b \in J$$

gives a mean on $J$. This mean was considered in [3] for arbitrary weight function $p$ and $f = e_n$ where $e_n$ is defined by

$$e_n(x) = \begin{cases} x^n, & \text{if} \quad n \neq 0 \\ \ln x, & \text{if} \quad n = 0. \end{cases}$$

More means of type $M(f,p)$ are given in [2], but only for special cases of functions $f$.

A general example of mean which can be defined in this way is the extended mean considered in [4]:

$$E_{r,s}(a,b) = \left( \frac{r}{s} \cdot \frac{b^s - a^s}{b^r - a^r} \right)^{\frac{1}{s-r}}, \; s \neq 0, \; r \neq s.$$

We have $E_{r,s} = M(e_{s-r}, e_{r-1})$.

The following is proved in [6].

**Lemma 1.1.** *If the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone, the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, and the composed function $g \circ f^{-1}$ is convex, then the inequality*

$$M(f,p) < M(g,p)$$

*holds for every positive function $p$.*

The means $A$, $G$ and $L$ can be obtained as means $M(e_n, 1)$ for $n = 1$, $n = -2$ and $n = -1$ respectively. So the relations between them follow from the above result. However, $H = M(e_1, e_{-3})$, thus the inequality $H < G$ cannot be proved on this way.

A special case of integral mean was defined in [5]. Let $p$ be a strictly increasing real function having an increasing derivative $p'$ on $J$. Then $M'_p$ given by

$$M'_p(a,b) = \int_a^b \frac{x \cdot p'(x) dx}{p(b) - p(a)}, \; a, b \in J$$

defines a mean. In fact we have $M'_p = M(e_1, p')$.

In this paper we use the result of the above lemma to modify the definition of the mean $M(f, p)$. Moreover, we find that an analogous property also holds for the weight function. We apply these properties for proving relations between some means.

## 2. The new integral mean

We define another integral mean using two functions as above, but only one integral. Let $f$ and $p$ be two strictly monotone functions on $J$. Then $N(f, p)$ defined by

$$N(f,p)(a,b) = f^{-1}\left(\int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1-t) \cdot p(b)]dt\right)$$

is a symmetric mean on $J$. Making the change of the variable

$$t = \frac{[p(b) - s]}{[p(b) - p(a)]}$$

we obtain the simpler representation

$$N(f,p)(a,b) = f^{-1}\left(\int_{p(a)}^{p(b)} \frac{(f \circ p^{-1})(s)ds}{p(b) - p(a)}\right).$$

Denoting $f \circ p^{-1} = g$, the mean $N(f, p)$ becomes

$$N'(g,p)(a,b) = p^{-1} \circ g^{-1}\left(\int_{p(a)}^{p(b)} \frac{g(x)dx}{p(b) - p(a)}\right).$$

Using it we can obtain again the extended mean $E_{r,s}$ as $N'(e_{s/r-1}, e_r)$.

Also, if the function $p$ has an increasing derivative, by the change of the variable

$$s = p(x)$$

the mean $N(f, p)$ reduces at $M(f, p')$. For such a function $p$ we have $N(e_1, p) = M'_p$. Thus $M'_p$ p can also be generalized for non differentiable functions $p$ at

$$M_p(a,b) = \int_0^1 p^{-1}[t \cdot p(a) + (1-t) \cdot p(b)]dt, \ \forall\ a, b \in J$$

or

$$M_p(a,b) = \int_{p(a)}^{p(b)} \frac{p^{-1}(s)ds}{p(b)-p(a)},\ \forall\ a,b \in J,$$

which is simpler for computations.

**Example 2.1.** For $n \neq -1, 0$, we get

$$M_{e_n}(a,b) = \frac{n}{n+1} \cdot \frac{b^{n+1}-a^{n+1}}{b^n - a^n},\ \text{for } a,b>0,$$

which is a special case of the extended mean. We obtain the arithmetic mean $A$ for $n=1$, the logarithmic mean $L$ for $n=0$, the geometric mean $G$ for $n=-1/2$, the inverse of the logarithmic mean $G^2/L$ for $n=-1$, and the harmonic mean $H$ for $n=-2$.

**Example 2.2.** Analogously we have

$$M_{\exp}(a,b) = \frac{b \cdot e^b - a \cdot e^a}{e^b - e^a} - 1 = E(a,b),\ a,b \geq 0$$

which is an exponential mean introduced by the authors in [7]. We can also give a new exponential mean

$$M_{1/\exp}(a,b) = \frac{a \cdot e^b - b \cdot e^a}{e^b - e^a} + 1 = (2A-E)(a,b),\ a,b \geq 0.$$

**Example 2.3.** Some trigonometric means such as

$$M_{\sin}(a,b) = \frac{b \cdot \sin b - a \cdot \sin a}{\sin b - \sin a} - \tan\frac{a+b}{2},\ a,b \in [0,\pi/2],$$

$$M_{\arcsin}(a,b) = \frac{\sqrt{1-b^2}-\sqrt{1-a^2}}{\arcsin a - \arcsin b},\ a,b \in [0,1],$$

$$M_{\tan}(a,b) = \frac{b \cdot \tan b - a \cdot \tan a + \ln(\cos b/\cos a)}{\tan b - \tan a},\ a,b \in [0,\pi/2)$$

and

$$M_{\arctan}(a,b) = \frac{\ln\sqrt{1+b^2} - \ln\sqrt{1+a^2}}{\arctan b - \arctan a},\ a,b \geq 0,$$

can be also obtained.

## 3. Main results

In [5] it was shown that the inequality $M'_p > A$ holds for each function $p$ (assumed to be strictly increasing and with strictly increasing derivative). We can prove more general properties. First of all, the result from Lemma 1.1 holds also in this case with the same proof.

**Theorem 3.1.** *If the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone, the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, and the composed function $g \circ f^{-1}$ is convex, then the inequality*

$$N(f,p) < N(g,p)$$

*holds for every monotone function $p$.*

**Proof.** Using a simplified variant of Jensen's integral inequality for the convex function $g \circ f^{-1}$ (see [1]), we have

$$(g \circ f^{-1}) \left( \int_0^1 (f \circ p^{-1})[t \cdot p(a) + (1-t) \cdot p(b)] dt \right)$$
$$\leq \int_0^1 (g \circ f^{-1}) \circ (f \circ p^{-1})[t \cdot p(a) + (1-t) \cdot p(b)] dt.$$

Applying the increasing function $g^{-1}$ we get the desired inequality. $\square$

We can now also prove a similar result with respect to the function $p$.

**Theorem 3.2.** *If $p$ is a strictly monotone real function on $J$ and $q$ is a strictly increasing real function on $J$, such that $q \circ p^{-1}$ is strictly convex, then*

$$N(f,p) < N(f,q) \text{ on } J,$$

*for each strictly monotone function $f$.*

**Proof.** Let $a, b \in J$ and denote $p(a) = c$, $p(b) = d$. As $q \circ p^{-1}$ is strictly convex, we have

$$(q \circ p^{-1})[tc + (1-t)d] < t(q \circ p^{-1})(c) + (1-t)(q \circ p^{-1})(d), \ \forall\, t \in (0,1).$$

As $q$ is strictly increasing, this implies

$$p^{-1}[t \cdot p(a) + (1-t) \cdot p(b)] < q^{-1}[t \cdot q(a) + (1-t) \cdot q(b)], \ \forall\, t \in (0,1).$$

If the function $f$ is increasing, the inequality is preserved by the composition with it. Integrating on $[0,1]$ and then composing with $f^{-1}$, we obtain the desired result. If the function $f$ is decreasing, so also is $f^{-1}$ and the result is the same. □

**Corollary 3.3.** *If the function $q$ is strictly convex and strictly increasing then*

$$M_q > A.$$

**Proof.** We apply the second theorem for $p = f = e_1$, taking into account that $M_{e_1} = A$. □

**Remark 3.4.** If we replace the convexity by the concavity and/or the increase by the decrease, we get in the above theorems the same/the opposite inequalities.

**Example 3.1.** Taking log, sin respectively arctan as function $q$, we get the inequalities

$$L,\ M_{\sin},\ M_{\arctan} < A.$$

**Example 3.2.** However, if we take exp, arcsin respectively tan as function $q$, we have

$$E,\ M_{\arcsin},\ M_{\tan} > A.$$

**Example 3.3.** Taking $p = e_n$, $q = e_m$ and $f = e_1$, from Theorem 3.2 we deduce that for $m \cdot n > 0$ we have

$$M_{e_n} < M_{e_m},\ \text{if } n < m.$$

As special cases we have

$$M_{e_n} > A,\ \text{for } n > 1,$$

$$L < M_{e_n} < A,\ \text{for } 0 < n < 1,$$

$$G < M_{e_n} < L,\ \text{for } -1/2 < n < 0,$$

$$H < M_{e_n} < G,\ \text{for } -2 < n < -1/2,$$

and

$$M_{e_n} < H, \text{ for } n < -2.$$

Applying the above theorems we can also study the monotonicity of the extended means.

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# 3.9 On upper Hermite-Hadamard inequalities for geometric-convex and log-convex functions

## 1. Introduction

Let $I \subset \mathbb{R}$ be a nonvoid interval. A function $f : I \to (0, +\infty)$ is called **log-convex** (or logarithmically convex), if the function $g : I \to \mathbb{R}$, defined by $g(x) = \ln f(x)$, $x \in I$ is convex; i.e. satisfies

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) \tag{1.1}$$

for all $x, y \in I$, $\lambda \in [0, 1]$.

Inequality (1.1) may be rewritten for the function $f$, as

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}, \tag{1.2}$$

for $x, y \in I$, $\lambda \in [0, 1]$.

If one replaces the weighted arithmetic mean $\lambda x + (1-\lambda)y$ of $x$ and $y$ with the weighted geometric mean, i.e. $x^{\lambda}y^{1-\lambda}$, then we get the concept of **geometric-convex** function $f : I \subset (0, +\infty) \to (0, +\infty)$

$$f(x^{\lambda}y^{1-\lambda}) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}, \tag{1.3}$$

for $x, y \in I$, $\lambda \in [0, 1]$.

These definitions are well-known in the literature, we quote e.g. [7] for an older and [4] for a recent monograph on this subject.

Also, the well-known Hermite-Hadamard inequalities state that for a convex function $g$ of (1.1) one has

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{g(a)+g(b)}{2}, \tag{1.4}$$

for any $a, b \in I$.

We will call the right side of (1.4) as the **upper Hermite-Hadamard inequality**.

By applying the weighted geometric mean-arithmetic mean inequality

$$a^{\lambda}b^{1-\lambda} \leq \lambda a + (1-\lambda)b, \tag{1.5}$$

the following properties easily follow:

**Lemma 1.** *(i) If $f : I \to (0,\infty)$ is log-convex, then it is convex;*

*(ii) If $f : I \subset (0,\infty) \to (0,\infty)$ is increasing and log-convex, then it is geometric convex.*

**Proof.** We offer for sake of completeness, the simple proof of this lemma.

(i) One has by (1.2) and (1.5):

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda} \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in I$, $\lambda \in [0,1]$.

(ii) $f(x^{\lambda}y^{1-\lambda}) \leq f(\lambda x + (1-\lambda)y)$ by (1.5) and the monotonicity of $f$. Now, by (1.2) we get (1.3).

Let $L(a,b)$ denote the logarithmic mean of two positive real numbers and $b$, i.e.

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \text{ for } a \neq b;\ L(a,a) = a. \tag{1.6}$$

In 1997, Gill, Pearce and Pečarić [1] have proved the following upper Hermite-Hadamard type inequality:

**Theorem 1.1.** *If $f : [a,b] \to (0,+\infty)$ is log-convex, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L(f(a), f(b)), \tag{1.7}$$

*where $L$ is defined by (1.6).*

Recently, Xi and Qi [6] proved the following result:

**Theorem 1.2.** *Let $a, b > 0$ and $f : [a,b] \to (0,+\infty)$ be increasing and log-convex. Then*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)dx \leq L(af(a), bf(b)). \tag{1.8}$$

Prior to [6], Iscan [2] published the following result:

**Theorem 1.3.** *Let $a, b > 0$ and $f : [a, b] \to (0, \infty)$ be integrable and geometric-convex function. Then*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \le L(f(a), f(b)). \tag{1.9}$$

In case of $f$ increasing and log-convex, (1.9) is stated in [6], too. However, by Lemma 1(ii), clearly Theorem 1.3 is a stronger version.

In what follows, we shall offer refinements of (1.8) and (1.9). In fact, in almost all cases, inequality (1.7) is the strongest from the above.

## 2. Main results

First we prove that the result of Theorem 1.2 holds true in fact for geometric-convex functions:

**Theorem 2.1.** *Relation (1.8) holds true when $f$ is integrable geometric convex function.*

**Proof.** First remark that when $f$ is geometric-convex, the same is true for the function $g(x) = xf(x)$, $x \in I$. Indeed, one has

$$\begin{aligned} g(x^\lambda y^{1-\lambda}) &= x^\lambda y^{1-\lambda} f(x^\lambda y^{1-\lambda}) \le x^\lambda y^{1-\lambda} (f(x))^\lambda (f(y))^{1-\lambda} \\ &= (xf(x))^\lambda (yf(y))^{1-\lambda} = (g(x))^\lambda (g(y))^{1-\lambda}, \end{aligned}$$

for all $x, y \in I$, $\lambda \in [0, 1]$. Therefore, by (1.3), $g$ is geometric convex.

Apply now inequality (1.9) for $xf(x)$ in place of $f(x)$. Relation (1.8) follows.

In what follows, we shall need the following auxiliary result:

**Lemma 2.1.** *Suppose that $b > a > 0$ and $q \ge p > 0$. Then one has*

$$L(pa, qb) \ge L(p, q)L(a, b), \tag{2.1}$$

*where $L$ denotes the logarithmic mean, defined by (1.6).*

**Proof.** Two proofs of this result may be found in [5]. Relation (2.1) holds true in a general setting of the Stolarksy means, see [3] (Theorem 3.8).

We offer here a proof of (2.1) for the sake of completeness. As

$$L(a,b) = \int_0^1 b^u a^{1-u} du, \tag{2.2}$$

applying the Chebysheff integral inequality

$$\frac{1}{y-x}\int_x^y f(t)dt \cdot \frac{1}{y-x}\int_x^y g(t)dt < \frac{1}{y-x}\int_x^y g(t)f(t)dt, \tag{2.3}$$

where $x < y$ and $f, g : [x, y] \to \mathbb{R}$ are strictly monotonic functions of the same type; to the particular case

$$[x, y] = [0, 1];\ f(t) = b^t a^{1-t} = a\left(\frac{b}{a}\right)^t$$

and

$$g(t) = q^t p^{1-t} = p\left(\frac{q}{p}\right)^t$$

for $b > a$ and $q > p$; by (2.2), relation (2.1) follows. For $p = q$ one has equality in (2.1).

One of the main results of this paper is stated as follows:

**Theorem 2.2.** *Let $b > a > 0$ and suppose that $f : [a, b] \to \mathbb{R}$ is log-convex. Suppose that $f(b) \geq f(a)$. Then one has*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L(f(a), f(b)) \leq \frac{\ln b - \ln a}{b-a} \cdot L(af(a), bf(b)). \tag{2.4}$$

**Proof.** The first inequality of (2.4) holds true by Theorem 1.1. Applying now Lemma 2.1, by $q = f(b) \geq f(a) = p$ and $b > a$, one has

$$L(f(a), f(b))L(a, b) \leq L(af(a), bf(b)).$$

As this is exactly the second inequality of (2.4), the proof of Theorem 2.2 is finished.

**Remark 2.1.** The weaker inequality of (2.4) is the result of Theorem 1.2, in an improved form (in place of increasing $f$, it is supposed only $f(b) \geq f(a)$).

When $f(b) > f(a)$, there is strict inequality in the right side of (2.4).

**Theorem 2.3.** *Let $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ log-convex function. Suppose that $\frac{f(b)}{b} \geq \frac{f(a)}{a}$. Then one has*

$$\frac{1}{b-a}\int_a^b \frac{f(x)}{x}dx \leq L\left(\frac{f(a)}{a}, \frac{f(b)}{b}\right) \leq \frac{\ln b - \ln a}{b-a} \cdot L(f(a), f(b)). \quad (2.5)$$

**Proof.** First remark that $\frac{f(x)}{x}$ is log-convex function, too, being the product of the log-convex functions $\frac{1}{x}$ and $f(x)$. Thus, applying Theorem 1.1 for $\frac{f(x)}{x}$ in place of $f(x)$, we get the first inequality of (2.5).

The second inequality of (2.5) may be rewritten as

$$L\left(\frac{f(a)}{a}, \frac{f(b)}{b}\right) L(a, b) \leq L(f(a), f(b)),$$

and this is a consequence of Lemma 2.1 applied to $p = \frac{f(a)}{a}$, $q = \frac{f(b)}{b}$.

**Remark 2.2.** Inequality (2.5) offers a refinement of (1.9) whenever

$$\frac{f(b)}{b} \geq \frac{f(a)}{a}.$$

When here is strict inequality, the last inequality of (2.5) will be strict, too.

**Lemma 2.2.** *Suppose that $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ is a real function such that $g(x) = \frac{f(x)}{x}$ is increasing in $[a, b]$. Then*

$$\int_a^b \frac{f(x)}{x}dx \leq \frac{1}{A}\int_a^b f(x)dx, \quad (2.6)$$

*where $A = A(a, b) = \frac{a+b}{2}$ denotes the arithmetic mean of $a$ and $b$.*

**Proof.** Using Chebysheff's inequality (2.3) on $[x, y] = [a, b]$,

$$f(t) := \frac{f(t)}{t}; \; g(t) := t,$$

which have the same type of monotonicity. Since

$$\frac{1}{b-a} \int_a^b t dt = \frac{a+b}{2} = A,$$

relation (2.6) follows.

The following theorem gives another refinement of (1.9):

**Theorem 2.4.** *Let $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ log-convex, such that the function $x \mapsto \dfrac{f(x)}{x}$ is increasing on $[a, b]$. Then*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{L}{A} \cdot L(f(a), f(b)) < L(f(a), f(b)), \tag{2.7}$$

*where $L = L(a, b)$ denotes the logarithmic mean of $a$ and $b$.*

**Proof.** By (2.6) we can write

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \left( \frac{b-a}{\ln b - \ln a} \right) \cdot \frac{1}{A} \cdot \left( \frac{1}{b-a} \int_a^b f(x) dx \right).$$

As

$$\frac{b-a}{\ln b - \ln a} = L \quad \text{and} \quad \frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)),$$

by (1.7), the first inequality of (2.7) follows. The last inequality of (2.7) follows by the classical relation (see e.g. [3])

$$L < A. \tag{2.8}$$

**Remark 2.2.** As inequality (1.7) holds true with reversed sign of inequality, whenever $f$ is log-concave (see [1]), (2.8) may be proved by an application for the log-concave function $f(x) = x$.

A counterpart to Lemma 2.1 is provided by:

**Lemma 2.3.** *If* $\frac{q}{p} \geq \frac{b}{a} \geq 1$, *then*

$$L(pa, qb) \leq L(p, q)A(a, b). \tag{2.9}$$

**Proof.** By letting $\frac{q}{p} = u$, $\frac{b}{a} = v$, inequality (2.9) may be rewritten as

$$\frac{uv-1}{\ln(uv)} \leq \frac{u+1}{2} \cdot \frac{v-1}{\ln v}, \;\; u \geq v \geq 1. \tag{2.10}$$

If $v = 1$, then (2.9) is trivially satisfied, so suppose $v > 1$.

Consider the application

$$k(u) = (v-1)(u+1)\ln(uv) - 2(uv-1)\ln v, \;\; u \geq v.$$

One has

$$k(v) = 0 \quad \text{and} \quad k'(u) = (v-1)\left(\ln u + 1 + \frac{1}{u}\right) - (v+1)\ln v.$$

Here $h(u) = \ln u + 1 + \frac{1}{u}$ has a derivative

$$h'(u) = \frac{u-1}{u^2} > 0,$$

so $h$ is strictly increasing, implying $h(u) \geq h(v)$, One gets

$$k'(u) \geq (v-1)\left(\ln v + 1 + \frac{1}{v}\right) - (v+1)\ln v = \frac{v^2 - 1 - \ln(v^2)}{v} > 0,$$

on base of the classical inequality

$$\ln t \leq t - 1, \tag{2.11}$$

where equality occurs only when $t = 1$.

The function $k$ being strictly increasing, we get $k(u) \geq k(v) = 0$, so inequality (2.9) follows.

Now, we will obtain a refinement of (1.9) for geometric convex functions:

**Theorem 2.5.** *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a geometric convex function such that the application $x \mapsto \dfrac{f(x)}{x}$ is increasing. Then one has the inequalities*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{1}{A(a,b)} \cdot L(af(a), bf(b)) \leq L(f(a), f(b)). \tag{2.12}$$

**Proof.** By Lemma 2.2 and Theorem 2.1, we can write

$$\begin{aligned}\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx &\leq \frac{1}{A(a,b)}\left(\frac{1}{\ln b - \ln a}\int_a^b f(x)dx\right) \\ &\leq \frac{L(af(a), bf(b))}{A(a,b)}.\end{aligned} \tag{2.13}$$

Now, applying Lemma 2.3 for $q = f(b)$, $p = f(a)$, by (2.9) we get

$$L(af(a), bf(b)) \leq L(f(a), f(b))A(a,b), \tag{2.14}$$

so the second inequality of (2.12) follows by the second inequality of (2.13).

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## 3.10 On certain identities for means, III

### 1. Introduction

Let $a, b > 0$ be positive real numbers. The power mean of order $k \in \mathbb{R} \setminus \{0\}$ of $a$ and $b$ is defined by

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}.$$

Denote

$$A = A_1(a,b) = \frac{a+b}{2},$$

$$G = G(a,b) = A_0(a,b) = \lim_{k\to\infty} A_k(a,b) = \sqrt{ab}$$

the arithmetic, resp. geometric means of $a$ and $b$.

The identric, resp. logarithmic means of $a$ and $b$ are defined by

$$I = I(a,b) = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a;$$

and

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \quad \text{for } a \neq b; \quad L(a,a) = a.$$

Consider also the weighted geometric mean $S$ of $a$ and $b$, the weights being $a/(a+b)$ and $b/(a+b)$ :

$$S = S(a,b) = a^{a/(a+b)} \cdot b^{b/(a+b)}. \tag{1}$$

We note that some authors use the notation $Z$ in place of $S$, (see [20], [5]) studied for the first time in 1990 by the first author [7], and then in 1993 [9], 1997 [10], and most recently in [4]. As one has the identity

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)},$$

discovered by the first author in [9], the mean $S$ is connected to the identric mean $I$.

Though here we are concerned with means of two arguments, we note that, the extension of $S$ to $n$ arguments is introduced in the first author's paper [14], where it is proved also the double-inequality from Theorem 3 of paper [5].

Other means of $a, b > 0$ which occur in this paper are

$$H = H(a,b) = A_{-1}(a,b) = \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}},$$

$$Q = Q(a,b) = A_2(a,b) = \sqrt{\frac{a^2+b^2}{2}},$$

as well as "Seiffert's mean" (see [16], [13])

$$P = P(a,b) = \frac{(a-b)}{2\arcsin\left(\dfrac{a-b}{a+b}\right)} \text{ for } a \neq b, \quad P(a,a) = a;$$

or an "exponential mean" (see [19], [8], [15])

$$E = E(a,b) = \left(ae^a - be^b\right) / \left(e^a - e^b\right) - 1 \text{ for } a \neq b; \quad E(a,a) = a.$$

As one has the identity (see [8])

$$E(a,b) = \log I(e^a, e^b), \tag{2}$$

the mean $E$ is also strongly related to the identric mean $I$.

This paper is a continuation of the former works [9], [12] where the importance of certain identities has been emphasized. For example, the identity

$$\log \frac{I}{G} = \frac{A-L}{L} \tag{3}$$

due to H.-J. Seiffert (see [17], [9]), where $I = I(a,b)$ for $a \neq b$, etc. In view of (2), relation (3) gives the identity (see [9])

$$E - A = \frac{A(e^a, e^b)}{L(e^a, e^b)} - 1. \tag{4}$$

As $A(x,y) > L(x,y)$ for any $x \neq y$; a corollary of (4) is Toader's inequality (see [10])

$$E > A \tag{5}$$

For this method, and many related inequalities, see the papers [8], [15].

Another method for the comparison of means is based on certain series representations. For example, one has (see [9])

$$\log \frac{A(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k} \left( \frac{b-a}{b+a} \right)^{2k}, \tag{6}$$

$$\log \frac{I(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left( \frac{b-a}{b+a} \right)^{2k}, \tag{7}$$

or (see [10])

$$\log \frac{S(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \left( \frac{b-a}{b+a} \right)^{2k}, \tag{8}$$

$$\log \frac{S(a,b)}{A(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k(k-1)} \left( \frac{b-a}{b+a} \right)^{2k}. \tag{9}$$

The representation

$$\log \frac{\sqrt{2A^2+G^2}}{I\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \left( \frac{1}{2k+1} - \frac{1}{3^k} \right) \left( \frac{b-a}{b+a} \right)^{2k}, \tag{10}$$

with $A = A(a,b)$ etc., appears in [11], while

$$\frac{L(a,b)}{G(a,b)} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \frac{\log a - \log b}{2} \right)^{2k} \tag{11}$$

is proved in paper [2].

In [16] it is proved that

$$\frac{A(a,b)}{G(a,b)} = \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} \left( \frac{b-a}{b+a} \right)^{2k}, \tag{12}$$

$$\frac{A(a,b)}{P(a,b)} = \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \binom{2k}{k} \left(\frac{b-a}{b+a}\right)^{2k}, \tag{13}$$

where $\binom{2k}{k}$ denotes a binomial coefficient.

In what follows, we shall deduce common proofs of these and similar identities. Some corollaries related to certain inequalities will be pointed out, too.

## 2. Series representations of integral means

The first main result is the following

**Theorem A.** *Let us suppose that, $f$ is a continuous function on $[a,b]$ and assume that all derivatives $f^{(l)}\left(\frac{a+b}{2}\right)$ $(l = 1,2,3,\ldots)$ exist at $\frac{a+b}{2}$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \left(\frac{b-a}{2}\right)^{2k} f^{(2k)}\left(\frac{a+b}{2}\right) \tag{14}$$

*and*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2} - \sum_{k=1}^{\infty} \frac{2k}{(2k+1)!} \left(\frac{b-a}{2}\right)^{2k} f^{(2k)}\left(\frac{a+b}{2}\right). \tag{15}$$

*One has also*

$$\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \left(\frac{b-a}{2}\right)^{2k} f^{(2k)}\left(\frac{a+b}{2}\right) \tag{16}$$

**Proof.** Writing Taylor's expansion for the function $f$ about the point $t = m$, we get

$$f(t) = f(m) + \sum_{r=1}^{\infty} \frac{(t-m)^r}{r!} f^{(r)}(m) \tag{17}$$

if one assumes that $f^{(l)}(m)$ exist for any $l = 1, 2, \ldots$ By integrating (17) on $t \in [a, b]$ we get

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = f(m) + \frac{1}{b-a}\int_a^b \frac{f^{(r)}(m)}{(r+1)!}\left[(b-m)^{r+1} - (a-m)^{r+1}\right] \tag{18}$$

Let now $m = \dfrac{a+b}{2}$ in (18). Since

$$(b-m)^{r+1} - (a-m)^{r+1} = \left(\frac{b-a}{2}\right)^{r+1} \cdot [1 + (-1)^r],$$

which is clearly zero for odd $r$; while for even $r = 2k$ it is $\left(\dfrac{b-a}{2}\right)^{2k}$, we get identity (14).

Now, letting again $m = \dfrac{a+b}{2}$ in (4) and $t = a, t = b$, respectively, we have

$$f(a) = f\left(\frac{a+b}{2}\right) + \sum_{r=1}^{\infty}\left(\frac{b-a}{2}\right)^r \cdot \frac{1}{r!}(-1)^r \cdot f^{(r)}\left(\frac{a+b}{2}\right);$$

$$f(b) = f\left(\frac{a+b}{2}\right) + \sum_{r=1}^{\infty}\left(\frac{b-a}{2}\right)^r \cdot \frac{1}{r!} \cdot f^{(r)}\left(\frac{a+b}{2}\right),$$

so after addition of these two equalities we get relation (16), by remarking that, as above $1 + (-1)^r = 0$ for $r$ =odd; and $= 2$ for $r = 2k =$ even.

Identity (15) is a consequence of relations (14) and (16), by eliminating $f\left(\dfrac{a+b}{2}\right)$. $\square$

**Remark 1.** Identity (14) appears also in [2], where it is applied for the proof of relation (11). Here we will deduce this relation by another method.

**Theorem B.** *If $f$ is continuous on $[a, b]$, and all derivatives $f^{(l)}(a)$ and $f^{(l)}(b)$ exist ($l = 1, 2, \ldots$), then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(b-a)^k}{(k+1)!}\left[f^{(k)}(a) + (-1)^k f^{(k)}(a)\right] \tag{19}$$

**Proof.** Applying relation (18) to $m = a$ and $m = b$, after addition we easily get relation (19). We omit the details. $\square$

## 3. Applications

**Theorem 1.** *Relations (6) and (7) are true, and one has also*

$$\log \frac{A(a,b)}{I(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \left(\frac{b-a}{b+a}\right)^{2k}; \tag{20}$$

$$\frac{A(a,b)}{L(a,b)} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k}. \tag{21}$$

**Proof.** Let $f(x) = \log x$ in relation (16). It is immediate that

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!/x^k,$$

so applying (16) to this function we get relation (6). Since

$$\frac{1}{b-a} \int_a^b \log x \mathrm{d}x = \log I(a,b),$$

the application of (15) gives identity (7). By a simple substraction, from (6) and (7) we get (20) by remarking that

$$\frac{1}{2k} - \frac{1}{2k+1} = \frac{1}{2k(2k+1)}.$$

Relation (21) is a consequence of (14), for the function

$$f(x) = \frac{1}{x}.$$

Indeed, as $f^{(k)}(x) = (-1)^k k!/x^k$, and remarking that

$$\frac{1}{b-a} \int_a^b \frac{1}{x} \mathrm{d}x = \frac{1}{L(a,b)},$$

we obtain

$$\frac{1}{L} = \frac{1}{A} + \frac{1}{A} \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k},$$

with $L = L(a,b)$ etc. $\square$

**Remark 2.** By (21) and (7) follows at once identity (3).

**Theorem 2.** *One has*

$$\frac{A(a,b)}{H(a,b)} = \sum_{k=0}^{\infty} \left(\frac{b-a}{b+a}\right)^{2k}; \tag{22}$$

$$A\left(\frac{1}{H} - \frac{1}{L}\right) = \sum_{k=1}^{\infty} \frac{2k}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k}. \tag{23}$$

$$\frac{A}{2}\left(\frac{1}{H} + \frac{1}{L}\right) = \sum_{k=0}^{\infty} \frac{k+1}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k}. \tag{24}$$

**Proof.** Let $f(x) = \dfrac{1}{x}$ in (14). We get

$$\frac{1}{H} - \frac{1}{A} = \frac{1}{A} \sum_{k=1}^{\infty} \left(\frac{b-a}{b+a}\right)^{2k},$$

so (22) follows. Relation (23) follows from (15) for the same function $f(x) = \dfrac{1}{x}$. We note that, (23) follows also from (22) and (21) by substraction and remarking that

$$1 - \frac{1}{2k+1} = \frac{2k}{2k+1}.$$

Identity (24) follows by addition of (22) and (21). $\square$

**Corollary 1.** *Let $a' = 1-a, b' = 1-b$, where $0 < a, b \leq \dfrac{1}{2}; a \neq b$. Put $A' = A'(a,b) = A(a',b'); G' = G'(a,b) = G(a',b')$, etc. Then one has the following Ky Fan type inequalities:*

$$A'\left(\frac{1}{H'} - \frac{1}{L'}\right) < A\left(\frac{1}{H} - \frac{1}{L}\right), \tag{25}$$

$$A'\left(\frac{1}{H'} + \frac{1}{L'}\right) < A\left(\frac{1}{H} + \frac{1}{L}\right). \tag{26}$$

**Proof.** Put $u = \dfrac{a-b}{a+b}$, $v = \dfrac{a'-b'}{a'+b'}$. Then the given conditions imply $|v| \le |u|$, so inequalities (25) and (26) are consequences of the representations (23), resp. (24). $\square$

**Remark 3.** Inequality (25) appears also in [3]. Since $A' > A$, as $A' = 1-A$ and $A < \dfrac{1}{2}$, (25) is not a consequence of the known inequality

$$\frac{1}{H'} - \frac{1}{L'} < \frac{1}{H} - \frac{1}{L}.$$

**Theorem 3.** *One has*

$$\log \frac{Q(a,b)}{G(a,b)} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \left( \frac{b-a}{b+a} \right)^{4k-2}, \tag{27}$$

*and relations (8) and (9) are also true.*

**Proof.** Applying (16) for $f(x) = x \log x$, after some computations we get identity (9). Now, by taking into account (6), from (9) we can deduce relation (8).

However, we shall use here method of proof for (8) and (9), which incorporates also the proof of (27).

Applying the identity (21) for $a = 1-z, b = 1+z$, where $|z| < 1$, we get, as $A(1-z, 1+z) = 1$ and $L(1-z, 1+z) = \dfrac{2z}{\log(1+z) - \log(1-z)}$:

$$\frac{1}{2z} \log \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} \frac{1}{2k+1} z^{2k}, \text{ i.e.}$$

$$\frac{1}{2} \log \frac{1+z}{1-z} = \sum_{k=1}^{\infty} \frac{1}{2k-1} z^{2k-1}. \tag{28}$$

Putting $z = \left( \dfrac{x-y}{x+y} \right)^2$, and remarking that $\dfrac{1+z}{1-z} = \dfrac{x^2+y^2}{2xy}$, we get identity (27) for $Q(x,y)$ in place of $Q(a,b)$ etc.

Applying (28) for $z = \dfrac{x-y}{x+y}$, we get

$$\frac{1}{2} \log \frac{x}{y} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \left( \frac{x-y}{x+y} \right)^{2k-1}. \tag{29}$$

We note that (29) follows also from identity (21) by writing

$$1+\sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{y-x}{y+x}\right)^{2k}=\sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{y-x}{y+x}\right)^{2k-2}.$$

As

$$\frac{S(x,y)}{G(x,y)}=\frac{(x^x y^y)^{1/(x+y)}}{x^{1/2}y^{1/2}}=\left(\frac{x}{y}\right)^{(x-y)/2(x+y)},$$

by multiplication of (29) with $\frac{x-y}{x+y}$, we get identity (8). Subtracting identities (6) and (8), we get (9). □

**Corollary 2.** *As*

$$\left(\frac{b-a}{b+a}\right)^{4k-2}\leq\left(\frac{b-a}{b+a}\right)^{2k},$$

*with equality only for $k=1$, we get from (27) and (8) that*

$$Q<S \tag{30}$$

This is better then the left side of Theorem 2 of [5]. In a recent paper [4] it is shown also that

$$S<sqrt(2)Q \tag{31}$$

where the constants 1 and $sqrt(2)$ in (28) and (29) are best possible. For another method of proof of (30), see [10]

**Remark 4.** In paper [7] the first author proved identity (1) and the following identity:

$$\log\frac{I^2(\sqrt{a},\sqrt{a})}{I(a,b)}=\frac{G-L}{L}, \tag{32}$$

where $G=G(a,b)$, etc. Letting $a\to a^2, b\to b^2$ in (32) and remarking that

$$L(a^2,b^2)=L(a,b)\cdot A(a,b),$$

we obtain the identity

$$\log\frac{S}{I}=1-\frac{G^2}{A\cdot L}=1-\frac{H}{L}. \tag{33}$$

**Theorem 4.** *One has*

$$\frac{A(a,b)}{G(a,b)} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \frac{\log a - \log b}{2} \right)^{2k} \tag{34}$$

*and identity (11) holds also true.*

**Proof.** Apply (16) to $f(x) = e^x$. One gets the identity

$$\frac{e^a + e^b}{2e^{(a+b)/2}} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \frac{b-a}{2} \right)^{2k}. \tag{35}$$

As here $a, b$ are arbitrary real numbers (not necessarily positive), we may let $a = \log x, b = \log y$; with $x, y > 0$. We get from (35) identity (34) for $A(x,y)$ in place of $A(a,b)$, etc.

Applying (14) for the same function $e^x$, we get

$$\frac{e^b - e^a}{(b-a)e^A} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \frac{b-a}{2} \right)^{2k}, \tag{36}$$

where $a \neq b$ are real numbers. Selecting $a = \log x, b = \log y$, identity (11) follows. $\square$

**Corollary 3.** *One has*

$$\frac{3L - (2G + A)}{3G} = -\sum_{k=1}^{\infty} \frac{1}{(2k)!} \left( \frac{1}{3} - \frac{1}{2k+1} \right) \left( \frac{\log a - \log b}{2} \right)^{2k}. \tag{37}$$

**Proof.** This follows at once from the representations (11) and (34), by remarking that the left side of (37) may be written as

$$\frac{L}{G} - \frac{1}{3} \cdot \frac{A}{G} - \frac{2}{3}. \qquad \square$$

**Remark 5.** As $\dfrac{1}{3} - \dfrac{1}{2k+1} \geq 0$ for $k \geq 1$, relation (37) implies the famous inequality of Pólya-Szegö ([18]) and Carlson ([1]):

$$L < \frac{2G + A}{3} \tag{38}$$

In fact, identity (37) shows the true meaning of inequality (38).

**Corollary 4.** *If $c \geq d$ and $ad - bc > 0$, then*

$$\frac{L(a,b)}{L(c,d)} > \frac{G(a,b)}{G(c,d)}. \tag{39}$$

**Proof.** First remark that for $c = d$, inequality (39) becomes

$$L(a,b) > G(a,b),$$

which is well-known. Assume now that $c > d$ and $ad > bc$. Then, as

$$\log a - \log b = \log \frac{a}{b}, \quad \log c - \log d = \log \frac{c}{d},$$

and

$$\log \frac{a}{b} > \log \frac{c}{d}$$

by $\frac{a}{b} > \frac{c}{d} > 1$, by the identity (11) follows inequality (39). □

**Remark 6.** Inequality (39) is proved in [6] under the assumption

$$a \geq b \geq c \geq d > 0 \quad \text{and} \quad ad - bc > 0 \tag{$*$}$$

Clearly, if $a = b > 0$, then $d > c$, contradicting $c \geq d$. Thus in $(*)$ one must have $a > b$. If $c = d$, then $L(c,d) = G(c,d)$ so (39) becomes trivially $L(a,b) > G(a,b)$. Also, inequality $b \geq c$ is not necessary. For example, $\frac{a}{b} > \frac{c}{d} > 1$ holds with $b < c$ in $\frac{7}{2} > \frac{3}{1} > 1$.

**Theorem 5.** *One has*

$$\log \frac{\sqrt{2A^2 + G^2}}{G\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \left(1 - \frac{1}{3^k}\right) \left(\frac{b-a}{b+a}\right)^{2k}, \tag{40}$$

*and identity (10) also holds true.*

**Proof.** Putting $\left(\frac{b-a}{b+a}\right)^2 = u$ in relation (6), by $1 - u = \frac{G^2}{A^2}$ we get

$$\log(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}, \tag{41}$$

where $0 < u < 1$. This is in fact known series expansion of $\log(1-u)$. In order to prove (40), remark that

$$\log\left[\frac{2}{3}\left(\frac{A}{G}\right)^2 + \frac{1}{3}\right] = \log\left(\frac{2}{3}\cdot\frac{1}{1-z^2} + \frac{1}{3}\right),$$

where $z = \dfrac{b-a}{b+a}$.

Now,

$$\begin{aligned}\log\left(\frac{2}{3}\cdot\frac{1}{1-z^2} + \frac{1}{3}\right) &= \log(3-z^2) - \log(1-z^2) - \log 3\\ &= \log\left(1-\left(\frac{z}{\sqrt{3}}\right)\right) - \log(1-z^2)\\ &= \sum_{k=1}^{\infty}\frac{z^{2k}}{k} - \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{\sqrt{3}}\right)^{2k}\\ &= \sum_{k=1}^{\infty}\frac{z^{2k}}{k}\left(1-\frac{1}{3^k}\right),\end{aligned}$$

where we have applied two times relation (41). This proves identity (40). □

Relation (10) follows as a combination of (40) and (7).

**Corollary 5.**

$$\log\frac{\sqrt{2A^2+G^2}}{I\sqrt{3}} > \frac{1}{45}\left(\frac{b-a}{b+a}\right)^4 > 0. \tag{42}$$

**Remark 7.** The weaker inequality of (42) implies a result of [11], namely:

$$3I^2 < 2A^2 + G^2. \tag{43}$$

**Theorem 6.** *Relations (12) and (13) hold true.*

**Proof.** Let $f(t) = \left(\dfrac{1}{t(a+b-t)}\right)^{1/2}$ in (14). After certain elementary integration we get that $\dfrac{1}{b-a}\displaystyle\int_a^b f(t)\mathrm{d}t = P(a,b)$ for $a \neq b$.

As

$$2f'(t) = f(t)\left(\frac{1}{2A-t} - \frac{1}{t}\right),$$

where $A = \dfrac{a+b}{2}$, by induction it can be proved that

$$f^{(2k)}(A) = \frac{\binom{2k}{k}(2k)!}{4^k} \cdot \frac{1}{A^{2k+1}}.$$

Thus from identity (14) we can deduce relation (13). Relation (12) follows by (16) applied for the same function $f(t)$. □

**Corollary 6.**

$$\frac{1}{P} < \frac{1}{3G} + \frac{2}{3A} \tag{44}$$

**Proof.** By (12) and (13) we get

$$\frac{A}{P} < 1 + \frac{1}{3}\left(\frac{A}{G} - 1\right),$$

as for $k \geq 1$ one has $2k+1 \geq 3$ and for $k = 0$ the respective terms of the sums are 1. Thus (44) follows. □

**Remark 8.** By using other methods, many similar inequalities are proved in [13].

**Theorem 7.** *One has*

$$\log \frac{2A+G}{G} = \sum_{k=0}^{\infty} \frac{2}{2k+1}\left(\frac{1}{1+\sqrt{1-z^2}}\right)^{2k+1}, \tag{45}$$

$$\log \frac{2A+G}{3G} = \sum_{k=0}^{\infty} \frac{2}{2k+1}\left(\frac{1-\sqrt{1-z^2}}{1+2\sqrt{1-z^2}}\right)^{2k+1}, \tag{46}$$

$$\log \frac{2G+A}{3G} = \sum_{k=0}^{\infty} \frac{2}{2k+1}\left(\frac{1-\sqrt{1-z^2}}{1+5\sqrt{1-z^2}}\right)^{2k+1}, \tag{47}$$

*where* $z = \dfrac{b-a}{b+a}$ *and* $A = A(a,b)$, *etc.*

**Proof.** Since $\left(\dfrac{A}{G}\right)^2 = \dfrac{(a+b)^2}{4ab} = \dfrac{1}{1-z^2}$, we get $\dfrac{A}{G} = \dfrac{1}{\sqrt{1-z^2}}$. Thus,

$$\frac{2A+G}{G} = 2 \cdot \frac{A}{G} + 1 = 2 \cdot \frac{1}{\sqrt{1-z^2}} + 1 = \frac{2+\sqrt{1-z^2}}{\sqrt{1-z^2}}.$$

Now,

$$\log\left(\frac{2A+G}{G}\right) = \log\left(2+\sqrt{1-z^2}\right) - \log\left(\sqrt{1-z^2}\right).$$

Applying identity (21), i.e.

$$\frac{a+b}{2} \cdot \frac{\log b - \log a}{b-a} = 1 + \sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{b-a}{b+a}\right)^{2k} \qquad (*)$$

to $b = 2+\sqrt{1-z^2}$, $a = \sqrt{1-z^2}$, we get:

$$\frac{1+\sqrt{1-z^2}}{2}\left[\log(2+\sqrt{1-z^2}) - \log\sqrt{1-z^2}\right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{1}{1+\sqrt{1-z^2}}\right)^{2k},$$

so

$$\frac{1+\sqrt{1-z^2}}{2} \log\frac{2A+G}{G} = 1 + \sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{1}{1+\sqrt{1-z^2}}\right)^{2k},$$

which implies relation (45).

Similarly, one has

$$\frac{2A+G}{3G} = \frac{2}{3} \cdot \frac{A}{G} + \frac{1}{3} = \frac{2+\sqrt{1-z^2}}{3\sqrt{1-z^2}},$$

so applying $(*)$ for $b = 2+\sqrt{1-z^2}$, $a = 3\sqrt{1-z^2}$, we get, after some simple computations, relation (46).

Applying $(*)$ to $b = 2\sqrt{1-z^2}+1$, $a = 3\sqrt{1-z^2}$, and remarking that

$$\frac{2G+A}{3G} = \frac{2\sqrt{1-z^2}+1}{3\sqrt{1-z^2}},$$

we can deduce relation (47). $\square$

**Remark 9.** A similar relation is the following:

$$\log \frac{2A+G}{3A} = -\sum_{k=0}^{\infty} \frac{2}{2k+1} \left( \frac{1-\sqrt{1-z^2}}{5+\sqrt{1-z^2}} \right)^{2k+1}. \tag{48}$$

**Remark 10.** Applying identity (28) for $z = \dfrac{(a-b)^2}{3a^2+2ab+3b^2}$, since

$$\frac{1+z}{1-z} = \frac{Q^2}{A^2},$$

we get the identity

$$\log \frac{Q}{A} = -\sum_{k=1}^{\infty} \frac{1}{2k-1} \left( \frac{b-a}{\sqrt{3a^2+2ab+3b^2}} \right)^{4k-2}. \tag{49}$$

**Corollary 7.**

$$\log \frac{Q}{A} < \sum_{k=1}^{\infty} \frac{1}{(2k-1)2^{2k-1}} \left( \frac{b-a}{b+a} \right)^{4k-2}.$$

**Proof.** Apply (49) and the inequality

$$\sqrt{3a^2+2ab+3b^2} > \sqrt{2}(a+b). \qquad \square$$

**Theorem 8.** *One has*

$$\log \frac{I}{G} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(b-a)^k}{k(k+1)} \left[ \frac{(-1)^{k-1}}{a^k} - \frac{1}{b^k} \right], \tag{50}$$

$$\frac{1}{L} - \frac{1}{H} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(b-a)^k}{k+1} \left[ \frac{(-1)^k}{a^{k+1}} + \frac{1}{b^{k+1}} \right], \tag{51}$$

$$L = A + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(\log a - \log b)^k}{(k+1)!}\left[b + (-1)^k a\right]. \tag{52}$$

**Proof.** Let $f(x) = \log x$, $f(x) = \frac{1}{x}$, $f(x) = e^x$ respectively in (19) of Theorem B. For example, for $f(x) = e^x$, we get

$$\frac{e^b - e^a}{b - a} = \frac{e^a + e^b}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(b-a)^k}{(k+1)!}\left[e^a + (-1)^k e^b\right].$$

Then replace $a \to \log a$, $b \to \log b$ in order to deduce identity (52). $\square$

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## 3.11 On some exponential means, II

### 1. Introduction

All the means that appear in this paper are functions $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the property that

$$\min(a, b) \le M(a, b) \le \max(a, b),\ \forall\ a, b > 0.$$

Of course $M(a, a) = a, \forall a > 0$. As usual $A, G, L, I, A_p$ denote the arithmetic, geometric, logarithmic, identric, respectively power means of two positive numbers, defined by

$$A = A(a, b) = \frac{a+b}{2}, \quad G = G(a, b) = \sqrt{ab},$$

$$L = L(a, b) = \frac{b-a}{\log b - \log a}, I = I(a, b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$A_p = A_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p},\ p \neq 0.$$

In [16], the first part of this paper, we have studied the exponential mean

$$E = E(a, b) = \frac{be^b - ae^a}{e^b - e^a} - 1$$

introduced in [23]. Another exponential mean was defined in [19] by

$$\overline{E} = \overline{E}(a, b) = \frac{ae^b - be^a}{e^b - e^a} + 1.$$

It is the complementary of $E$, according to a definition from [4], i.e.

$$\overline{E} = 2A - E. \tag{1}$$

A basic inequality proved in [23] is

$$E > A, \tag{2}$$

which gives the new inequality

$$\overline{E} < A.$$

More general means have been studied in [14], [17] and [19]. For example, letting $f(x) = e^x$ in formula (5) from [14], we recapture (2). We note that by selecting $f(x) = \log x$ in the formula (8) from [14], and then $f(x) = 1/x$, we get the standard inequalities

$$G < L < I < A \tag{3}$$

(for history see for example [7]).

In what follows, for any mean $M$ we will denote by $\mathcal{M}$ the new mean given by

$$\mathcal{M}(x, y) = \log M(e^x, e^y),\ \ x, y > 0.$$

As we put $a = e^x, b = e^y$ and then take logarithms, we call this procedure the exp-log method. The method will be applied also to some inequalities for deriving new inequalities. For example, in [16] we proved that

$$E = \mathcal{I},$$

and so (3) becomes

$$A < \mathcal{L} < E < \mathcal{A}. \tag{4}$$

In [16] was also shown that

$$A + \mathcal{A} - \mathcal{L} < E < 2\mathcal{L} - A,$$

and

$$\mathcal{A}_{2/3} < E < \mathcal{A}_{\log 2}$$

(see also [6] and [22]). In [9], the first author improved the inequality (2) by

$$E > \frac{A + 2\mathcal{A}}{3} > A.$$

This is based on the following identity proved there

$$(E-A)(a,b)=\frac{A(e^a,e^b)}{L(e^a,e^b)}-1. \tag{5}$$

We get the same result using the known result

$$I>\frac{2A+G}{3}>\left(A^2G\right)^{1/3}$$

and the exp-log method.

The aim of this paper is to obtain other inequalities related to the above means.

## 2. Main results

1. After some computations, the inequality (2) becomes

$$\frac{e^b-e^a}{b-a}<\frac{e^a+e^b}{2}. \tag{6}$$

This follows at once from the Hadamard inequality

$$\frac{1}{b-a}\int_a^b f(t)dt<\frac{f(a)+f(b)}{2},$$

applied to the strictly convex function $f(t)=e^t$. We note that by the second Hadamard inequality, namely

$$\frac{1}{b-a}\int_a^b f(t)dt>f\left(\frac{a+b}{2}\right),$$

for the same function, one obtains

$$\frac{e^b-e^a}{b-a}>e^{\frac{a+b}{2}}, \tag{7}$$

which has been proposed as a problem in [3].

The relation (4) improves the inequality (6), which means $\mathcal{A} > \mathcal{L}$, and (7), which means $\mathcal{L} > A$. In fact, by the above remarks one can say that

$$E > A \iff \mathcal{A} > \mathcal{L}. \tag{8}$$

2. In [23] was proven that $E$ is not comparable with $A_\lambda$ for $\lambda > 5/3$. Then in [17] we have shown, among others, that

$$A(a,b) < E(a,b) < A(a,b) \cdot e^{|b-a|/2}.$$

Now, if $|b-a|$ becomes small, clearly $e^{|b-a|/2}$ approaches to 1, i.e. the conjecture $E > A_\lambda$ of [23] cannot be true for any $1 < \lambda \leq 5/3$.

We get another double inequality from (1) and (2)

$$A < E < 2A.$$

These inequalities cannot be improved. Indeed, for $1 < \lambda < 2$, we have

$$\lim_{x\to\infty} [E(1,x) - \lambda A(1,x)] = \infty,$$

but

$$E(1,1) - \lambda A(1,1) = 1 - \lambda < 0,$$

thus $E$ is not comparable with $\lambda A$.

On the other hand,

$$\overline{E}(a,b) = \frac{e^b(a+1) - e^a(b+1)}{e^b - e^a} = (a+1)(b+1) \cdot \frac{f(b) - f(a)}{e^b - e^a},$$

where $f(x) = e^x/(x+1)$. By Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{e^b - e^a} = \frac{f'(c)}{e^c}, c \in (a,b).$$

Since

$$\frac{f'(c)}{e^c} = \frac{c}{(c+1)^2} \leq \frac{1}{4},$$

we get

$$0 < 2A - E \le \frac{(a+1)(b+1)}{4}.$$

3. By using the series representation

$$\log \frac{I}{G} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left( \frac{b-a}{b+a} \right)^{2k},$$

(see [9] and [21]), we can deduce the following series representation

$$(E - A)(a,b) = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left( \frac{e^b - e^a}{e^b + e^a} \right)^{2k}. \tag{9}$$

By (6), $\dfrac{\left|e^b - e^a\right|}{e^b + e^a} < \dfrac{|b-a|}{2}$, thus we get the estimate

$$(E - A)(a,b) < \sum_{k=1}^{\infty} \frac{1}{2k+1} \left( \frac{b-a}{2} \right)^{2k}.$$

The series is convergent at least for $|b-a| < 2$. Writing

$$\frac{A(e^a, e^b)}{L(e^a, e^b)} = e^{\mathcal{A}(a,b) - \mathcal{L}(a,b)},$$

the identity (5) implies the relation

$$E - A = e^{\mathcal{A} - \mathcal{L}} - 1. \tag{10}$$

This gives again the equivalence (8). But one can obtain also a stronger relation by writing $e^x > 1 + x + x^2/2$, for $x > 0$. Thus (10) gives

$$E - A > \mathcal{A} - \mathcal{L} + \frac{1}{2} (\mathcal{A} - \mathcal{L})^2.$$

4. Consider the inequality proved in [10]

$$\frac{2}{e} A < I < A.$$

By the exp-log method, we deduce

$$\log 2 - 1 + \mathcal{A} < E < \mathcal{A}. \tag{11}$$

From the inequality

$$I < \frac{2}{e}(A+G) = \frac{4}{e}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$

given in [5], we have, by the same method,

$$E(x,y) < 2\log 2 - 1 + 2\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right). \tag{12}$$

Relation (12) may be compared with the left side of (11). Take now the relation

$$L < L(A,G) = \frac{A-G}{\log \frac{A}{G}}$$

from [5]. Since

$$A - G = \frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2,$$

one obtains

$$\mathcal{A} - A < \frac{1}{2e^{\mathcal{L}}}\left(e^{x/2} - e^{y/2}\right)^2.$$

The relation

$$L^3 > \left(\frac{A+G}{2}\right)^2 G,$$

from [13], gives similarly

$$3\mathcal{L}(x,y) > A(x,y) + 4\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right),$$

while the inequality

$$\log \frac{I}{L} > 1 - \frac{G}{L},$$

from [7], offers the relation

$$E - \mathcal{L} > 1 - e^{A-\mathcal{L}}.$$

5. The exp-log method applied to the inequality

$$L > \sqrt{GI},$$

given in [2] and [11], implies

$$\mathcal{L} > \frac{A+E}{2} > \frac{2A+\mathcal{A}}{3}.$$

On the other side, the inequality

$$I > \sqrt{AL}$$

proven in [11], gives on the same way the inequality

$$E > \frac{\mathcal{A}+\mathcal{L}}{2}. \tag{13}$$

After all we have the double inequality

$$\frac{\mathcal{A}+\mathcal{L}}{2} < E < 2\mathcal{L} - A.$$

6. Consider now the inequality

$$3I^2 < 2A^2 + G^2,$$

from [20]. It gives

$$\log 3 + 2E < \log\left(e^{2A} + 2e^{2\mathcal{A}}\right).$$

Similarly

$$I > \frac{2A+G}{3},$$

given in [8], implies

$$\log 3 + E > \log\left(2e^{A} + e^{\mathcal{A}}\right). \tag{14}$$

In fact, the relation

$$I > \frac{A+L}{2},$$

from [7] gives

$$\log 2 + E > \log\left(e^{\mathcal{L}} + e^{\mathcal{A}}\right), \tag{15}$$

but this is weaker than (14), as follows from [8]. The inequalities (13) and (15) can be combined as

$$E > \log\left(\frac{e^{\mathcal{L}} + e^{\mathcal{A}}}{2}\right) > \frac{\mathcal{L} + \mathcal{A}}{2},$$

where the second inequality is a consequence of the concavity of the logarithmic function. We notice also that, by

$$L + I < A + G,$$

given in [1], one can write

$$e^{\mathcal{L}} + e^{E} < e^{\mathcal{A}} + e^{A}.$$

7. In [9] was proved the inequality

$$I\left(a^2, b^2\right) < \frac{A^4(a,b)}{I^2(a,b)}.$$

By the exp-log method, we get

$$E(2x, 2y) < 4\mathcal{A}(x,y) - 2E(x,y). \tag{16}$$

It is interesting to note that by the equality

$$\log \frac{I^2\left(\sqrt{a}, \sqrt{b}\right)}{I(a,b)} = \frac{G(a,b)}{L(a,b)} - 1,$$

given in [7], we have the identity

$$2E\left(\frac{x}{2}, \frac{y}{2}\right) - E(x,y) = e^{A(x,y) - \mathcal{L}(x,y)} - 1. \tag{17}$$

Putting $x \to \dfrac{x}{2}$, $y \to \dfrac{y}{2}$ in (16), and taking into account (17), we can write

$$2E(x,y) + e^{A(x,y) - \mathcal{L}(x,y)} - 1 < 4\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right).$$

This may be compared to (12).

8. We consider now applications of the special Gini mean

$$S = S(a,b) = \left(a^a b^b\right)^{1/(a+b)}$$

(see [15]). Its attached mean (by the exp-log method)

$$\mathcal{S}(x,y) = \frac{xe^x + ye^y}{e^x + e^y} = \log S(e^x, e^y),$$

is a special case of

$$M_f(x,y) = \frac{xf(x) + yf(y)}{f(x) + f(y)}$$

which was defined in [18]. Using the inequality

$$\left(\frac{S}{A}\right)^2 < \left(\frac{I}{G}\right)^3$$

from [15], we get

$$2\mathcal{S} - 2\mathcal{A} < 3E - 3A.$$

The inequalities

$$\frac{A^2}{I} < S < \frac{A^4}{I^3} < \frac{A^2}{G}$$

given in [15] imply

$$2\mathcal{A} - E < \mathcal{S} < 4\mathcal{A} - 3E < 2\mathcal{A} - A.$$

These offer connections between the exponential means $E$ and $\mathcal{S}$.

Let now the mean

$$U = U(a,b) = \frac{1}{3}\sqrt{(2a+b)\,(a+2b)}.$$

In [12] it is proved that

$$G < \sqrt[4]{U^3 G} < I < \frac{U^2}{A} < U < A.$$

By the exp-log method, we get

$$A < \frac{1}{4}\,(3\mathcal{U} + A) < E < 2\mathcal{U} - \mathcal{A} < \mathcal{U} < \mathcal{A}.$$

These relations offer a connection between the means $E$ and $\mathcal{U}$.

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## 3.12 On global bounds for generalized Jensen's inequality

### 1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be a convex function, and $x_i \in [a,b]$ for $i = 1,2,\ldots,n$. Let $\underline{p} = \{p_i\}$, $\sum_{i=1}^{n} p_i = 1$, $p_i > 0$ $(i = \overline{1,n})$ be a sequence of positive weights. Put $\underline{x} = \{x_i\}$. Then the Jensen functional $J_f(\underline{p}, \underline{x})$ is defined by

$$J_f(\underline{p}, \underline{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

In a recent paper [7] the following global bounds have been proved:

**Theorem 1.** *Let $f, \underline{p}, \underline{x}$ be defined as above, and let $p, q \geq 0$, $p+q = 1$. Then*

$$0 \leq J_f(\underline{p}, \underline{x}) \leq \max_{p}[pf(a) + qf(b) - f(pa + qb)]. \tag{1}$$

The left side of (1) is the classical Jensen inequality. Both bounds of $J_f(\underline{p}, \underline{x})$ in (1) are global, as they depend only on $f$ and the interval $[a,b]$.

As it is shown in [7], the upper bound in relation (1) refines many earlier results, and in fact it is the best possible bound. In what follows, we will show that, this result has been discovered essentially by the present author in 1991 [4], and in fact this is true in a general framework for positive linear functionals defined on the space of all continuous functions defined on $[a,b]$.

In paper [4], as a particular case of a more general result, the following is proved:

**Theorem 2.** *Let $f, \underline{p}, \underline{x}$ as above. Then one has the double inequality:*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$$

$$\leq \left(\sum_{i=1}^{n} p_i x_i\right)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{2}$$

The right side of (2) follows from the fact that the graph of $f$ is below the graph of line passing through the points $(a, f(a))$, $(b, f(b))$:

$$f(x) \leq (x-a)\frac{f(b)}{b-a} + (b-x)\frac{f(a)}{b-a}.$$

By letting $x = x_i$, and multiplying both sides with $p_i$, after summation we get the right side of (2) (the left side is Jensen's inequality).

Now, remark that the right side of (2) can be written also as

$$f(a)\left[\frac{b-\sum_{i=1}^{n} p_i x_i}{b-a}\right] + f(b)\left[\frac{\sum_{i=1}^{n} p_i x_i - a}{b-a}\right].$$

Therefore, by denoting

$$\frac{b-\sum_{i=1}^{n} p_i x_i}{b-a} = p \quad \text{and} \quad \frac{\sum_{i=1}^{n} p_i x_i - a}{b-a} = q,$$

we get $p \geq 0$, $p+q=1$ and $\sum_{i=1}^{n} p_i x_i = pa + qb$. Thus, from (2) we get

$$0 \leq J_f(\underline{p}, \underline{x}) \leq pf(a) + qf(b) - f(pa+qb) \tag{3}$$

and this immediately gives Theorem 1.

## 2. An extension

Let $C[a,b]$ denote the space of all continuous functions defined on $[a,b]$, and let $L : C[a,b] \to \mathbb{R}$ be a linear and positive functional defined on $C[a,b]$ i.e. satisfying

$$L(f_1+f_2) = L(f_1) + L(f_2), \quad L(\lambda f) = \lambda L(f) \quad (\lambda \in \mathbb{R})$$

and $L(f) \geq 0$ for $f \geq 0$.

Define $e_k(x) = x^k$ for $x \in [a, b]$ and $k = 0, 1, 2, \ldots$.

The following result has been discovered independently by A. Lupaş [2] and J. Sándor [4]:

**Theorem 3.** *Let $f$ be convex and $L$, $e_k$ as above and suppose that $L(e_0) = 1$. Then we have the double inequality*

$$f(L(e_1)) \leq L(f) \leq L(e_1)\left[\frac{f(b) - f(a)}{b - a}\right] + \frac{bf(a) - af(b)}{b - a}. \tag{4}$$

We note that the proof of (4) is based on basic properties of convex functions (e.g. $f \in C[a, b]$). Particularly, the right side follows on similar lines as shown for the right side of (2).

Define now the generalized Jensen functional as follows:

$$J_f(L) = L(f) - f(L(e_1)).$$

Then the following extension of Theorem 1 holds true:

**Theorem 4.** *Let $f$, $L$, $p$, $q$ be as above. Then*

$$0 \leq J_f(L) \leq \max_p [pf(a) + qf(b) - f(pa + qb)] = T_f(a, b). \tag{5}$$

**Proof.** This is similar to the method shown in the case of Theorem 2. Remark that the right side of (4) can be rewritten as

$$f(a)p + f(b)q,$$

where

$$p = \frac{b - L(e_1)}{b - a} \quad \text{and} \quad q = \frac{L(e_1) - a}{b - a}.$$

As $e_1(x) = x$ and $a \leq x \leq b$, we get $a \leq L(e_1) \leq b$, the functional $L$ being a positive one. Thus $p \geq 0$, $q \geq 0$ and $p + q = 1$. Moreover, $L(e_1) = pa + qb$; so relation (5) is an immediate consequence of (4).

By letting

$$L(f) = \sum_{i=1}^{n} p_i f(x_i),$$

which is a linear and positive functional, we get $J_f(L) = J_f(\underline{p}, \underline{x})$, so Theorem 1 is reobtained.

Let now $k : [a, b] \to \mathbb{R}$ be a strictly positive, integrable function, and $g : [a, b] \to [a, b]$ such that $f[g(x)]$ is integrable on $[a, b]$. Define

$$L_g(f) = \frac{\displaystyle\int_a^b k(x) f[g(x)] dx}{\displaystyle\int_a^b k(x) dx}.$$

It is immediate that $L_g$ is a positive linear functional, with $L_g(e_0) = 1$. Since

$$L(e_1) = \frac{\displaystyle\int_a^b k(x) g(x) dx}{\displaystyle\int_a^b k(x) dx},$$

by denoting

$$J_f(k, g) = \frac{\displaystyle\int_a^b k(x) f[g(x)] dx}{\displaystyle\int_a^b k(x) dx} - f\left(\frac{\displaystyle\int_a^b k(x) g(x) dx}{\displaystyle\int_a^b k(x) dx}\right),$$

we can deduce from Theorem 4 a corollary. Moreover, as in the discrete case, the obtained bound is best possible:

**Theorem 5.** *Let $f, k, g$ as above, and let $p, q \geq 0$, $p + q = 1$. Then*

$$0 \leq J_f(k, g) \leq T_f(a, b). \tag{6}$$

*The upper bound in (6) is best possible.*

**Proof.** Relation (6) is a particular case of (5) applied to $L_g$ and $J_f(k, g)$ above.

In order to prove that the upper bound in (6) is best possible, let $p_0 \in [0, 1]$ be the point at which the maximum $T_f(a, b)$ is attained (see [7]). Let $c \in [a, b]$ be defined as follows:

$$\int_a^c k(x) dx = p_0 \int_a^b k(x) dx. \tag{7}$$

If $p_0 = 0$ then put $c = a$; while for $p_0 = 1$, put $c = b$. When $p_0 \in (0, 1)$ remark that the application

$$h(t) = \int_a^t k(x)dx - p_0 \int_a^b k(x)dx$$

has the property $h(a) < 0$ and $h(b) > 0$; so there exist $t_0 = c \in (a, b)$ such that $h(c) = 0$, i.e. (7) is proved.

Now, select $g(x)$ as follows:

$$g(x) = \begin{cases} a, & \text{if} \quad a \leq x \leq c \\ b, & \text{if} \quad c \leq x \leq b. \end{cases}$$

Then

$$\int_a^b k(x)g(x)dx / \int_a^b k(x)dx = a \int_a^c k(x)dx / \int_a^b k(x)dx$$

$$+b \int_a^b k(x)dx / \int_a^b k(x)dx = ap_0 + bq_0,$$

where $q_0 = 1 - p_0$.

On the other hand,

$$\int_a^b k(x)f[g(x)]dx / \int_a^b k(x)dx = f(a) \int_a^c k(x)dx / \int_a^b k(x)dx$$

$$+f(b) \int_c^b k(x)dx / \int_a^b k(x)dx = p_0 f(a) + q_0 f(b).$$

This means that

$$J_f(k, g) = p_0 f(a) + q_0 f(b) - f(ap_0 + bq_0) = T_f(a, b).$$

Therefore, the equality is attained at the right side of (6), which means that this bound is best possible.

## 3. Applications

a) The left side of (6) is the generalized form of the famous Jensen integral inequality

$$f\left(\frac{\displaystyle\int_a^b k(x)g(x)dx}{\displaystyle\int_a^b k(x)dx}\right) \le \frac{\displaystyle\int_a^b k(x)f[g(x)]dx}{\displaystyle\int_a^b k(x)dx}, \tag{8}$$

with many application in various fields of Mathematics.

For $f(x) = -\ln x$, this has a more familiar form.

Now, the right side of (4) applied to $L = L_g$ gives the inequality

$$\frac{\displaystyle\int_a^b k(x)f[g(x)]dx}{\displaystyle\int_a^b k(x)dx} \le \frac{b-u}{b-a}f(a) + \frac{u-a}{b-a}f(b), \tag{9}$$

where

$$u = L(e_1) = \frac{\displaystyle\int_a^b k(x)g(x)dx}{\displaystyle\int_a^b k(x)dx}.$$

Inequalities (8) and (9) offer an extension of the famous Hadamard inequalities (or Jensen-Hadamard, or Hermite-Hadamard inequalities) (see e.g. [1], [3], [4])

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{10}$$

Applying (8) and (9) for $g(x) = x$, we get from (8) and (9):

$$f(v) \le \frac{\displaystyle\int_a^b k(x)f(x)dx}{\displaystyle\int_a^b k(x)dx} \le \frac{(b-v)f(a)+(v-a)f(b)}{b-a}, \tag{11}$$

where

$$v = \frac{\int_a^b xk(x)dx}{\int_a^b k(x)dx}.$$

When $k(x) \equiv 1$, inequality (11) reduces to (10).

b) Let $a, b > 0$ and

$$G = G(a,b) = \sqrt{ab};$$

$$L = L(a,b) = \frac{b-a}{\ln b - \ln a} \quad (a \neq b), \quad L(a,a) = a,$$

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad (a \neq b), \quad I(a,a) = a$$

be the well-known geometric, logarithmic and identric means.

In our paper [5] the following generalized means have been introduced (assume $a \neq b$):

$$\ln I_k(a,b) = \int_a^b k(x)\ln x dx \Big/ \int_a^b k(x)dx,$$

$$A_k(a,b) = \int_a^b xk(x)dx \Big/ \int_a^b k(x)dx,$$

$$L_k(a,b) = \int_a^b k(x)dx \Big/ \int_a^b k(x)/x dx,$$

$$G_k^2(a,b) = \int_a^b k(x)dx \Big/ \int_a^b k(x)/x^2 dx.$$

Clearly, $I_1 \equiv I$, $A_1 \equiv A$, $L_1 \equiv L$, $G_1 \equiv G$.

Applying inequality (6) for $f(x) = -\ln x$, and using the fact that in this case $T_f(a,b) = \ln \dfrac{L \cdot I}{G^2}$ (see [7]), we get the inequalities

$$0 \leq \ln\left(\frac{\int_a^b k(x)g(x)dx}{\int_a^b k(x)dx}\right) - \frac{\int_a^b k(x)\ln g(x)dx}{\int_a^b k(x)dx} \leq \ln\frac{L \cdot I}{G^2}. \tag{12}$$

For $g(x) = x$, with the above notations, we get

$$1 \leq \frac{A_k}{I_k} \leq \frac{L \cdot I}{G^2}. \tag{13}$$

Applying the right side of inequality (11) for the same function

$$f(x) = -\ln x$$

we get

$$\frac{A_k}{L} \leq 1 + \ln\left(\frac{I \cdot I_k}{G^2}\right), \tag{14}$$

where we have used the remark that

$$\ln(e \cdot I) = \frac{b\ln b - a\ln a}{b-a} \quad \text{and} \quad \ln G^2 - \ln(e \cdot I) = \frac{b\ln a - a\ln b}{b-a}.$$

Note that the more complicated inequality (14) is a slightly stronger than the right side of (13), as by the classical inequality $\ln x \leq x - 1$ $(x > 0)$ one has

$$\ln\left(\frac{I \cdot I_k}{G^2}\right) + 1 \leq \frac{I \cdot I_k}{G^2},$$

so

$$\frac{A_k}{L} \leq 1 + \ln\left(\frac{I \cdot I_k}{G^2}\right) \leq \frac{I \cdot I_k}{G^2}.$$

These inequalities seem to be new even in the case $k(x) \equiv 1$. For $k(x) = e^x$ one obtains the exponential mean $A_{e^x} = E$, where

$$E(a,b) = \frac{be^b - ae^a - 1}{b-a}.$$

The mean $I_{e^x}$ has been called as the "identric exponential mean" in [6], where other inequalities for these means have been obtained.

c) Applying inequality (6) for $g(x) = \ln x$, $f(x) = e^x$, we get

$$0 \leq A_k - I_k \leq \frac{e^b - e^a}{b-a}\ln\left(\frac{e^b - e^a}{b-a}\right) + \frac{be^a - ae^b}{b-a} - \frac{e^b - e^a}{b-a}, \tag{15}$$

where the right hand side is $T_f(a,b)$ for $f(x) = e^x$. This may be rewritten also as

$$0 \le A_k(a,b) - I_k(a,b) \le 2[A(x,y) - L(x,y)] - L(x,y) \ln \frac{I(x,y)}{L(x,y)}, \quad (15')$$

where $e^a = x$, $e^b = y$.

As in [5] it is proved that $\ln \dfrac{I}{L} \ge \dfrac{L-G}{L}$, the right side of $(15')$ implies

$$0 \le A_k(a,b) - I_k(a,b) \le 2A(x,y) + G(x,y) - 3L(x,y). \qquad (15'')$$

d) Finally, applying (11) for $f(x) = x \ln x$ and $k(x)$ replaced with $k(x)/x$, we can deduce

$$\ln L_k \le \ln I_k \le 1 + \ln I - \frac{G^2}{L \cdot L_k}, \qquad (16)$$

where the identity

$$\frac{b \ln b - a \ln a}{b-a} = \ln I + 1$$

has been used. We note that for $k(x) \equiv 1$, inequality (16) offers a new proof of the classical relations

$$G \le L \le I.$$

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# Chapter 4

# Means and their Ky Fan type inequalities

"In some sense all insights come suddenly, usually
in some impure form which is clarified later."

(G. Faltings)

"The elegance of a mathematical theorem is directly proportional to the number of independent ideas one can see in the theorem and inversely proportional to the effort it takes to see them."

(G. Pólya)

## 4.1 On an inequality of Ky Fan

### 1

In the famous book [3] one can find the following "unpublished result due to Ky Fan":

If $x_i \in \left(0, \dfrac{1}{2}\right]$, $(i = 1, 2, \ldots, n)$, then

$$\left[\frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n}(1 - x_i)}\right]^{1/n} \le \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1 - x_i)} \tag{1}$$

with equality only if $x_1 = x_2 = \ldots = x_n$.

This inequality can be established by forward and backward induction ([5], [3]) a method used by Cauchy to prove the inequality between the arithmetic and geometric means. In [7] N. Levinson has published the following beautiful generalization of (1):

Let $x_i \in \left(0, \dfrac{1}{2}\right]$, $(i = 1, 2, \ldots, n)$ and suppose that the function $f$ has a nonnegative third derivative on $\left(0, \dfrac{1}{2}\right)$. Then

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f(A_n) \le \frac{1}{n}\sum_{i=1}^{n} f(1 - x_i) - f(A_n') \tag{2}$$

where the notations are introduced below.

For further extensions of Levinson's result, see T. Popoviciu [11] and P.S. Bullen [4]. Recently, H. Alzer [1], by answering a question asked by C.-L. Wang: "Are there more proofs of inequality (1) in addition to the one by Levinson and the original, unpublished one?", has obtained two new proof of Ky Fan's inequality. Our aim is to add one more proof of (1) to be the above list, by showing that (1) is equivalent with an inequality of P. Henrici [6], and to obtain some connected results.

## 2

Let $x_i \in (0, 1)$, $(i = 1, 2, \ldots, n)$. We denote by

$$A_n(x) = A_n, \quad G_n(x) = G_n \quad \text{and} \quad H_n(x) = H_n$$

(resp. $A'_n$, $G'_n$ and $H'_n$) the arithmetic, geometric and harmonic means of $x_1, \ldots, x_n$ (resp. $1-x_1, \ldots, 1-x_n$), i.e.

$$\begin{aligned} A_n &= \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \qquad H_n = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{x_i}}, \\ A'_n &= \frac{1}{n}\sum_{i=1}^{n}(1-x_i), \quad G'_n = \left[\prod_{i=1}^{n}(1-x_i)\right]^{1/n}, \quad H'_n = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{1-x_i}}. \end{aligned} \tag{3}$$

In 1956 P. Henrici [6] proved the following result:

Let $a_i \geq 1$, $(i = 1, 2, \ldots, n)$ and denote

$$P_n = \sum_{i=1}^{n} \frac{1}{1+a_k}, \quad Q_n(a) = \frac{n}{1+\sqrt[n]{a_1 \ldots a_n}}.$$

Then

$$P_n(a) \geq Q_n(a), \text{ with equality only if } a_1 = \ldots = a_n. \tag{4}$$

For $0 \leq a_i \leq 1$, $(i = 1, 2, \ldots, n)$, we have

$$P_n(a) \leq Q_n(a). \tag{5}$$

Now we prove the surprising result that (4) and (1) are equivalent. Indeed, suppose first that $x_i \in \left(0, \frac{1}{2}\right]$ and select

$$a_i = \frac{1-x_i}{x_i} = \frac{1}{x_i} - 1, \ (i = 1, 2, \ldots, n)$$

in (4). Clearly $a_i \geq 1$, so we get:

$$A_n \geq \frac{G_n}{G_n + G'_n} \quad \text{or} \quad A_n G'_n \geq G_n(1 - A_n).$$

Since $A_n + A'_n = 1$ (see (3)) this means that

$$\frac{A_n}{A'_n} \geq \frac{G_n}{G'_n} \tag{6}$$

that is, inequality (1). Conversely, suppose (1) is true with $x_i \in \left(0, \frac{1}{2}\right]$ and put

$$x_i = \frac{1}{1+a_i}, \ (i = 1, 2, \dots, n)$$

in (1). Then $a_i \geq 1$ and after some elementary transformations we get (4).

**Remark 2.1.** An interesting simple proof for (4) (and (5)) can be obtained by the well-known Sturm method ([13], [5]): supposing that not all the $a$'s are equal, e.g. $a_1 < G_n(a)$, $a_2 > G_n(a)$, replace $a_1$ by $G_n(a)$ and $a_2$ by $a_1 a_2 / G_n(a)$. Then $Q_n(a)$ remains unchanged while $P_n(a)$ decreases, etc. On the base of the above simple equivalence, perhaps would be more convenient to call (1) as the "Henrici-Fan" inequality.

# 3

Applying (4) for $a_i = (1-x_i)^k / x_i^k$, $k \geq 1$, we get:

$$\frac{A_{n,k}}{A'_{n,k}} \geq \frac{G_n^k}{G_n'^k} \tag{7}$$

where

$$A_{n,k} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^k}{x_i^k + (1-x_i)^k}$$

and $A'_{n,k}$ is obtained from $A_{n,k}$ by replacing $x_i$ with $1-x_i$. This generalizes inequality (6).

For another generalization we consider an extension of (4), namely:

If $b_i \geq 1$, $a_i > 0$ $(i = 1, 2, \dots, n)$ and $\sum_{i=1}^{n} a_i = 1$, then

$$\sum_{i=1}^{n} \frac{a_i}{1+b_i} \geq \frac{1}{1 + \prod_{i=1}^{n} b_i^{a_i}} \tag{8}$$

To prove this relation, we apply Jensen's inequality

$$f\left(\sum_{i=1}^{n} a_i t_i\right) \le \sum_{i=1}^{n} a_i f(t_i)$$

for the function $f : [0, \infty) \to \mathbb{R}$, defined by $f(t) = (1 + e^t)^{-1}$ which is convex, since

$$f''(t) = \frac{e^t(e^t - 1)}{(1 + e^t)^3} \ge 0.$$

From the inequality

$$\sum_{i=1}^{n} \frac{a_i}{1 + e^{t_i}} \ge \frac{1}{1 + e^{\sum a_i t_i}},$$

by replacing $e^t = b_i \ge 1$, we get the proposed inequality (8).

Let now $b_i = 1/x_i - 1$ in (8). Since $0 < x_i \le 1/2$, clearly $b_i \ge 1$. Because

$$1 - \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i (1 - x_i),$$

a simple computation gives:

$$\frac{\prod_{i=1}^{n} x_i^{a_i}}{\prod_{i=1}^{n} (1 - x_i)^{a_i}} \le \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i (1 - x_i)} \tag{9}$$

where, as we have supposed,

$$\sum_{i=1}^{n} a_i = 1.$$

For $a_i = 1/n$ $(i = 1, \ldots, n)$, we can reobtain (6).

## 4

An inequality of W. Sierpinski [12], [9] says that

$$(H_n(a))^{n-1}A_n(a) \leq (G_n(a))^n \leq (A_n(a))^{n-1}H_n(a). \tag{10}$$

Set $a_i = (1-x_i)/x_i$ for $\delta < x_i < 1$ $(i = 1, 2, \dots, n)$. One obtains:

$$\left(\frac{1-H'_n}{H'_n}\right)^{n-1} \cdot \frac{H_n}{1-H_n} \geq \left(\frac{G_n}{G'_n}\right)^n \geq \left(\frac{H_n}{1-H_n}\right)^{n-1} \cdot \frac{1-H'_n}{H'_n} \tag{11}$$

where we have used the following relations:

$$\sum_{i=1}^{n} \frac{x_i}{1-x_i} = \frac{n(1-H'_n)}{H'_n}, \quad \sum_{i=1}^{n} \frac{1-x_i}{x_i} = \frac{n(1-H_n)}{H_n}. \tag{12}$$

It would be interesting to compare this double inequality with the following ones:

$$\frac{\left(\frac{A_n}{A'_n}\right)^{n-1} H_n}{H'_n} \geq \left(\frac{G_n}{G'_n}\right)^n \geq \frac{\left(\frac{H_n}{H'_n}\right)^{n-1} A_n}{A'_n} \tag{13}$$

obtained recently by H. Alzer [2]. (Here $x_i \in \left(0, \frac{1}{2}\right]$).

A related result can be obtained with the use of the function

$$g : (0,1) \to \mathbb{R},\ g(x) = \frac{x}{1-x},$$

which is convex:

$$\frac{A_n}{A'_n} \leq \frac{1-H'_n}{H'_n},\ x_i \in (0,1) \tag{14}$$

This is complementary to (6). We note that in [14] is proved that

$$\frac{G_n}{G'_n} \geq \frac{H_n}{H'_n} \text{ for } x_i \in \left(0, \frac{1}{2}\right] \tag{15}$$

From (14) and (15) it follows also $H_n + H'_n \leq 1$ which can be proved by other ways, too (e.g. by $H_n + H'_n \leq G_n + G'_n \leq 1$).

**5**

Let us introduce now the notations

$$M_n = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^{1/3} \quad \text{and} \quad M'_n = \left(\frac{1}{n}\sum_{i=1}^{n}(1-x_i)^3\right)^{1/3}$$

and assume that $f : [a,b] \to \mathbb{R}$, $a < b$, has a continuous third derivative on $[a,b]$. Denote

$$m_3(f) = \min\{f^{(3)}(x) : \ x \in [a,b]\}, \quad M_3(f) = \max\{f^{(3)}(x) : \ x \in [a,b]\}$$

and introduce the functions $f_1, f_2$ by

$$f_1(t) = f(t) - \frac{t^3}{6} \cdot m_3(f), \quad f_2(t) = \frac{t^3}{6} \cdot M_3(f) - f(t), \quad t \in [a,b].$$

Then, obviously, $f_1, f_2 \in C^3[a,b]$ with

$$f_1^{(3)}(x) = f^{(3)}(x) - m_3(f) \geq 0, \quad f_2^{(3)}(x) = M_3(f) - f^{(3)}(x) \geq 0.$$

Supposing that $[a,b] \subset \left(0, \frac{1}{2}\right]$, we can apply relation (2) for these two functions, thus giving the following improvement:

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f(A_n) + \frac{m_3(f)}{6} D_n \leq \frac{1}{n}\sum_{i=1}^{n} f(1-x_i) - f(A'_n)$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f(A_n) + \frac{M_3(f)}{6} D_n \tag{16}$$

where

$$D_n = (M_n'^3 - A_n'^3) - (M_n^3 - A_n^3) \geq 0$$

which is a consequence of (2) applied with $f(x) = x^3$. Since $D_n \geq 0$, (16) is indeed an improvement of (2).

Selecting $f(t) = \ln t$, $0 < a < b$, $x_i \in [a,b] \subset \left(0, \frac{1}{2}\right]$, $i = 1, 2, \ldots, n$, we can derive the following refinement of Fan's inequality:

$$\frac{G_n}{G'_n} \cdot \exp\frac{D_n}{3b^3} \leq \frac{A_n}{A'_n} \leq \frac{G_n}{G'_n} \cdot \exp\frac{D_n}{3a^3}. \tag{17}$$

Letting $f(t) = 1/t$ in (16), we get an additive analogue of the Ky Fan inequality:

$$\frac{1}{H_n} - \frac{1}{A_n} - \frac{1}{a^4}D_n \leq \frac{1}{H'_n} - \frac{1}{A'_n} \leq \frac{1}{H_n} - \frac{1}{A_n} - \frac{1}{b^4}D_n. \qquad (18)$$

## 6

Choose $n = m+1$, $x_1 = \ldots = x_m = x$ and $x_{m+1} = y$ in (5). This gives us

$$\left[\frac{mx+y}{m-(mx+y)}\right]^{m+1} \geq \frac{x^m y}{(1-x)^m(1-y)},\ x, y \in \left(0, \frac{1}{2}\right]. \qquad (19)$$

Set $x = (x_1 + \ldots + x_m)/m$, $y = x_{m+1}$ in (19). After simple calculations we get

$$\left(\frac{A_{m+1}G'_{m+1}}{A'_{m+1}G_{m_1}}\right)^{m+1} \geq \left(\frac{A_m G'_m}{A'_m G_m}\right)^m \geq 1 \qquad (20)$$

providing a "Popoviciu-type" inequality (see [5], [8]).

D.S. Mitrinović and P.M. Vasić [10] have obtained the following result connected with Henrici's inequality:

If $a_1 \ldots a_m \geq 1$ and $a_{m+1} \geq (a_1 \ldots a_m)^{-1/(m+2)}$ $(m \geq 1)$, then

$$P_{m+1}(a) - Q_{m+1}(a) \geq P_m(a) - Q_m(a), \qquad (21)$$

with $P_m(a)$ and $Q_m(a)$ defined as in 2. If $a_i \geq 1$ $(i = 1, \ldots, m+1)$, then clearly the conditions are satisfied, so by setting $a_i = (1-x_i)/x_i$, we obtain:

$$(m+1)(G_m + G'_m)(A_{m+1}G'_{m+1} - A'_{m+1}G_{m+1})$$

$$\geq m(G_{m+1} + G'_{m+1})(A_m G'_m - A'_m G_m) \qquad (22)$$

giving a "Rado-type" inequality ([5], [8]). This contains also a refinement of (6). This note is a version of our paper [15], published in 1990.

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## 4.2 A refinement of the Ky Fan inequality

### 1

Let $x_1, \ldots, x_n$ be a sequence of positive real numbers lying in the open interval $]0,1[$, and let $A_n$, $G_n$ and $H_n$ denote their arithmetic, geometric and harmonic mean, respectively, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad H_n = \frac{n}{\displaystyle\sum_{i=1}^{n}\frac{1}{x_i}}.$$

Further, let $A_n'$, $G_n'$ and $H_n'$ denote the arithmetic, geometric and harmonic mean, respectively, of $1-x_1, \ldots, 1-x_n$, i.e.

$$A_n' = \frac{1}{n}\sum_{i=1}^{n}(1-x_i), \quad G_n = \left(\prod_{i=1}^{n}(1-x_i)\right)^{1/n}, \quad H_n = \frac{n}{\displaystyle\sum_{i=1}^{n}\frac{1}{1-x_i}}.$$

The arithmetic-geometric mean inequality $G_n \leq A_n$ (and its weighted variant) played an important role in the development of the theory of inequalities. Because of its importance, many proofs and refinements have been published. In 1961, a remarkable new counterpart of the AM-GM inequality was published in the famous book [7]:

**Theorem 1.** *If $x_i \in ]0,1/2]$ for all $i \in \{1,\ldots,n\}$, then*

$$\frac{G_n}{G_n'} \leq \frac{A_n}{A_n'}, \tag{1}$$

*with equality holding if and only if $x_1 = \ldots = x_n$.*

Inequality (1), which is due to Ky Fan, has evoked the interest of several mathematicians, and different proofs as well as many extensions, sharpenings, and variants have been published. For proofs of (1) the reader is referred to [3], [6], [16], [17]. Refinements of (1) are proved in [1], [5], [6], [18], while generalizations can be found in [9], [11], [19], [21].

For converses and related results see [2], [4], [14]. See also the survey paper [6].

In 1984, Wang and Wang [20] established the following counterpart of (1):

$$\frac{H_n}{H'_n} \leq \frac{G_n}{G'_n}. \tag{2}$$

For extension to weighted means and other proofs of (2) see, for instance, [6] and [18].

In 1990, J. Sándor [15, relation (33)] proved the following refinement of (1) in the case of two arguments (i.e. $n = 2$):

$$\frac{G}{G'} \leq \frac{I}{I'} \leq \frac{A}{A'}, \tag{3}$$

where $G = G_2$, $G' = G'_2$ etc. and $I$ denotes the so-called identric mean of two numbers:

$$I(x_1, x_2) = \frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{1/(x_2 - x_1)}, \quad \text{if} \quad x_1 \neq x_2$$

$$I(x, x) = x.$$

Here $I'(x_1, x_2) = I(1 - x_1, 1 - x_2)$ and $x_1, x_2 \in ]0, 1/2]$.

In what follows, inequality (3) will be extended to the case of $n$ arguments, thus giving a new refinement of Ky Fan inequality (1).

## 2

Let $n \geq 2$ be a given integer, and let

$$A_{n-1} = \{(\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i \geq 0,\ i = 1, \ldots, n-1,\ \lambda_1 + \ldots + \lambda_{n-1} \leq 1\}$$

be the Euclidean simplex. Given $X = (x_1, \ldots, x_n)$ ($x_i > 0$ for all $i \in \{1, \ldots, n\}$), and a probability measure $\mu$ on $A_{n-1}$, for a continuous strictly monotone function $f\ :]0, \infty[\to \mathbb{R}$, the following functional mean of $n$ arguments can be introduced:

$$M_f(X; \mu) = f^{-1}\left(\int_{A_{n-1}} f(X \cdot \lambda) d\mu(\lambda)\right), \tag{4}$$

where

$$X \cdot \lambda = \sum_{i=1}^{n} x_i \lambda_i$$

denotes the scalar product,

$$\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1} \quad \text{and} \quad \lambda_n = 1 - \lambda_1 - \ldots - \lambda_{n-1}.$$

For $\mu = (n-1)!$ and $f(t) = 1/t$, the unweighted logarithmic mean

$$L(x_1, \ldots, x_n) = \left( (n-1)! \int_{A_{n-1}} \frac{1}{X \cdot \lambda} d\lambda_1 \ldots d\lambda_{n-1} \right)^{-1} \tag{5}$$

is obtained. For properties and an explicit form of this mean, the reader is referred to [13].

For $f(t) = \log t$ we obtain a mean, which can be considered as a generalization of the identric mean

$$I(X; \mu) = \exp\left( \int_{A_{n-1}} \log(X \cdot \lambda) d\mu(\lambda) \right). \tag{6}$$

Indeed, it is immediately seen that for the classical identric mean of two arguments one has

$$I(x_1, x_2) = \exp\left( \int_0^1 \log(tx_1 + (1-t)x_2) dt \right).$$

For $\mu = (n-1)!$ we obtain the unweighted (and symmetric) identric mean of $n$ variables

$$I(x_1, \ldots, x_n) = \exp\left( (n-1)! \int_{A_{n-1}} \log(X \cdot \lambda) d\lambda_1 \ldots d\lambda_{n-1} \right), \tag{7}$$

in analogy with (5). It should be noted that (7) is a special case of (4), which has been considered in [13]. The mean (4) even is a special case of the B.C. Carlson's function $M$ (see [8, p. 33]). For an explicit form of $I(x_1, \ldots, x_n)$ see [12].

Let $n \geq 2$, let $\mu$ be a probability measure of $A_{n-1}$, and let $i \in \{1, \ldots, n\}$. The $i$th weight $w_i$ associated to $\mu$ is defined by

$$w_i = \int_{A_{n-1}} \lambda_i d\mu(\lambda), \text{ if } 1 \leq i \leq n-1, \tag{8}$$

$$w_n = \int_{A_{n-1}} (1 - \lambda_1 - \ldots - \lambda_{n-1}) d\mu(\lambda),$$

where $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}$. Obviously, $w_i > 0$ for all $i \in \{1, \ldots, n\}$ and $w_1 + \ldots + w_n = 1$. Moreover, if $\mu = (n-1)!$, then $w_i = 1/n$ for all $i \in \{1, \ldots, n\}$.

We are now in a position to state the main result of the paper, a weighted improvement of the Ky Fan inequality.

**Theorem 2.** *Let $n \geq 2$, let $\mu$ be a probability measure on $A_{n-1}$ whose weights $w_1, \ldots, w_n$ are given by (8), and let $x_i \in ]0, 1/2]$ $(i = 1, \ldots, n)$. Then*

$$\frac{\prod_{i=1}^{n} x_i^{w_i}}{\prod_{i=1}^{n} (1 - x_i)^{w_i}} \leq \frac{I(x_1, \ldots, x_n; \mu)}{I(1 - x_1, \ldots, 1 - x_n; \mu)} \leq \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i (1 - x_i)}. \tag{9}$$

**Proof.** First remark that the function $\phi : ]0, 1/2] \to \mathbb{R}$ defined by

$$\phi(t) = \log t - \log(1 - t)$$

is concave. Consequently

$$\sum_{i=1}^{n} w_i \phi(x_i) \leq \int_{A_{n-1}} \phi(X \cdot \lambda) d\mu(\lambda) \leq \phi\left(\sum_{i=1}^{n} w_i x_i\right). \tag{10}$$

This inequality has been established in [10]. From (10), after a simple computation we deduce (9). □

**3**

**Remark.** For $\mu = (n-1)!$, inequality (9) reduces to the following unweighted improvement of the Ky Fan inequality, which generalizes (3):

$$\frac{G_n}{G'_n} \leq \frac{I_n}{I'_n} \leq \frac{A_n}{A'_n}.$$

Here $I_n = I(x_1, \ldots, x_n)$, while $I'_n = I(1-x_1, \ldots, 1-x_n)$.

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## 4.3 A converse of Ky Fan's inequality

Let $x_i \in \left(0, \frac{1}{2}\right]$, $i = \overline{1,n}$, and let $A_n, G_n, H_n$ denote the arithmetic, geometric, resp. harmonic means of these numbers. Put $A'_n, G'_n, H'_n$ for the corresponding means of the numbers $1 - x_i$. The famous inequality of Ky Fan (see [1]) states that

$$\frac{G_n}{G'_n} \le \frac{A_n}{A'_n}. \tag{1}$$

Suppose that $m > 0$ and $x_i \in \left[m, \frac{1}{2}\right]$. Then the following converse of (1) is true:

**Theorem 1.**

$$\frac{A_n}{A'_n} \le \frac{G_n}{G'_n} \exp\left[(A_n - G_n)\frac{1}{m(1-m)}\right]. \tag{2}$$

**Proof.** We shall obtain a slightly stronger relation. Let us define

$$f(x) = \frac{x}{1-x} \exp\left\{\left(1 - \frac{x}{m}\right)\frac{1}{m}\right\}, \quad \text{where} \quad x \in \left[m, \frac{1}{2}\right].$$

Then

$$\frac{f'(x)}{f(x)} = \frac{1}{x(1-x)} - \frac{1}{m(1-m)} \le 0 \text{ for } x \ge m,$$

since the function $g(x) = x(1-x)$ is strictly increasing on $\left[0, \frac{1}{2}\right]$. Thus the function $f$ is non-increasing on $\left[m, \frac{1}{2}\right]$. Suppose that $m \le x_1 \le x_2 \le \ldots \le x_n \le \dfrac{1}{2}$. Then, since $m \le G_n \le A_n$, we have $f(A_n) \le f(G_n)$ so that

$$\frac{A_n}{A'_n} \le \frac{G_n}{1-G_n} \exp\left[(A_n - G_n)\frac{1}{m(1-m)}\right]. \tag{3}$$

**Theorem 2.** $(G_n + G'_n)^n \le (G_{n-1} + G'_{n-1})^{n-1} \le 1$ *for all* $n \ge 2$.

**Proof.** Let us consider the application $f : (0,1) \to \mathbb{R}$, defined by

$$f(x) = (x_1 \ldots x_{n-1} x)^{1/n} + [(1-x_1)\ldots(1-x_{n-1})(1-x)]^{1/n}.$$

We have

$$f'(x) = \frac{1}{n}(x_1 \ldots x_{n-1})^{\frac{1}{x}} x^{\frac{1}{n}-1} - \frac{1}{n}[(1-x_1)\ldots(1-x_{n-1})]^{\frac{1}{n}}(1-x)^{\frac{1}{n}-1},$$

so $f'(x) = 0$ iff $x = x_0 = \dfrac{G_{n-1}}{G_{n-1} + G'_{n-1}}$. Since

$$f''(x) = \frac{1}{n}\left(\frac{1}{n} - 1\right)(x_1 \dots x_{n-1})^{\frac{1}{n}} x^{\frac{1}{n}-2}$$

$$+\frac{1}{n}\left(\frac{1}{n} - 1\right)[(1-x_1)\dots(1-x_{n-1})]^{\frac{1}{n}}(1-x)^{\frac{1}{n}-2} \le 0$$

we observe that $f$ is a concave function. It is well known (see e.g. [1]) that then $x_0$ must be a maximum point on $(0,1)$, implying $f(x_n) \le f(x_0)$. After some simple calculations this gives

$$G_n + G'_n \le (G_{n-1} + G'_{n-1})^{\frac{n-1}{n}},$$

i.e. the first relation of Theorem.

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# 4.4 On certain new Ky Fan type inequalities for means

## 1

Let $x_k > 0$, $k = 1, 2, \ldots, n$ and put

$$A_n = A_n(x) = \frac{1}{n}\sum_{k=1}^{n} x_k,$$

$$G_n = G_n(x) = \sqrt[n]{\prod_{k=1}^{n} x_k}$$

for the arithmetic, respective geometric means of $x = (x_1, x_2, \ldots, x_n)$. If

$$1 - x = (1 - x_1, 1 - x_2, \ldots, 1 - x_n)$$

we denote, as usual,

$$A_n' = A_n'(x) = A_n(1 - x), \quad G_n' = G_n'(x) = G_n(1 - x)$$

for $0 < x_k < 1$, $k = 1, 2, \ldots, n$. The famous Ky Fan inequality states that for all $x_k \in \left(0, \frac{1}{2}\right]$, $k = 1, 2, \ldots, n$ one has:

$$\frac{A_n}{A_n'} \geq \frac{G_n}{G_n'}. \tag{1}$$

In 1990 [4] we have proved the surprising fact that inequality (1) is equivalent to an inequality of Henrici [3] from 1956:

$$\sum_{k=1}^{n} \frac{1}{1 + a_k} \geq \frac{n}{1 + \sqrt[n]{\prod_{k=1}^{n} x_k}} \tag{2}$$

for $a_k \geq 1$, $k = 1, 2, \ldots, n$. Indeed, if $x_k \in \left(0, \frac{1}{2}\right]$, select $a_k = \frac{1}{x_k} - 1 \geq 1$ in (2). We get from (2) that

$$A_n \geq \frac{G_n}{G_n + G_n'} \quad \text{or} \quad A_n G_n' \geq G_n(1 - A_n) = G_n A_n'.$$

Thus, relation (1) follows. Conversely, if (1) is true with $x_k \in \left(0, \frac{1}{2}\right]$, then by letting $x_k = \frac{1}{1 + a_k}$, where $a_k \geq 1$, from (2) after some transformations we get inequality (2).

For weighted variants of (1), as well as some Radó or Popoviciu type Ky Fan inequalities, see [4], [5]. See also [6].

## 2

Let $x_k > 0$, $k = 1, 2, \ldots, n$ and $\alpha \in (0, 1]$. We introduce the following notations

$$1 - \alpha x = (1 - \alpha x_1, 1 - \alpha x_2, \ldots, 1 - \alpha x_n)$$

and

$$A_n^\alpha = A_n^\alpha(x) = A_n(1 - \alpha x), \quad G_n^\alpha = G_n^\alpha(x) = G_n(1 - \alpha x)$$

for $0 < x_k < \frac{1}{\alpha}$, $k = 1, 2, \ldots, n$.

The function $f(x) = \frac{1}{\alpha + e^x}$ is convex, because

$$f''(x) = \frac{e^x(e^x - \alpha)}{(\alpha + e^x)^3} > 0$$

and from Jensen's inequality, we get:

$$\sum_{k=1}^{n} \frac{1}{\alpha + e^{\ln a_k}} \geq \frac{n}{\alpha + e^{\frac{1}{n}\sum\limits_{k=1}^{n} \ln a_k}}. \tag{4}$$

If $\ln a_k = x_k$, $k = 1, 2, \ldots, n$, we obtain the inequality:

$$\sum_{k=1}^{n} \frac{1}{\alpha + x_k} \geq \frac{n}{\alpha + \sqrt[n]{\prod_{k=1}^{n} x_k}} \tag{3}$$

If $x_k \in \left(0, \dfrac{1}{1+\alpha}\right]$, $k = 1, 2, \ldots, n$, select $x_k = \dfrac{1}{x_k} - \alpha$, $k = 1, 2, \ldots, n$ in (3). We get from (3), that

$$A_n \geq \frac{G_n}{\alpha G_n + G_n^{\alpha}}$$

on the surprising inequality

$$\frac{A_n}{G_n} \geq \frac{A_n^{\alpha}}{G_n^{\alpha}} \tag{5}$$

which is a generalization of Ky Fan inequality (for $\alpha = 1$, we get relation (1)).

## 3

If $x_k, p_k > 0$, $k = 1, 2, \ldots, n$ and

$$A_n(p, x) = \frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}, \quad G_n(p, x) = \left(\prod_{k=1}^{n} x_k^{p_k}\right)^{\frac{1}{\sum_{k=1}^{n} p_k}},$$

then we introduce the following notation

$$A_n^{\alpha}(p, x) = A_n(p; 1 - \alpha x) \quad \text{and} \quad G_n^{\alpha}(p, x) = G_n(p; 1 - \alpha x),$$

where $\alpha \in (0, 1]$, $x_k \in \left(0, \dfrac{1}{\alpha}\right)$, $k = 1, 2, \ldots, n$.

Using the weighted version of Jensen's inequality in (4) we get

$$\sum_{k=1}^{n} \frac{p_k}{\alpha + x_k} \geq \frac{\sum\limits_{k=1}^{n} p_k}{\alpha + \left(\prod\limits_{k=1}^{n} x_k^{p_k}\right)^{\frac{1}{\sum\limits_{k=1}^{n} p_k}}} \tag{5}$$

Indeed, if $x_k \in \left(0, \dfrac{1}{\alpha+1}\right]$, $k = 1, 2, \ldots, n$ select $x_k = \dfrac{1}{x_k} - \alpha$, $k = 1, 2, \ldots, n$ in (5). We get from (5) that

$$A_n(p, x) \geq \frac{G_n(p, x)}{\alpha G_n(p, x) + G_n^{\alpha}(p, x)}$$

or the inequality

$$\frac{A_n(p, x)}{G_n(p, x)} \geq \frac{A_n^{\alpha}(p, x)}{G_n^{\alpha}(p, x)} \tag{6}$$

which is a new generalization of Ky Fan inequality (for $\alpha = 1$, $p = 1$ we get relation (1)).

## 4

In 1970 Klamkin and Newman [2], by extending certain Weierstrass type inequalities, have shown that (their notation):

$$\prod_{k=1}^{n} (1 - A_k) \geq \left( \frac{1 - \dfrac{S_1}{n}}{\left(\dfrac{S_1}{n}\right)^{a^2}} \right)^{n} \prod_{k=1}^{n} A_k^{a^2} \tag{7}$$

where $0 < A_k \leq \dfrac{a}{a+1}$, $S_1 = \sum\limits_{k=1}^{n} A_k$.

To simplify this inequality, put $A_k = x_k$, $k = 1, 2, \ldots, n$ and use the notations of section 1. After some simplifications, we get the inequality

$$\frac{A_n^{a^2}}{A_n'} \geq \frac{G_n^{a^2}}{G_n'} \tag{8}$$

for $0 < x_k \leq \dfrac{a}{a+1}$, $k = 1, 2, \ldots, n$.

This is an extension of the Ky Fan inequality, as for $a = 1$ we get exactly relation (1).

## 5

In 1990 H. Alzer [1] proved some other Weierstrass type inequalities. One of his results states that

$$\prod_{k=1}^{n} \frac{x_k}{1-x_k} \leq \frac{1 + 2\sum\limits_{k=1}^{n} x_k}{1 + 2\sum\limits_{k=1}^{n}(1-x_k)} \tag{9}$$

for $x_k \in \left(0, \dfrac{1}{2}\right]$, $k = 1, 2, \ldots, n$. By the notation of section 1, this may be rewritten also as

$$\left(\frac{G_n}{G_n'}\right)^n \leq \frac{1+2nA_n}{1+2nA_n'} \text{ for } 0 < x_k \leq \frac{1}{2},\ k = 1, 2, \ldots, n. \tag{10}$$

We note here that in fact one has

$$\left(\frac{A_n}{A_n'}\right)^n \leq \frac{1+2nA_n}{1+2nA_n'}. \tag{11}$$

Since this may be written also as

$$A_n^n + 2nA_n^n(A_n')^n \leq (A_n')^n + 2n(A_n')^n \cdot A_n^n$$

and this is true by $A_n \leq A_n' = 1 - A_n$ or $A_n \leq \dfrac{1}{2}$. Indeed $A_n^n \leq (A_n')^n$ and $2nA_n^n(A_n')^n \leq 2n(A_n')^n A_n$, since this last inequality is $A_n^{n-1} \leq (A_n')^{n-1}$ and for $n \geq 1$ this is true again. In view of (1), we can write

$$\left(\frac{G_n}{G_n'}\right)^n \leq \left(\frac{A_n}{A_n'}\right)^n \leq \frac{1+2nA_n}{1+2nA_n'}. \tag{12}$$

Another inequality from [1] states that for $x_k \in \left(0, \frac{1}{2}\right]$, $k = 1, 2, \dots, n$

$$\frac{1 + \sum_{k=1}^{n} x_k}{1 + \sum_{k=1}^{n} (1 - x_k)} \le \frac{1 + \prod_{k=1}^{n} x_k}{1 + \prod_{k=1}^{n} (1 - x_k)} \tag{13}$$

We can write (13) as

$$\frac{1 + nA_n}{1 + nA'_n} \le \frac{1 + G_n^n}{1 + (G'_n)^n} \tag{14}$$

Now, remark that

$$\frac{1 + 2nA_n}{1 + 2nA'_n} \le \frac{1 + nA_n}{1 + nA'_n}$$

by $A'_n \ge A_n$, so by (12) and (14) the following chain of inequalities holds true

$$\left(\frac{G_n}{G'_n}\right)^n \le \left(\frac{A_n}{A'_n}\right)^n \le \frac{1 + 2nA_n}{1 + 2nA'_n} \le \frac{1 + nA_n}{1 + nA'_n} \le \frac{1 + G_n^n}{1 + (G'_n)^n}. \tag{15}$$

By the well-known inequality

$$\prod_{k=1}^{n} (a_k + 1) \le 2^{n-1} \left( \prod_{k=1}^{n} a_k + 1 \right), \ a_k \ge 1, \ k = 1, 2, \dots, n \tag{16}$$

with the notations $a_k = \frac{1}{x_k} - 1$, where $x_k \in \left(0, \frac{1}{2}\right]$, $k = 1, 2, \dots, n$, we can deduce the inequality

$$G_n^n + (G'_n)^n \ge \frac{1}{2^{n-1}} \tag{17}$$

We note that, as $G_n \le \frac{1}{2} \le G'_n$, this is not trivial. As

$$\frac{1 + G_n^n}{1 + (G'_n)^n} \le \frac{1 + 2G_n}{1 + G_n^n + (G'_n)^n}$$

(more generally $\dfrac{1+x}{1+y} \le \dfrac{1+2x}{1+2y}$ for $x \le y$, here $x = G_n^n$, $y = (G_n')^n$), by (17) we get

$$\frac{1+G_n^n}{1+(G_n')^n} \le \frac{1+2G_n^n}{1+\dfrac{1}{2^{n-1}}} \le 1 \tag{18}$$

completing the chain from (15).

## 6

The reverse of Henrici's inequality (see [3]) states that for $0 < b_k \le 1$, $k = 1, 2, \ldots, n$, one has

$$\sum_{k=1}^{n} \frac{1}{1+b_k} \le \frac{n}{1+\sqrt[n]{\prod_{k=1}^{n} b_k}} \tag{19}$$

Put $b_k = 2x_k$, where $x_k \in \left(0, \dfrac{1}{2}\right]$, $k = 1, 2, \ldots, n$. Then we get:

$$1 + 2G_n(x) \le H_n(2x+1) \tag{20}$$

where $2x+1 = (2x_1+1, \ldots, 2x_n+1) = (a_1, a_2, \ldots, a_n)$ and

$$H(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{k=1}^{n} \dfrac{1}{a_k}}$$

denotes the harmonic mean of $a_k > 0$, $k = 1, 2, \ldots, n$.

Now, by the Chrystal inequality (see [7]), one can write

$$\sqrt[n]{\prod_{k=1}^{n}(2x_k+1)} \ge \sqrt[n]{\prod_{k=1}^{n} 2x_k} + 1$$

or

$$G_n(2x+1) \ge 2G_n(x) + 1 \tag{21}$$

so as $H_n(2x+1) \leq G_n(2x+1)$, relation (20) is a refinement of (21):

$$1 + 2G_n(x) \leq H_n(2x+1) \leq G_n(2x+1). \tag{22}$$

However, we note that (21) holds true for all $x_k > 0$ while the stronger inequality (20) only for $0 < x_k \leq \dfrac{1}{2}$, $k = 1, 2, \ldots, n$.

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## 4.5 An extension of Ky Fan's inequalities

Let $x_k$ $(k = \overline{1,n})$ be positive real numbers. The arithmetic respectively geometric means of $x_k$ are

$$A = A(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n},$$

$$G = G(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}.$$

Let $f : I \to \mathbb{R}$ ($I$ interval) and suppose that $x_k \in (a, b)$. Define the functional arithmetic, respectively geometric means, by

$$A_f = A_f(x_1, \ldots, x_n) = \frac{f(x_1) + \ldots + f(x_n)}{n}$$

and

$$G_f = G_f(x_1, \ldots, x_n) = \sqrt[n]{f(x_1) \ldots f(x_n)}.$$

Clearly, $A_f$ and $G_f$ are means in the usual sense, if

$$\min\{x_1, \ldots, x_n\} \le A_f \le \max\{x_1, \ldots, x_n\}$$

and

$$\min\{x_1, \ldots, x_n\} \le G_f \le \max\{x_1, \ldots, x_n\}.$$

For example, when $I = (0, +\infty)$ and $f(x) = x$; $A_f \equiv A$, $G_f \equiv G$; when $I = (0, 1)$ and $f(x) = 1 - x$, $A_f = A'$, $G_f = G'$, are indeed means in the above sense.

The following famous relations are well-known:

$$G \le A, \text{ for } x_k > 0 \ (k = \overline{1,n}) \tag{1}$$

$$\frac{G}{G'} \le \frac{A}{A'}, \text{ for } x_k > \left(0, \frac{1}{2}\right] \tag{2}$$

The first is the arithmetic-geometric inequality, while the second is the Ky-Fan inequality (see e.g. [2], [3], [4]). Now, even if $A_f$ and $G_f$ are not means in the usual sense, the following extension of (2) may be true:

$$\frac{G}{G_f} \le \frac{A}{A_f} \tag{3}$$

This inequality (with other notations) is stated in OQ. 633, in [1].

We now prove (3) for certain particular $f$.

**Theorem.** *Let* $id : \mathbb{R} \to \mathbb{R}$, $id(x) = x$ *and suppose that* $f : I \to \mathbb{R}$ *satisfies the following conditions:* $f$ *and* $\ln \dfrac{id}{f}$ *are concave functions.*

*Then inequality (3) holds true.*

**Proof.** By concavity of $\ln \dfrac{id}{f}$ one can write:

$$\ln \frac{\dfrac{x_1 + \ldots + x_n}{n}}{f\left(\dfrac{x_1 + \ldots + x_n}{n}\right)} \geq \left[\ln \frac{x_1}{f(x_1)} + \ldots + \ln \frac{x_n}{f(x_n)}\right] \frac{1}{n},$$

i.e.

$$\ln \frac{A}{f(A)} \geq \ln \frac{G}{G_f}.$$

Therefore

$$\frac{G}{G_f} \leq \frac{A}{f(A)} \tag{4}$$

Now, since $f$ is concave, one has

$$A_f = \frac{f(x_1) + \ldots + f(x_n)}{n} \leq f\left(\frac{x_1 + \ldots + x_n}{n}\right) = f(A),$$

and by (4) this gives (3).

**Remark 1.** Let $I = \left(0, \dfrac{1}{2}\right]$ and $f(x) = 1 - x$, then $g(x) = \ln \dfrac{x}{1-x}$ has a derivative

$$g'(x) = \frac{1}{x} + \frac{1}{1-x},$$

so

$$g''(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = \frac{x^2 - (1-x)^2}{x^2(1-x)^2} = \frac{2x-1}{x^2(1-x)^2} \leq 0.$$

Therefore $f$ and $\ln \dfrac{id}{f}$ are concave functions, and (4) gives Ky Fan's inequality (2). One has equality for $x_k = \dfrac{1}{2}$ $(k = \overline{1,n})$.

**Remark 2.** There are many functions $f : I \to \mathbb{R}$ such that $f$ and $\ln \frac{id}{f}$ are simultaneously concave. Put e.g. $f(x) = \ln x$. Then

$$g(x) = \ln \frac{x}{\ln x} = \ln x - \ln \ln x.$$

One has

$$g''(x) = \frac{-\ln^2 x + \ln x + 1}{x^2 \ln^2 x} \leq 0$$

if

$$\ln x \geq \frac{1+\sqrt{5}}{2}, \quad \text{i.e.} \quad x \geq e^{\frac{1+\sqrt{5}}{2}} = x_0.$$

(Take $I = [x_0, +\infty)$).

**Remark 3.** Without concavity of $f$, holds true (4).

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## 4.6 Notes on certain inequalities by Hölder, Lewent and Ky Fan

### 1. Historical notes

In 1888 Rogers (see [10]) proved that for $x_i > 0$, $\alpha_i > 0$ $(i = \overline{1,n})$

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leq \left( \frac{\sum\limits_{i=1}^{n} \alpha_i x_i}{\sum\limits_{i=1}^{n} \alpha_i} \right)^{\sum\limits_{i=1}^{n} \alpha_i} \tag{1}$$

F. Sibirani [15] reported in 1907 that the proof of (1) was already known. Namely, it was published by D. Besso [3] in 1879. We note that, Besso's original article was reprinted in 1907, but never included with a review in JFM ("Jahrbuch der Fortschritte der Mathematik"); see [16].

It is known that, Hölder concludes inequality (1) as a special case of

$$\varphi \left( \frac{\sum\limits_{i=1}^{n} \alpha_i x_i}{\sum\limits_{i=1}^{n} \alpha_i} \right) \leq \frac{\sum\limits_{i=1}^{n} \alpha_i \varphi(x_i)}{\sum\limits_{i=1}^{n} \alpha_i}, \tag{2}$$

where $\varphi$ has an increasing derivative; see [7], [6], [11]. The real importance of this inequality, for continuous, mid-convex ("Jensen-convex") functions $\varphi$ was discovered, however by Jensen [8].

It is little (or only fragmentarily) known today that, Hölder's result in the case of equal weights (e.g., $\alpha_i = \frac{1}{n}$, $\varphi'' \geq 0$) was proved much earlier by Grolous [5]. He applied the so-called "method of centers" (see e.g. [11]) in his proof.

Finally, we wish to mention here the names of the reviewers contributing to JFM, related to the above mentioned articles. These were

M. Hamburger, E. Lampe, J. Glaisher, P. Stäckel, R. Hoppe, H. Valentiner, F. Müller, and F. Lewent. It seems that, they did not publish in the area of mathematical inequalities, the only exception being [9].

## 2. Lewent's and Ky Fan's inequalities

By using the power-series method, in 1908 Lewent [9] proved the relation

$$\frac{1+\sum_{i=1}^{n}\alpha_i x_i}{1-\sum_{i=1}^{n}\alpha_i x_i} \leq \prod_{i=1}^{n}\left(\frac{1+x_i}{1-x_i}\right)^{\alpha_i}, \tag{3}$$

where

$$x_i \in [0,1), \quad i=1,2,\dots,n; \text{ and } \sum_{i=1}^{n}\alpha_i = 1. \tag{4}$$

We note that, this follows also by inequality (2) applied to the function

$$\varphi(t) = \log\frac{1+t}{1-t}, \quad t \in [0,1).$$

The famous Ky Fan inequality (see e.g. [1], [2], [12], [13], [14]) states that if $a_i \in \left(0, \frac{1}{2}\right]$ $(i=1,2,\dots,n)$ and $A_n(a) = A_n$, $G_n(a) = G_n$ denote the arithmetic, resp. geometric means of $a = (a_1,\dots,a_n)$; by putting $A'_n = A_n(1-a)$, $G'_n = G_n(1-a)$, where $1-a = (1-a_1,\dots,1-a_n)$; then one has

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n} \tag{5}$$

We want to point out now that, by a method of Sándor ([13], I) (who applied an inequality of Henrici to deduce (5)), Lewent's inequality implies Ky Fan's inequality (5). Indeed, let $\alpha_i = \frac{1}{n}$, and put $x_i = 1-2a_i$ $(i=\overline{1,n})$ in (3). As $a_i \in \left(0, \frac{1}{2}\right]$, clearly $x_i \in [0,1)$. A simple transformation yields relation (5), and we are done.

**Remark.** Let $A_n^+ = A_n(1+a)$, $G_n^+ = G_n(1+a)$, where $1+a = (1+a_1,\dots,1+a_n)$. By letting $\alpha_i = \frac{1}{n}$, $x_i = a_i$, inequality (3) may be written equivalently also as

$$\frac{G'_n}{G_n^+} \le \frac{A'_n}{A_n^+}, \tag{6}$$

where $0 \le a_i < 1$, $i = 1, 2, \dots, n$; and $G'_n = G'_n(a)$ etc. For such inequalities, see also [4] and [12] (II).

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# 4.7 On certain new means and their Ky Fan type inequalities

## 1. Introduction

Let $x = (x_1, \dots, x_n)$ be an $n$-tuple of positive numbers. The unweighted arithmetic, geometric and harmonic means of $x$, denoted by $A = A_n$, $G = G_n$, $H = H_n$, respectively, are defined as follows

$$A = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad H = n/\left(\sum_{i=1}^{n}\frac{1}{x_i}\right).$$

Assume $0 < x_i < 1$, $1 \le i \le n$ and define $x' := 1-x = (1-x_1, \dots, 1-x_n)$. Throughout the sequel the symbols $A' = A'_n$, $G' = G'_n$ and $H' = H'_n$ will stand for the unweighted arithmetic, geometric and harmonic means of $x'$.

The arithmetic-geometric mean inequality $G_n \le A_n$ (and its weighted variant) played an important role in the development of the theory of inequalities. Because of its importance, many proofs and refinements have been published. The following remarkable inequality is due to Ky Fan:

If $x_i \in \left(0, \frac{1}{2}\right]$ $(1 \le i \le n)$, then

$$\frac{G}{G'} \le \frac{A}{A'} \tag{1}$$

with equality only if $x_1 = \dots = x_n$. The paper by H. Alzer [1] (who obtained many results related to (1)) contains a very good account up to 1995 of the Ky Fan type results (1). For example, in 1984 Wang and Wang [11] established the following counterpart of (1):

$$\frac{H}{H'} \le \frac{G}{G'} \tag{2}$$

Let $I = I(x_1, x_2) = \frac{1}{e}(x_2^{x_2}/x_1^{x_1})^{1/(x_2-x_1)}$ $(x_1 \ne x_2)$, $I(x, x) = x$ denote the so-called identric mean of $x_1, x_2 > 0$. In 1990 J. Sándor [8] proved the following refinement of (1) in the case of two arguments (i.e. $n = 2$):

$$\frac{G}{G'} \leq \frac{I}{I'} \leq \frac{A}{A'}, \tag{3}$$

where $I' = I'(x_1, x_2) = I(1 - x_1, 1 - x_2)$.

We note that, inequality (14) in Rooin's paper [6] is exactly (3).

In 1999 Sándor and Trif [10] have introduced an extension of the identric mean to $n$ arguments, as follows. For $n \geq 2$, let

$$E_{n-1} = \{(\lambda_1, \dots, \lambda_{n-1}) : \ \lambda_i \geq 0, \ 1 \leq i \leq n-1, \ \lambda_1 + \dots + \lambda_{n-1} \leq 1\}$$

be the Euclidean simplex. Given any probability measure $\mu$ on $E_{n-1}$, for a continuous strictly monotone function $f : (0, \infty) \to \mathbb{R}$, the following functional means of $n$ arguments can be introduced:

$$M_f(x; \mu) = f^{-1}\left(\int_{E_{n-1}} f(x\lambda) d\mu(\lambda)\right), \tag{4}$$

where

$$x\lambda = \sum_{i=1}^{n} x_i \lambda_i$$

denotes the scalar product,

$$\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in E_{n-1} \quad \text{and} \quad \lambda_n = 1 - \lambda_1 - \dots - \lambda_{n-1}.$$

For $\mu = (n-1)!$ and $f(t) = 1/t$, one obtains the unweighted logarithmic mean, studied by A. P. Pittenger [5]. For $f(t) = \ln t$, however we obtain a mean

$$I = I(x) = \exp\left(\int_{E_{n-1}} \ln(x\lambda) d\mu(\lambda)\right) \tag{5}$$

which may be considered as a generalization of the identric mean. Indeed, it is immediately seen that

$$I(x_1, x_2) = \exp\left(\int_0^1 \ln(tx_1 + (1-t)x_2) dt\right),$$

in concordance with (5), which for $\mu = (n-1)!$ gives the unweighted (and symmetric) identric mean of $n$ arguments:

$$I = I_n = I_n(x_1, \dots, x_n) = \exp\left((n-1)! \int_{E_{n-1}} \ln(x\lambda) d\lambda_1 \dots d\lambda_{n-1}\right) \quad (6)$$

Let $I' = I'_n = I_n(1-x)$ in (5) for $\mu = (n-1)!$. Then Sándor and Trif [10] proved that relation (3) holds true for any $n \geq 2$ $\left(x_i \in \left(0, \frac{1}{2}\right]\right)$. The weighted versions hold also true.

In 1990 J. Sándor [7] discovered the following additive analogue of the Ky Fan inequality (1): If $x_i \in \left(0, \frac{1}{2}\right]$ $(1 \leq i \leq n)$, then

$$\frac{1}{H'} - \frac{1}{H} \leq \frac{1}{A'} - \frac{1}{A} \quad (7)$$

In 2002, E. Neuman and J. Sándor [2] proved the following refinement of (7):

$$\frac{1}{H'} - \frac{1}{H} \leq \frac{1}{L'} - \frac{1}{L} \leq \frac{1}{A'} - \frac{1}{A}, \quad (8)$$

where $L$ is the (unweighted) logarithmic mean, obtained from (4) for $f(t) = 1/t$, i.e.

$$L = L_n = L_n(x_1, \dots, x_n) = \left((n-1)! \int_{E_{n-1}} \frac{1}{x\lambda} d\lambda_1 \dots d\lambda_{n-1}\right)^{-1}, \quad (9)$$

and $L' = L(1-x)$.

For $n = 2$ this gives the logarithmic mean of two arguments,

$$L(x_1, x_2) = \frac{x_2 - x_1}{\ln x_2 - \ln x_1} \quad (x_1 \neq x_2), \quad L(x, x) = 1.$$

We note that for $n = 2$, relation (8) is exactly inequality (27) in Rooin's paper [6].

Alzer ([1]) proved another refinement of Sándor inequality, as follows:

$$\frac{1}{H'} - \frac{1}{H} \leq \frac{1}{G'} - \frac{1}{G} \leq \frac{1}{A'} - \frac{1}{A} \quad (10)$$

In [2] we have introduced a new mean $J = J_n$ and deduced a new refinement of the Wang-Wang inequality:

$$\frac{H}{H'} \leq \frac{J}{J'} \leq \frac{G}{G'} \tag{11}$$

We note that in a recent paper, Neuman and Sándor [4] have proved the following strong improvements of Alzer's inequality (10):

$$\frac{1}{H'} - \frac{1}{H} \leq \frac{1}{J'} - \frac{1}{J} \leq \frac{1}{G'} - \frac{1}{G} \leq \frac{1}{I'} - \frac{1}{I} \leq \frac{1}{A'} - \frac{1}{A} \tag{12}$$

(where $J' = J(1-x)$ etc.).

## 2. New means and Ky Fan type inequalities

### 2.1

The results obtained by J. Rooin [6] are based essentially on the following

**Lemma 1.** *Let $f$ be a convex function defined on a convex set $C$, and let $x_i \in C$, $1 \leq i \leq n$. Define $F : [0,1] \to \mathbb{R}$ by*

$$F(t) = \frac{1}{n}\sum_{i=1}^{n} f[(1-t)x_i + tx_{n+1-i}], \quad t \in [0,1].$$

*Then*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq F(t) \leq \frac{f(x_1) + \cdots + f(x_n)}{n},$$

*and the similar double inequality holds for $\int_0^1 F(t)dt$.*

**Proof.** By the definition of convexity, one has

$$f[(1-t)x_i + tx_{n+1-i}] \leq (1-t)f(x_i) + tf(x_{n+1-i}),$$

and after summation, remarking that

$$\sum_{i=1}^{n}[f(x_{n+1-i}) - f(x_i)] = 0,$$

we get the right-side inequality. On the other hand, by Jensen's discrete inequality for convex functions,

$$F(t) \geq f\left(\frac{1}{n}\sum_{i=1}^{n}[(1-t)x_i + tx_{n+1-i}]\right) = f\left(\frac{x_1+\cdots+x_n}{n}\right),$$

giving the left-side inequality. By integrating on $[0,1]$, clearly the same result holds true.

**2.2**

Now define the following mean of $n$ arguments:

$$K = K_n = K_n(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n} I(x_i, x_{n+1-i})\right)^{1/n} \tag{13}$$

Letting $f(x) = -\ln x$ for $x \in (0,+\infty)$, and remarking that

$$\int_0^1 \ln[(1-t)a + tb]dt = \ln I(a,b),$$

Lemma 1 gives the following new refinement of the arithmetic-geometric inequality:

$$G \leq K \leq A, \tag{14}$$

which holds true for any $x_i > 0$ $(i = \overline{1,n})$.

Selecting $f(x) = \ln \dfrac{1-x}{x}$ for $C = \left(0, \dfrac{1}{2}\right]$, and remarking that

$$\int_0^1 \ln\{1 - [(1-t)a + tb]\}dt = \ln I(x_1', x_2') = \ln I'(x_1, x_2),$$

we get the following Ky Fan-type inequality:

$$\frac{G}{G'} \leq \frac{K}{K'} \leq \frac{A}{A'} \tag{15}$$

This is essentially inequality (13) in [6] (discovered independently by the author).

**2.3**

Let now $f(x) = \dfrac{1}{x}$ for $x \in (0, \infty)$. Since $f$ is convex, and

$$\int_0^1 \frac{1}{(1-t)a+tb}dt = \frac{1}{L(a,b)},$$

Lemma 1 gives

$$H \leq R \leq A, \tag{16}$$

where

$$R = R_n = R_n(x_1, \ldots, x_n) = n / \sum_{i=1}^{n} \frac{1}{L(x_i, x_{n+1-i})} \tag{17}$$

This is a refinement - involving the new mean $R$ - of the harmonic-arithmetic inequality.

Letting $f(x) = \dfrac{1}{x} - \dfrac{1}{1-x}$ for $x \in \left(0, \dfrac{1}{2}\right]$, the above arguments imply the relations

$$\frac{1}{A} - \frac{1}{A'} \leq \frac{1}{R} - \frac{1}{R'} \leq \frac{1}{H} - \frac{1}{H'}, \tag{18}$$

where $x_i \in \left(0, \dfrac{1}{2}\right]$, and $R' = R'_n = R_n(1-x_1, \ldots, 1-x_n)$. Relation (18) coincides essentially with (26) of Rooin's paper [6].

**2.4**

Let

$$S = S_n(x_1, \ldots, x_n) = (x_1^{x_1} \ldots x_n^{x_n})^{1/(x_1+\cdots+x_n)} \tag{19}$$

For $n = 2$, this mean has been extensively studied e.g. in [8], [9], [3]. Applying the Jensen inequality for the convex function $f(x) = x \ln x$ $(x > 0)$, we get $A \leq S$. On the other hand, remarking that $S$ is a weighted geometric mean of $x_1, \ldots, x_n$ with weights

$$\alpha_1 = x_1/(x_1 + \cdots + x_n), \ldots, \alpha_n = x_n/(x_1 + \cdots + x_n),$$

by applying the weighted geometric-arithmetic inequality

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \le \alpha_1 x_1 + \dots + \alpha_n x_n,$$

we can deduce $S \le Q$, where

$$Q = Q_n(x_1, \dots, x_n) = \frac{x_1^2 + \dots + x_n^2}{x_1 + \dots + x_n}.$$

Therefore, we have proved that

$$A \le S \le Q \tag{20}$$

In [8] it is shown that

$$\int_a^b x \ln x dx = \frac{b^2 - a^2}{4} \ln I(a^2, b^2) \tag{21}$$

Denote $J(a,b) = (I(a^2,b^2))^{1/2}$ and put $J'(a,b) = J(1-a, 1-b)$. By applying Lemma 1, we get

$$A \le T \le S, \tag{22}$$

where the mean $T$ is defined by

$$T = T_n(x_1, \dots, x_n) = \left[ \prod_{i=1}^{n} (J(x_i, x_{n+1-i}))^{\frac{A(x_i, x_{n+1-i})}{n}} \right]^{1/A} \tag{23}$$

Letting now

$$f(x) = x \ln x - (1-x) \ln(1-x), \quad x \in \left(0, \frac{1}{2}\right],$$

by

$$f''(x) = \frac{1-2x}{x(1-x)} \ge 0$$

we can state that $f$ is convex, so by Lemma 1 and by (21) we can write, for $x_i \in \left(0, \frac{1}{2}\right]$:

$$A^A / A'^{A'} \le T^A / T'^{A'} \le S^A / S'^{A'}, \tag{24}$$

where the mean $T$ is defined by (23), while $T' = T(1-x)$. Since for $n = 2$, $T \equiv J$, for means of two arguments (24) gives a Ky Fan-type inequality involving $A, I, S$.

**2.5**

Relation (23) shows that $T$ is a generalization of the mean $J$ to $n$ arguments. In what follows we shall introduce another generalization, provided by the formula

$$U = U_n(x_1, \ldots, x_n)$$

$$= \left\{ \exp\left( (n-1)! \int_{E_{n-1}} (x\lambda)\ln(x\lambda) d\lambda_1 \ldots d\lambda_{n-1} \right) \right\}^{1/A} \tag{25}$$

Here the notations are as in the Introduction. Since, by (21),

$$\int_0^1 [(1-t)a + tb] \ln[(1-t)a+tb] dt = \frac{1}{b-a} \int_a^b x \ln x dx$$

$$= \frac{A}{2} \ln I(a^2, b^2) = \ln J^A,$$

for $n = 2$, we have $U \equiv J$, thus $U$ is indeed another generalization of the mean $J$.

Now, the following result is due to E. Neuman (see e.g. [2]).

**Lemma 2.** *Let $K$ be an interval containing $x_1, \ldots, x_n$, and suppose that $f : K \to \mathbb{R}$ is convex. Then*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le (n-1)! \int_{E_{n-1}} f(\lambda x) d\lambda_1 \ldots d\lambda_{n-1}$$

$$\le \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

Letting $K = \left(0, \frac{1}{2}\right]$, and $f(x) = x \ln x - (1-x)\ln(1-x)$ in Lemma 2, we can deduce for $x_i \in \left(0, \frac{1}{2}\right]$

$$A^A / A'^{A'} \le U^A / U'^{A'} \le S^A / S'^{A'} \tag{26}$$

Remark that for $n = 2$, inequalities (24) and (26) reduce to the same inequality, as in that case one has $T = J = U$. The mean $U$ separates also $A$ and $S$, since applying Lemma 2 for $f(x) = x \ln x$ $(x > 0)$, we have

$$A \leq U \leq S. \tag{27}$$

There remains an Open Problem, namely the comparability of the above defined means $T$ and $U$ for $n > 2$. Also, the connections of these means to $K$ and $R$, introduced in the preceding sections.

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## 4.8 On common generalizations of some inequalities

In what follows we shall prove a double inequality, which offers a common proof of many famous inequalities. For example, the arithmetic mean – geometric mean – harmonic mean inequality, the Ky Fan or the Wang-Wang inequalities will be consequences (see e.g. [4]).

**Theorem.** *Let $I \subset \mathbb{R}$ be an interval and $\overset{\circ}{I}$ its interior. Suppose that $f : I \to \mathbb{R}$ is continuous and differentiable on $\overset{\circ}{I}$, and that the derivative $f'$ is monotone increasing. Then for any positive integer $n$ and $x_i \in \overset{\circ}{I}$ $(i = \overline{1, n})$ one has*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{\sum\limits_{i=1}^{n} x_i f'(x_i)}{\sum\limits_{i=1}^{n} f'(x_i)}\right), \tag{1}$$

*where on the right side of (1) one assumes that $\sum\limits_{i=1}^{n} f'(x_i) \neq 0$.*

**Proof.** The left side of (1) is nothing else than the classical Jensen inequality. For its proof, for any $x, y \in I$ apply the Lagrange mean-value theorem on the interval $[x, y]$:

$$f(y) - f(x) = (y - x)f'(\xi),$$

where $\xi$ lies between $x$ and $y$. If $y > x$, then $x < \xi < y$ and by monotonicity of $f : f'(\xi) \leq f'(y)$, giving

$$f(y) \leq f(x) + (y - x)f'(y). \tag{2}$$

But inequality (2) holds true for $y < x$, too. Indeed, then we have

$$(y - x)f'(\xi) \leq (y - x)f'(y),$$

since $y-x<0$ and $f'(\xi)\ge f'(y)$. Thus inequality (2) holds true for any $x\in I$, $y\in \overset{\circ}{I}$. Let

$$x=x_i,\ y=\frac{1}{n}\sum_{i=1}^{n}x_i\ (i=\overline{1,n},\ x_i\in \overset{\circ}{I}).$$

Then, by (2) one has

$$f(y)\le f(x_i)+(y-x_i)f'(y_i),\ i=\overline{1,n}. \tag{3}$$

After summation, from (3) we get the left side of (1), as

$$\sum_{i=1}^{n}(y-x_i)f'(y)=f'(y)\left(ny-\sum_{i=1}^{n}x_i\right)=0.$$

The right side of (1) can be proved in a similar way, by first remarking that

$$f(y)\ge f(x)+(y-x)f'(x),\ (y\in I,\ x\in \overset{\circ}{I}). \tag{4}$$

This can be proved in a similar manner to (3). Let now

$$x=x_i,\ y=\frac{\sum_{i=1}^{n}x_if'(x_i)}{\sum_{i=1}^{n}f'(x_i)}$$

in (4). Since

$$\sum_{i=1}^{n}(y-x_i)f'(x_i)=0,$$

after summation we get the right side of (1).

**Remarks.** 1) If one assumes $f$ to be convex function (in place of monotonicity of $f'$), in relation (2) in place of $f'(y)$ we will take $f'_-(y)$ (i.e. the left side derivative) and $f'_+(x)$ in place of $f'(x)$ in (4). Then in (1) the left side remains the same, while the right side appears $f'_+(x_i)$ in place of $f'(x_i)$.

2) From the proof we get also that if $f'$ is strictly increasing (strictly convex), then equalities (in the left side of (1), or the right side) can occur only if $x_1=x_2=\ldots=x_n$.

## Applications

1) Let $I = (0, \infty)$, $f(x) = -\ln x$. Since

$$f'(x) = -\frac{1}{x}, \quad f''(x) = \frac{1}{x^2} > 0,$$

$f'$ will be strictly increasing. By computations we get from (1):

$$H_n \le G_n \le A_n, \tag{5}$$

where

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n = \sqrt[n]{\prod_{i=1}^{n} x_i}, \quad H_n = \frac{n}{\displaystyle\sum_{i=1}^{n}\frac{1}{x_i}}$$

are the arithmetic, geometric, resp. harmonic means of $x_i$ $(i = \overline{1,n})$.

2) Let $\left(0, \frac{1}{2}\right] \subset I$, and put $f(x) = \ln(1-x) - \ln x$. Since

$$f''(x) = \frac{1-2x}{x^2(1-x)^2} \ge 0,$$

we get that $f'$ is strictly increasing. Let us introduce the notations

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-x_i), \quad G'_n = \sqrt[n]{\prod_{i=1}^{n}(1-x_i)}, \quad H'_n = \frac{n}{\displaystyle\sum_{i=1}^{n}\frac{1}{1-x_i}},$$

where $x_i \in (0,1)$. By simple computations, we get from (1) the double inequality

$$\frac{H_n}{H'_n} \le \frac{G_n}{G'_n} \le \frac{A_n}{A'_n}, \tag{6}$$

where $A_n = A_n(x_i)$, $A'_n = A'_n(x_i)$, etc., and $x_i \in \left(0, \frac{1}{2}\right]$.

The right side inequality of (6) is known as the famous Ky Fan inequality [1], while the left side is the Wang-Wang inequality [4].

**Remark.** It is not difficult that the weighted version of (1) holds true, too, so the above inequalities (5) and (6) are valid also in the weighted case. Other proofs and refinements of these inequalities may be found in papers [2] and [3].

This paper is an English version of [5].

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# Chapter 5

# Stolarsky and Gini means

"The ideas chosen by my unconscious are those which reach my consciousness, and I see that they are those which agree with my aesthetic sense."

(J. Hadamard)

"In both theorems... there is a very high degree of unexpectedness, combined with inevitability and economy."

(G.H. Hardy)

## 5.1 On reverse Stolarsky type inequalities for means

### 1. Introduction

Let $a, b > 0$ be positive real numbers. The logarithmic, resp. identric means of $a$ and $b$ are defined by

$$L = L(a, b) = \frac{b - a}{\log b - \log a} \text{ for } a \neq b; \quad L(a, a) = a \tag{1}$$

and

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a \tag{2}$$

Let

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k} \tag{3}$$

denote the power mean of order $k$, where $k \neq 0$ is a real number. Denote

$$A = A_1(a,b) = \frac{a+b}{2} \quad \text{and} \quad G = G(a,b) = \lim_{k \to 0} A_k(a,b) = \sqrt{ab} \tag{4}$$

the arithmetic, resp. geometric means of $a$ and $b$.

The means (1)-(4) have been extensively investigated. In particular, many remarkable inequalities or identities for these means have been proved. For a survey of results, see e.g. [1-3], [5], [7], [12], [13-15]. Consider also the weighted geometric mean $S$ of $a$ and $b$, the weights being $a/(a+b)$ and $b/(a+b)$:

$$S = S(a,b) = a^{a/(a+b)} b^{b/(a+b)} \tag{5}$$

As we have the identity (see [9])

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}, \tag{6}$$

the mean $S$ is strongly related to the identric mean. For properties of this mean, see e.g. [7], [8], [12].

Finally, the Heronian mean will be denoted by $He$ (see [3]), where

$$He = He(a,b) = \frac{a + \sqrt{ab} + b}{3} = \frac{2A + G}{3} \tag{7}$$

Now, we quote some inequalities of interest in what follows.

In 1980 K.B. Stolarsky [16] proved that for all $a \neq b$ one has

$$A_{2/3} < I, \tag{8}$$

and that $2/3$ is optimal; i.e. the constant $2/3$ cannot be replaced by a greater constant $t > 2/3$ such that $A_t < I$ for all $a \neq b$.

In 1991 J. Sándor [8] proved that for $a \neq b$

$$He < I, \tag{9}$$

while in [12] it is shown that

$$A_2 < S \tag{10}$$

both being sharp in certain sense.

Clearly one has

$$L < I < A \tag{11}$$

and, as a counterpart of the right side of (11), in [10] it is shown that

$$\frac{2}{e}A < I, \tag{12}$$

while in [6] that

$$I < \frac{2}{e}(A + G) \tag{13}$$

The aim of this paper is to deduce certain reverses of type (12) for the inequalities (8)-(10). One of these relations will provide an improvement of inequality (13) and in fact the method will offer a new proof for inequalities (8)-(10).

## 2. Main results

The first theorem is well-known, but here we will give a new proof, which shows that the involved constants are optimal:

**Theorem 1.** *For all $a \neq b$ one has*

$$I < A < \frac{e}{2}I, \tag{14}$$

*where the constants* $1$ *and* $\frac{e}{2}$ *are best possible.*

**Proof.** Put $x = b/a > 1$, and consider the function

$$f_1(x) = \frac{A(x, 1)}{I(x, 1)}$$

An easy computation implies that the logarithmic derivative of $f_1(x)$ is

$$\frac{f_1'(x)}{f_1(x)} = \frac{2\log x}{(x-1)^2(x+1)}\left(\frac{x+1}{2} - \frac{x-1}{\log x}\right)$$

As

$$\frac{x+1}{2} - \frac{x-1}{\log x} = A(x,1) - L(x,1) > 0$$

by the weaker form of (11), we get $f_1'(x) > 0$ for $x > 1$. This shows that $f_1(x)$ is strictly increasing for $x > 1$, implying

$$f_1(x) > \lim_{x\to 1} f_1(x) = 1 \quad \text{and} \quad f_1(x) < \lim_{x\to\infty} f_1(x) = \frac{2}{e}.$$

This proves inequality (14), by the homogeneity of $A$ and $I$. Since the function $f_1(x)$ is continuous for $x > 1$, it is immediate that the constants 1 and $2/e$ cannot be improved.

**Theorem 2.** *For all $a \neq b$ one has*

$$A_{2/3} < I < \frac{2\sqrt{2}}{e} A_{2/3}, \tag{15}$$

*where the constants* 1 *and* $\dfrac{2\sqrt{2}}{e}$ *are best possible.*

**Proof.** Put $x^3 = b/a > 1$, and consider the application

$$f_2(x) = \frac{A_{2/3}(x^3,1)}{I(x^3,1)}$$

We have

$$\frac{f_2'(x)}{f_2(x)} = \frac{3x^2}{(x^3-1)^2} k(x),$$

where

$$k(x) = 3\log x - \frac{(x+1)(x^3-1)}{x(x^2+1)}.$$

Letting $t = x^3$ in the following inequality (see [4], p. 272):

$$\frac{\log t}{t-1} < \frac{1+t^{1/3}}{t+t^{1/3}}$$

we obtain $k(x) < 0$ for $x > 1$. Thus the application $f_2$ is strictly decreasing. As

$$\lim_{x\to 1} f_2(x) = 1, \quad \lim_{x\to\infty} f_2(x) = \frac{2\sqrt{2}}{e},$$

the result follows. The function $f_2(x)$ being continuous and strictly decreasing, we easily get the optimality of the constants 1 and $2\sqrt{2}/e$.

**Theorem 3.** *For all $a \neq b$ one has*

$$He < A_{2/3} < \frac{3}{2\sqrt{2}} He, \tag{16}$$

*where the constants are best possible.*

**Proof.** As above, let $x^3 = b/a > 1$, and let

$$f_3(x) = \frac{He(x^3, 1)}{A_{2/3}(x^3, 1)}$$

Logarithmic differentiation gives

$$\frac{f_3'(x)}{f_3(x)} = -\frac{3}{2} \cdot \frac{x^{1/2}(x-1)(x^{1/2}-1)^2}{2(x^3 + x^{3/2} + 1)(x^2+1)} < 0,$$

so $f_3(x)$ is strictly decreasing for $x > 1$. The result follows.

**Corollary.** *For all $a \neq b$ one has*

$$He < I < \frac{3}{e} He, \tag{17}$$

*where the constants* 1 *and* $\frac{3}{e}$ *are best possible.*

**Proof.** Inequality (17) follows by a combination of relations (15) and (16). As $He/I = (He/A_{2/3})(A_{2/3}/I)$, and the product of two positive strictly decreasing functions is also strictly decreasing, the sharpness of (17) follows.

**Remark 1.** It is not difficult to see that

$$\sqrt{2} \cdot A_{2/3} > A + G, \tag{18}$$

which shows that the left side of (15) offers an improvement of inequality (13).

**Theorem 4.** *For all $a \neq b$ one has*

$$A_2 < S < sqrt(2) A_2, \tag{19}$$

*where the constants* 1 *and* $sqrt(2)$ *are best possible.*

**Proof.** Put $x = b/a > 1$, and let

$$f_4(x) = \frac{A_2(x,1)}{S(x,1)} = \left(\frac{x^2+1}{2}\right)^{1/2} \cdot \frac{1}{x^{x/(x+1)}}$$

Hence

$$\frac{f_4'(x)}{f_4(x)} = \frac{h(x)}{(x+1)^2(x^2+1)},$$

where $h(x) = x^2 - 1 - (x^2+1)\log x$.

Since

$$\frac{x-1}{\log x} = L(x,1) < A(x,1) = \frac{x+1}{2} < \frac{x^2+1}{x+1},$$

we get $h(x) < 0$ for $x > 1$, yielding that $f_4(x)$ is a strictly decreasing function for $x > 1$. As $\lim\limits_{x\to 1} f_4(x) = 1$ and $\lim\limits_{x\to\infty} f_4(x) = \dfrac{1}{sqrt(2)}$, relation (18) follows.

**Remark 2.** There are known also improvements of other types to the Stolarsky inequality (8). For example, in [5] it is shown that for $a \neq b$

$$A_{2/3} < \sqrt{I_{5/6} \cdot I_{7/6}} < I,$$

where $I_t = I_t(a,b) = (I(a^t, b^t))^{1/t}$ $(t \neq 0)$.

For strong inequalities connecting the mean $I$ to other means (e.g. the arithmetic-geometric mean of Gauss), see [11]. For connections between $L$, $I$ and a Seiffert mean, see [14].

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## 5.2 Inequalities for certain means in two arguments

### 1

The means in two arguments are special [3] and have been intensively investigated. We mention here the geometric, logarithmic, identric and exponential, arithmetic, etc. means of two numbers. For $b > a > 0$ let

$$G = G(a,b) = \sqrt{ab},$$

$$L = L(a,b) = \frac{b-a}{\ln b - \ln a},$$

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)},$$

$$A = A(a,b) = \frac{a+b}{2}.$$

The following relations are known:

$$G < L < I < A \tag{1}$$

$$A + L < 2I \quad ([6]) \tag{2}$$

$$L + I < A + G \quad ([2]) \tag{3}$$

$$2A + G < 3I \quad ([7]) \tag{4}$$

$$GI < L^2 \quad ([1]) \tag{5}$$

$$2A < eI \quad ([8]) \tag{6}$$

$$L(a^2, b^2) = A(a,b) \cdot L(a,b) \tag{7}$$

$$\log \frac{A}{G} = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{b-a}{b+a}\right)^{2k} \quad ([9]) \tag{8}$$

$$\log \frac{I}{G} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{b-a}{b+a}\right)^{2k} \quad ([9]) \tag{9}$$

$$\log \frac{I}{G} = \frac{A}{L} - 1 \quad ([11]). \tag{10}$$

Consider also the weighted geometric mean $S$ of $a$ and $b$, the weights being

$$\frac{a}{a+b} \quad \text{and} \quad \frac{b}{a+b}: \quad S = S(a,b) = a^{a/(a+b)} \cdot b^{b/(a+b)}$$

Some properties of $S$ have been discussed in [6], [9] and [4]. Other relations connecting $G, L, I, A, S$ will be presented in this paper.

## 2

By comparing $S$ with the corresponding weighted harmonic and arithmetic means we obtain

$$A(a,b)S(a,b) < \frac{a^2+b^2}{a+b}. \tag{11}$$

Concerning the first inequality, the following relations are also true:

$$A^2(a,b) < I(a^2,b^2) < S^2(a,b) \quad ([6],\ [9]) \tag{12}$$

$$\log \frac{S}{A} = \sum_{k=1}^{\infty} \frac{1}{2k(k-1)} \left(\frac{b-a}{b+a}\right)^{2k} \quad ([4]). \tag{13}$$

From (8) and (13) we deduce the following relation similar to (8) and (9):

$$\log \frac{S}{G} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{b-a}{b+a}\right)^{2k}. \tag{14}$$

From (8) and (13) we deduce the following relation similar to (8) and (9):

$$\log \frac{S}{I} = 1 - \frac{G^2}{AL} \quad ([4]) \tag{15}$$

$$I(a^2,b^2) = S(a,b) \cdot I(a,b) \quad ([9]). \tag{16}$$

**3**

From (12) and (16) we deduce $A^2 < SI$. From (8) and (13) it follows $SG < A^2$. Thus we have obtained:

$$\frac{A^2}{I} < S < \frac{A^2}{G} \quad ([4]). \tag{17}$$

The following theorem contains certain refinements.

**Theorem 1.** *The following inequalities are valid:*

$$\frac{A^2}{I} < \frac{4A^2 - G^2}{3I} < S < \frac{A^4}{I^3} < \frac{A^2}{G} \tag{18}$$

$$AL + SI < 2A^2 < S^2 + G^2 \tag{19}$$

$$\frac{4A^2 - 2G^2}{e} < SI < \frac{A^2 L^2}{G^2}. \tag{20}$$

**Proof.** Apply (4) with $a^2, b^2$ instead of $a, b$. Then use (16), it follows that

$$SI > \frac{4A^2 - G^2}{3} > A^2.$$

On the other hand,

$$I(a^2, b^2) < \frac{A^4}{I^2} \quad ([9]),$$

this means that

$$S < \frac{A^4}{I^3}.$$

From (4) we deduce $I^3 > A^2 G$, hence $\frac{A^4}{I^3} < \frac{A^2}{G}$ and this proves (18).

The left side of (19) follows from (7) and (16) with the application of $a^2$ and $b^2$ in place of $a$ and $b$, respectively. In order to prove the right hand side of (19) divide each term by $a^2 < b^2$ and denote $t := \frac{b}{a} > 1$. The inequality to be proved becomes

$$2t^{2t/(t+1)} > t^2 + 1 \quad (t > 1) \tag{21}$$

Let

$$f(t) = \ln 2 + \frac{2t}{t+1} \ln t - \ln(t^2 + 1).$$

Simple calculations give

$$f'(t) = 2(t+1)^{-2}\left[\ln t - \frac{t^2-1}{t^2+1}\right] > 0$$

since by

$$L(t,1) < A(t,1) < \frac{t^2+1}{t+1}$$

it results

$$\ln t > \frac{t^2-1}{t^2+1}.$$

Thus, via $f(1) = 0$, we can deduce that $f(t) > 0$ for $t > 1$, yielding (21).

Finally, by using (5) and (6) with $a^2, b^2$ instead of $a, b$, it is easy to prove (20). $\square$

**Remark 1.** By remarking that

$$2A^2 - G^2 = A(a^2, b^2),$$

the right hand side of (19) can also be written as

$$A(a^2, b^2) < S^2(a, b) \tag{22}$$

improving, in view of (1), inequality (12).

## 4

Let $M$ be one of the means $L, I, A$ or $S$. Denote

$$M(t) = (M(a^t, b^t))^{1/t}$$

if $t \neq 0$; $M(0) = G(a, b)$. By examining each case, it is not difficult to verify that

$$M(-t)M(t) = M^2(0),\ t \in \mathbb{R}. \tag{23}$$

**Theorem 2.** *(i) The function $M(t)$ is increasing and continuous on $\mathbb{R}$.*

*(ii) For $t > 0$ we have*

$$S(2t) > S(t) > A(2t) > I(2t) > A(t) > I(t) > L(2t) > L(t). \tag{24}$$

*If $t < 0$, the inequalities in (24) are reversed.*

**Proof.** It is well known that (i) holds for $M = A$. It is a matter of calculus to prove (i) for $M = I$ and $M = S$, as well as the continuity of $L(t)$. It was proved in [10] that $L(t) > L(1)$ for $t > 1$. By using this fact and (23), it can be proved that $L(t)$ is an increasing function.

Let now $t > 0$. By (22) one has $S(1) > A(2)$ which implies $S(t) < A(2t)$. From (16) and $SI > A^2$ (see (17)) we can obtain $I(2) > A(1)$, giving $I(2t) > A(t)$. From (2) it follows $I^2 > AL$, which implies that $I(1) > L(2)$, and this in turn implies $I(t) > L(2t)$. By (1) we have $A(t) > I(t)$. Since $S(t)$ and $I(t)$ are increasing, (24) is proved. Due to (23), all the inequalities of (24) are reversed for $t < 0$.

## 5

Now we deal with series representations like (8), (9), (13), (14), deriving inequalities from them.

**Theorem 3.** *One has*

$$\left(\frac{S}{A}\right)^2 < \left(\frac{I}{G}\right)^3 \tag{25}$$

$$\frac{A^2 - G^2}{A^2} < \ln\frac{S}{G} < \frac{A^2 - G^2}{AG}. \tag{26}$$

**Proof.** From (9) and (13) and the elementary inequality

$$\frac{1}{2k(2k-1)} \le \frac{3}{2} \cdot \frac{1}{2k+1}$$

with equality only for $k = 1$, we get relation (25).

For (26) remark first that

$$\ln \frac{S}{G} = \frac{(b-a)(\ln b - \ln a)}{2(b+a)} < \frac{(b-a)^2}{2(a+b)\sqrt{ab}}$$
$$= \frac{A^2 - G^2}{AG}, \text{ since } L > G \text{ (see (1))}$$

On the other hand, relation (14) gives us

$$\ln \frac{S}{G} > \left(\frac{b-a}{b+a}\right)^2 = \frac{A^2 - G^2}{A^2}. \qquad \square$$

## 6

Inequality (4) can be written also as

$$\frac{I - G}{A - I} > 2.$$

A similar result holds for the mean $S$. More precisely, the following result is true:

**Theorem 4.**

$$\frac{S - G}{S - A} > \sqrt{2} \tag{27}$$

Before proving this theorem, we need an auxiliary result, interesting in itself:

**Lemma.** *For the logarithmic, harmonic, and geometric means the positive numbers $0 < a < b$ holds the inequality*

$$L + H > \sqrt{2}G \tag{28}$$

**Proof.** We have $H = \dfrac{G^2}{A}$. Denoting $x := \dfrac{\sqrt{b}}{\sqrt{a}} > 1$, after certain elementary transformations (28) becomes equivalent with

$$\ln x < \frac{x^4 - 1}{2\sqrt{2}x\left(x^2 - \sqrt{2}x + 1\right)}, \quad x > 1. \tag{29}$$

Consider the function

$$g(x) = \frac{\frac{\sqrt{2}}{4}(x^4 - 1)}{x\left(x^2 - \sqrt{2}x + 1\right) - \ln x}.$$

A simple (but tedious) computation shows that

$$2\sqrt{2}x^2\left(x^2 - \sqrt{2}x + 1\right)^2 (g'(x))$$

$$= x^6 - 4\sqrt{2}x^5 + 11x^4 - 8\sqrt{2}x^3 + 11x^2 - 4\sqrt{2}x + 1.$$

The right side of this expression, divided by $x^3$ and with the notation

$$\frac{x+1}{x} = t,$$

becomes

$$(x^3 + 1/x^3) - 4\sqrt{2}(x^2 + 1/x^2) + 11(x + 1/x) - 8\sqrt{2} = t\left(t - 2\sqrt{2}\right)^2.$$

In conclusion, $g(1) = 0$, $g'(x) > 0$ for $x > 0$, thus $g(x) > 0$ for $x > 1$, proving (29), thus (28). $\square$

**Remark 2.** Another inequality connecting the means $L, H$ and $G$, namely

$$\frac{3}{L} < \frac{1}{G} + \frac{2}{H}$$

has been proved in [6] (relation (37)).

**Proof of Theorem 4.** Denote $\frac{b}{a} = x^2$. Then (27) becomes equivalent (after certain simple computations, which we omit) to

$$h(x) := \ln\left(x^2 - \sqrt{2}x + 1\right) - \frac{2x^2}{1+x^2}\ln x - \ln\left(2 - \sqrt{2}\right) > 0 \text{ for all } x > 1.$$

By (28) (i.e. (29)), this implies $h'(x) > 0$ for $x > 1$, and since $h(1) = 0$, we conclude that $h(x) > 0$ for $x > 1$, and this proves (27). $\square$

**Corollary.**

$$2A^2 - G^2 < S^2 < \left(6 + 4\sqrt{2}\right)A^2 - \left(5 + 4\sqrt{2}\right)G^2 \tag{30}$$

**Proof.** The left side inequality is exactly the second part of (19). For the right side remark that (27) written in the form

$$S < \frac{A\sqrt{2} - G}{\sqrt{2} - 1}$$

implies

$$S^2 < \left(2/\left(3 - 2\sqrt{2}\right)\right) A^2 - \left(\left(2\sqrt{2} - 1\right)/\left(3 - 2\sqrt{2}\right)\right) G^2$$

by $A > G$. Now, observe that

$$2/\left(3 - 2\sqrt{2}\right) = 6 + 4\sqrt{2}, \quad \left(2\sqrt{2} - 1\right)/\left(3 - 2\sqrt{2}\right) = 5 + 4\sqrt{2}. \quad \square$$

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## 5.3 A note on certain inequalities for bivariate means

### 1. Introduction

Let $a, b$ be two distinct positive numbers. The power mean of order $k$ of $a$ and $b$ is defined by

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}, \quad k \neq 0$$

and

$$A_0 = \lim_{k \to 0} A_k = \sqrt{ab} = G(a,b).$$

Let $A_1 = A$ denote also the classical arithmetic mean of $a$ and $b$, and

$$He = He(a,b) = \frac{2A+G}{3} = \frac{a+b+\sqrt{ab}}{3}$$

the so-called Heronian mean.

In the recent paper [1] the following results have been proved:

$$A_k(a,b) > a^{1-k} I(a^k, b^k) \text{ for } 0 < k \leq 1,\ b > a \tag{1.1}$$

$$A_k(a,b) < I(a,b) \text{ for } 0 < k \leq \frac{1}{2} \tag{1.2}$$

$$He(a^k, b^k) < A_\beta(a^k, b^k) < \frac{3}{2^{1/\beta}} He(a^k, b^k) \text{ for } k > 0,\ \beta \geq \frac{2}{3} \tag{1.3}$$

and

$$A_k < S < 2^{1/k} \cdot A_k \text{ for } 1 \leq k \leq 2. \tag{1.4}$$

In the proofs of (1.1)-(1.4) the differential calculus has been used. Our aim will be to show that, relations (1.1)-(1.4) are easy consequences of some known results.

## 2. Main results

**Lemma 2.1.** *The function* $f_1(k) = \left(\dfrac{a^k + b^k}{2}\right)^{1/k} = A_k(a, b)$ *is a strictly increasing function of* $k$*; while* $f_2(k) = (a^k + b^k)^{1/k}$ *is a strictly decreasing function of* $k$*. Here* $k$ *runs through the set of real numbers.*

**Proof.** Through these results are essentially known in the mathematical folklore, we shall give here a proof.

Simple computations yield:

$$k^2 \frac{f_1'(k)}{f_1(k)} = \frac{x \ln x + y \ln y}{x + y} - \ln\left(\frac{x + y}{2}\right), \tag{2.1}$$

and

$$k^2 \frac{f_2'(k)}{f_2(k)} = \frac{x \ln x + y \ln y}{x + y} - \ln(x + y), \tag{2.2}$$

where $x = a^k > 0$, $y = b^k > 0$. Since the function $f(x) = x \ln x$ is strictly convex (indeed: $f''(x) = \dfrac{1}{x} > 0$) by

$$f\left(\frac{x + y}{2}\right) < \frac{f(x) + f(y)}{2},$$

relation (2.1) implies $f_1'(k) > 0$. Since the function $t \to \ln t$ is strictly increasing, one has $\ln x < \ln(x + y)$ and $\ln y < \ln(x + y)$; so

$$x \ln x + y \ln y < (x + y) \ln(x + y),$$

so relation (2.2) implies that $f_2'(t) < 0$. These prove the stated monotonicity properties.

**Proof of (1.1).** By the known inequality $I < A$ we have

$$I(a^k, b^k) < A(a^k, b^k) = \frac{a^k + b^k}{2}.$$

Now

$$\frac{a^k + b^k}{2} \leq a^{k-1} \left(\frac{a^k + b^k}{2}\right)^{1/k}$$

is equivalent with (for $1-k>0$)

$$\left(\frac{a^k+b^k}{2}\right)^{1/k}>a \text{ or } a^k+b^k>2a^k,$$

which is true for $b>a$. For $k=1$ the inequality becomes $I<A$.

**Proof of (1.2).** Since $A_k$ is strictly increasing, one has

$$A_k \le A_{1/2}=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2=\frac{A+G}{2}<I,$$

by a known result (see [3]) of the author:

$$I>\frac{2A+G}{3}>\frac{A+G}{2}. \tag{2.3}$$

**Proof of (1.3).** By the inequality $He<A_{2/3}$ (see [2]) one has

$$He(a^k,b^k)<A_{2/3}(a^k,b^k)\le A_\beta(a^k,b^k),$$

by the first part of Lemma 2.1.

Now, $2^{1/\beta}A_\beta(a^k,b^k)\le 2^{3/2}(a^k,b^k)$ by the second part of Lemma 2.1, and $A_{2/3}(a^k,b^k)<\dfrac{3}{2\sqrt{2}}He(a^k,b^k)$, by (see [2])

$$A_{2/3}<\frac{3}{2\sqrt{2}}He. \tag{2.4}$$

Since $2^{3/2}=2\sqrt{2}$, inequality (1.3) follows.

**Proof of (1.4).** In [2] it was proved that

$$A_2<S<\sqrt{2}A_2. \tag{2.5}$$

Now, by Lemma 2.1 one has, as $k\le 2$ that $A_k\le A_2<S$ and $\sqrt{2}A_2\le 2^{1/k}A_k$. Thus, by (2.5), relation (1.4) follows. We note that condition $1\le k$ is not necessary.

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# 5.4 A note on logarithmically completely monotonic ratios of certain mean values

## 1. Introduction

A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic (c.m. for short), if $f$ has derivatives of all orders and satisfies

$$(-1)^n \cdot f^{(n)}(x) \geq 0 \text{ for all } x > 0 \text{ and } n = 0, 1, 2, \ldots \tag{1}$$

J. Dubourdieu [3] pointed out that, if a non-constant function $f$ is c.m., then strict inequality holds in (1). It is known (and called as Bernstein theorem) that $f$ is c.m. iff $f$ can be represented as

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\mu(t), \tag{2}$$

where $\mu$ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [11]).

Completely monotonic functions appear naturally in many fields, like, for example, probability theory and potential theory. The main properties of these functions are given in [11]. We also refer to [4], [1], [2], where detailed lists of references can be found.

Let $a, b > 0$ be two positive real numbers. The power mean of order $k \in \mathbb{R} \setminus \{0\}$ of $a$ and $b$ is defined by

$$A_k = A_k(a, b) = \left( \frac{a^k + b^k}{2} \right)^{1/k}.$$

Denote

$$A = A_1(a, b) = \frac{a + b}{2}, \quad G = G(a, b) = A_0(a, b) = \lim_{k \to \infty} A_k(a, b) = \sqrt{ab}$$

the arithmetic, resp. geometric means of $a$ and $b$.

The identric, resp. logarithmic means of $a$ and $b$ are defined by

$$I = I(a,b) = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)} \text{ for } a \neq b; \quad I(a,a) = a;$$

and

$$L = L(a,b) = \frac{b-a}{\log b - \log a} \text{ for } a \neq b; \quad L(a,a) = a.$$

Consider also the weighted geometric mean $S$ of $a$ and $b$, the weights being $a/(a+b)$ and $b/(a+b)$ :

$$S = S(a,b) = a^{a/(a+b)} \cdot b^{b/(a+b)}.$$

As one has the identity (see [6])

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)},$$

the mean $S$ is connected with the identric mean $I$.

Other means which occur in this paper are

$$H = H(a,b) = A_{-1}(a,b) = \frac{2ab}{a+b}, \quad Q = Q(a,b) = A_2(a,b) = \sqrt{\frac{a^2+b^2}{2}},$$

as well as Seiffert's mean (see [10], [9])

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)} \text{ for } a \neq b, \quad P(a,a) = a.$$

In the paper [2] C.-P. Chen and F. Qi have considered the ratios

a) $\dfrac{A}{I}(x,x+1)$, b) $\dfrac{A}{G}(x,x+1)$, c) $\dfrac{A}{H}(x,x+1)$,

d) $\dfrac{I}{G}(x,x+1)$, e) $\dfrac{I}{H}(x,x+1)$, f) $\dfrac{G}{H}(x,x+1)$,

g) $\dfrac{A}{L}(x,x+1)$,

where

$$\frac{A}{I}(x,x+1)=\frac{A(x,x+1)}{I(x,x+1)}$$

etc., and proved that the logarithms of the ratios $a)-f)$ are c.m., while the ratio from $g)$ is c.m.

In [2] the authors call a function $f$ as logarithmically completely monotonic (l.c.m. for short) if the function $g=\log f$ is c.m. They notice that they proved earlier (in 2004) that if $f$ is l.c.m., then it is also c.m. We note that this result has been proved already in paper [4]:

**Lemma 1.** *If $f$ is l.c.m, then it is also c.m.*

The following basic property is well-known (see e.g. [4]):

**Lemma 2.** *If $a>0$ and $f$ is c.m., then $a\cdot f$ is c.m., too. The sum and the product of two c.m. functions is c.m., too.*

**Corollary 1.** *If $k$ is a positive integer and $f$ is c.m., then the function $f^k$ is c.m., too.*

Indeed, it follows by induction from Lemma 2 that, the product of a finite number of c.m. functions is c.m., too.

Particularly, when there are $k$ equal functions, Corollary 1 follows.

The aim of this note is to offer new proofs for more general results than in [2], and involving also the means $S,P,Q$.

## 2. Main results

First we note that, as one has the identity

$$H=\frac{G^2}{A},$$

we get immediately

$$\frac{A}{H}=\frac{A^2}{G^2},\quad \frac{G}{H}=\frac{A}{G}$$

so that as

$$\log\frac{A}{H}=2\log\frac{A}{G}\ \text{ and }\ \log\frac{G}{H}=\log\frac{A}{G},$$

by Lemma 2 the ratios $c$) and $f$) may be reduced to the ratio $a$).

Similarly, as

$$\frac{I}{H} = \frac{A}{G} \cdot \frac{I}{G},$$

the study of ratio $e$) follows (based again on Lemma 2) from the ratios $b$) and $d$).

As one has

$$\frac{A}{G} = \frac{A}{I} \cdot \frac{I}{G},$$

it will be sufficient to consider the ratios $a$) and $d$).

Therefore, in Theorem 1 of [2] we should prove only that $\frac{A}{I}(x, x+1)$ and $\frac{I}{G}(x, x+1)$ are l.c.m., and $\frac{A}{L}(x, x+1)$ is c.m.

A more general result is contained in the following:

**Theorem 1.** *For any $a > 0$ (fixed), the ratios*

$$\frac{A}{I}(x, x+a) \text{ and } \frac{I}{G}(x, x+a)$$

*are l.c.m., and the ratio*

$$\frac{A}{L}(x, x+a)$$

*is c.m. function.*

**Proof.** The following series representations are well-known (see e.g. [6], [9]):

$$\log \frac{A}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{3}$$

$$\log \frac{I}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k+1} \cdot \left(\frac{y-x}{y+x}\right)^{2k}. \tag{4}$$

By substraction, from (3) and (4) we get

$$\log \frac{A}{I}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{5}$$

where $\frac{A}{G}(x,y) = \frac{A(x,y)}{G(x,y)}$, etc.

By letting $y = x + a$ in (4), we get that

$$\log \frac{I}{G}(x, x+a) = \sum_{k=1}^{\infty} \frac{a^{2k}}{2k+1} \cdot \left(\frac{1}{2x+a}\right)^{2k}. \tag{6}$$

As $\frac{1}{2x+a}$ is c.m., by Corollary 1, $g(x) = \left(\frac{1}{2x+a}\right)^{2k}$ will be c.m., too. This means that

$$(-1)^n g^{(n)}(x) \geq 0 \text{ for any } x > 0,\ n \geq 0,$$

so by $n$ times differentiation of the series from (6), we get that

$$\log \frac{I}{G}(x, x+a)$$

is c.m., thus $\frac{I}{G}(x, x+a)$ is l.c.m.

The similar proof for $\frac{A}{I}(x, x+a)$ follows from the series representation (5).

Finally, by the known identity (see e.g. [6], [9])

$$\log \frac{I}{G} = \frac{A}{L} - 1 \tag{7}$$

we get the last part of Theorem 1. □

**Remark 1.** It follows from the above that $\frac{A}{G}(x, x+a)$, $\frac{A}{H}(x, x+a)$, $\frac{I}{H}(x, x+a)$, $\frac{G}{H}(x, x+a)$ are all l.c.m. functions.

**Theorem 2.** *For any $a > 0$, the ratios*

$$\frac{\sqrt{2A^2+G^2}}{I\sqrt{3}}(x, x+a),\ \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}}(x, x+a) \text{ and } \frac{Q}{G}(x, x+a)$$

*are l.c.m. functions.*

**Proof.** In paper [8] it is proved that

$$\log \frac{\sqrt{2A^2+G^2}}{I\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left( \frac{1}{2k+1} - \frac{1}{3^k} \right) \cdot \left( \frac{y-x}{y+x} \right)^{2k}, \tag{8}$$

while in [9] that

$$\log \frac{\sqrt{2A^2+G^2}}{G\sqrt{3}} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot \left( 1 - \frac{1}{3^k} \right) \cdot \left( \frac{y-x}{y+x} \right)^{2k}. \tag{9}$$

Letting $y = x + a$, by the method of proof of Theorem 1, the first part of Theorem 2 follows. Finally, the identity

$$\log \frac{Q}{G} = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left( \frac{y-x}{y+x} \right)^{4k-2} \tag{10}$$

appears in [9]. This leads also to the proof of l.c.m. monotonicity of the ratio $\dfrac{Q}{G}(x, x+a)$. $\square$

**Theorem 3.** *For any* $a > 0$, *the ratios*

$$\frac{L}{G}(x, x+a),\ -\frac{H}{L}(x, x+a) \text{ and } \frac{A}{P}(x, x+a)$$

*are c.m. functions.*

**Proof.** In [5] (see also [9] for a new proof) it is shown that

$$\frac{L}{G}(x, y) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \cdot \left( \frac{\log x - \log y}{2} \right)^{2k}. \tag{11}$$

Letting $y = x + a$ and remarking that the function

$$f(x) = \log(x+a) - \log x$$

is c.m., by Corollary 1, and by differentiation of the series from (11), we get that $\dfrac{L}{G}(x, x+a)$ is c.m.

The identity

$$\log \frac{S}{I} = 1 - \frac{H}{L} \tag{12}$$

appears in [9]. Since we have the series representations (see [7], [9])

$$\log \frac{S}{G}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \left(\frac{y-x}{y+x}\right)^{2k} \tag{13}$$

and

$$\log \frac{S}{A}(x, y) = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{14}$$

by using relation (4), we get $\log \frac{S}{G} - \log \frac{I}{G} = \log \frac{S}{I}$, so

$$\log \frac{S}{I}(x, y) = \sum_{k=1}^{\infty} \frac{2}{4k^2-1} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{15}$$

thus $\frac{S}{I}(x, x+a)$ is l.c.m., which by (12) implies that the ratio $-\frac{H}{L}$ is l.c.m. function.

Finally, Seiffert's identity (see [10], [9])

$$\log \frac{A}{P}(x, y) = \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \cdot \binom{2k}{k} \cdot \left(\frac{y-x}{y+x}\right)^{2k}, \tag{16}$$

implies the last part of the theorem. □

**Remark 2.** By (13), (14) and (15) we get also that $\frac{S}{G}(x, x+a)$, $\frac{S}{A}(x, x+a)$ and $\frac{S}{I}(x, x+a)$ are l.c.m. functions.

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## 5.5 On means generated by two positive functions

### 1

Let $f, g : (0, +\infty) \to (0, +\infty)$ be two positive functions of positive arguments. We shall consider the following means generated by these functions:

$$M_{f,g} := M_{f,g}(a,b) = \frac{af(a) + bg(b)}{f(a) + g(b)} \ (a, b > 0) \tag{1}$$

and

$$N_{f,g} := N_{f,g}(a,b) = \left(a^{f(a)} \cdot b^{g(b)}\right)^{1/(f(a)+g(b))} \ (a, b > 0) \tag{2}$$

First remark that $M$ and $N$ are indeed means, since

$$M_{f,g} = N_{f,g} = (a,a) = a \text{ and } a < M_{f,g}(a,b) < b$$

$$\Leftrightarrow a < b \text{ and } a < N_{f,g}(a,b) < b \Leftrightarrow a < b$$

for any $f, g$. Some particular cases of these general means are worthy to note:

1) Let $f(x) = g(x) = x^k$ $(k \in \mathbb{R})$. Then one obtains the means

$$M_k := M_k(a,b) = \frac{a^{k+1} + b^{k+1}}{a^k + b^k},$$

known also as the Lehmer means. For $k = 0$ one gets

$$M_0 = M_0(a,b) = \frac{a+b}{2} = A(a,b), \text{ the arithmetic mean,}$$

for $k = -1$ one has

$$M_{-1} = M_{-1}(a,b) = \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}} = H(a,b), \text{ the harmonic mean,}$$

for $k = 1$ one gets

$$M_1 = M_1(a,b) = \frac{a^2+b^2}{a+b};$$

for $k = -1/2$ one has

$$M_{-1/2} = \sqrt{ab} = G(a,b), \text{ the geometric mean.}$$

Similarly, for the means $N$ one obtains:

$$N_k := N_k(a,b) = \left(a^{a^k} b^{b^k}\right)^{1/(a^k+b^k)}.$$

Here

$$N_0 = \sqrt{ab} = G(a,b), \text{ the geometric mean;}$$

$$N_1 = (a^a b^b)^{1/(a+b)} = S(a,b),$$

a mean studied e.g. in [1], [4], [5].

For $k = -1$ one can deduce the following similar mean:

$$N_{-1} = (a^{1/a} b^{1/b})^{ab/(a+b)} = (a^b b^a)^{1/(a+b)} = S^*(a,b)$$

(which we denote here as the conjugate of the mean $S$).

## 2

But certainly, by selecting $f \neq g$, one can write certain unsymmetric means, as for $f(x) = x$, $g(x) = \sqrt{x}$:

$$M_{f,g} = \frac{a^2 + b\sqrt{b}}{a+\sqrt{b}}, \quad N_{f,g} = \left(a^a b^{\sqrt{b}}\right)^{1/\left(a+\sqrt{b}\right)}.$$

## 3

For $f(x) = g(x) = e^x$, one has

$$M_{f,g} = \frac{a^{e^a} + b^{e^b}}{e^a + e^b}$$

(studied also in [5]), while

$$N_{f,g} = \left(a^{(e^a)} b^{(e^b)}\right)^{1/(e^a+e^b)}.$$

**4**

Finally, when $f(x) = e^x$, $g(x) = e^{-x}$ one gets

$$M_{f,g} = \frac{ae^a + be^{-b}}{e^a + e^{-b}}, \quad N_{f,g} = \left(a^{(e^a)} a^{(e^{-b})}\right)^{1/(e^a+e^{-b})}$$

but we stop with the examples.

**Lemma 1.** *Let* $a_i > 0$, $\lambda_i > 0$ $(i = \overline{1,n})$, $\sum_{i=1}^{n} \lambda_i = 1$. *Then*

$$a_1^{\lambda_1} \dots a_n^{\lambda_n} \le \lambda_1 a_1 + \dots + \lambda_n a_n \tag{3}$$

*with equality only for* $a_1 = \dots = a_n = 1$.

**Proof.** (With an idea by F. Riesz [6]). Apply the logarithmic inequality

$$\ln x \le x - 1 \quad (x > 0, \text{ with equality for } x = 1) \tag{4}$$

to $x = \dfrac{a_i}{A}$, where $A = \lambda_1 a_1 + \dots + \lambda_n a_n$ $(i = \overline{1,n})$ and multiply (4) by $\lambda_i$. One gets

$$\lambda_i \ln \frac{a_i}{A} \le \frac{\lambda_i a_i}{A} - \lambda_i.$$

By summing on $i = \overline{1,n}$, one can deduce

$$\ln \frac{a_1^{\lambda_1} \dots a_n^{\lambda_n}}{A^{\lambda_1 + \dots + \lambda_n}} \le \frac{A}{A} - \sum_{i=1}^{n} \lambda_i, \quad \text{i.e.} \quad \ln \frac{a_1^{\lambda_1} \dots a_n^{\lambda_n}}{A} \le 0,$$

giving (3). One has equality only for $x = 1$, i.e. $\dfrac{a_i}{A} = 1$, implying

$$a_1 = a_2 = \dots = a_n = A.$$

**Theorem 1.** *For all* $f, g$ *one has*

$$N_{f,g} \le M_{f,g} \tag{5}$$

*with equality only for* $a = b$.

**Proof.** Remark that $N_{f,g}$ can be written as $a^{\lambda_1}b^{\lambda_2}$, where

$$\lambda_1 = \frac{f(a)}{f(a)+g(b)}, \quad \lambda_2 = \frac{g(b)}{f(a)+g(b)}.$$

Then $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, and by Lemma 1 ($n = 2$) we get

$$a^{\lambda_1}b^{\lambda_2} \leq \lambda_1 a + \lambda_2 b = \frac{af(a)+bg(b)}{f(a)+f(b)} = M_{f,g}(a,b).$$

One has equality for $a = b$, one base of Lemma 1. This proves Theorem 1.

**Remark.** Since for $a < b$ one has $M_{f,g}(a,b) < b$, (5) gives an improvement of relation $a < N_{f,g}(a,b) < b$ (right side), and also of $a < M_{f,g}(a) < b$ (left side). For particular $f, g$, relation (5) contains many inequalities.

## 5

The definition of $N_{f,g}$ remembers the identric mean $I$ defined by

$$I := I(a,b) = \frac{1}{e}(a^a/b^b)^{1/(a-b)} \ (a \neq b); \quad I(a,a) = a. \tag{6}$$

Therefore it is natural that following the above ideas to look for a generalization like

$$I_{f,g} := I_{f,g}(a,b) = c\left(a^{f(a)}/b^{g(b)}\right)^{1/(f(a)-g(b))} \tag{7}$$

where $c$ is a positive constant. For $f = g$ the following result can be proved:

**Theorem 2.** *Let us suppose that $f$ is differentiable. Then (7) defines a mean if and only if $f(x) = kx^{\lambda}$ and $c = e^{-1/\lambda}$ where $\lambda \neq 0$ is a real number, $k > 0$ constant.*

**Proof.** Let $0 < a < b$. Then $I_f$ is a mean if

$$a < c\left(a^{f(a)}/b^{f(b)}\right)^{1/(f(a)-f(b))} < b,$$

or written equivalently

$$\frac{1}{c} < \left(\frac{a}{b}\right)^{\frac{f(b)}{f(a)-f(b)}} \quad \text{and} \quad \left(\frac{a}{b}\right)^{\frac{f(a)}{f(a)-f(b)}} < \frac{1}{c}.$$

Now, remark that

$$\left(\frac{a}{b}\right)^{f(b)/(f(a)-f(b))} = \left[\left(\frac{a}{b}\right)^{\frac{a}{a-b}}\right]^{\frac{f(b)(a-b)}{a(f(a)-f(b))}}.$$

It is immediate that

$$\lim_{b\to a}\left(\frac{a}{b}\right)^{\frac{a}{a-b}} = e,$$

so by the above identity, on letting $b \to a$ we get

$$\frac{1}{c} \le e^{f(a)/af'(a)} \quad \text{and} \quad e^{f(a)/af'(a)} \le \frac{1}{c}, \tag{8}$$

i.e.

$$e^{f(a)/af'(a)} = \frac{1}{c},\ \forall\ a > 0.$$

This well-known differential equation, by integrating gives

$$f(a) = ka^{\lambda},\ \lambda \neq 0.$$

Conversely, if $f$ is given by this formula, then $I_f$ given by (7) defines a mean for $c = e^{-1/\lambda}$. Indeed, as above one can write

$$e^{\frac{1}{\lambda}} \le \left(\frac{a}{b}\right)^{\frac{b^{\lambda}}{a^{\lambda}-b^{\lambda}}}.$$

Denoting $\frac{a}{b} = x \in (0,1)$ and logarithming this becomes

$$\frac{1}{x^{\lambda}-1}\log x \ge \frac{1}{\lambda},$$

which is obviously true. Analogously, the second inequality becomes

$$\frac{x^{\lambda}}{x^{\lambda}-1}\lg x \le \frac{1}{\lambda}$$

which is also true.

## 6

We now consider certain particular means of the general means introduced above. In Example 1), we have denoted

$$S(a,b) = (a^a b^b)^{1/(a+b)}, \quad S^*(a,b) = (a^b b^a)^{1/(a+b)}.$$

The example from 3), was defined in [5] by $F$:

$$F(a,b) = \frac{ae^a + be^b}{e^a + e^b}.$$

Now the following identities were remarked (see [3], [5]).

$$F(a,b) = \log S(e^a, e^b) \tag{9}$$

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}, \tag{10}$$

where $I$ is the identric mean. Another identity for these means in ([3])

$$\log \frac{S}{I} = 1 - \frac{G^2}{AL} \tag{11}$$

where $L$ is the logarithmic mean. See [7] for related identities. By replacing $a \to 1/a$, $b \to 1/b$ in (6) one gets the mean

$$J(a,b) = \frac{1}{I\left(\frac{1}{a}, \frac{1}{b}\right)}$$

introduced in [1].

$$J(a,b) = e(a^b/b^a)^{1/(b-a)} \tag{12}$$

Now, remark that

$$S(a,b) \cdot S^*(a,b) = (a^{a+b} b^{a+b})^{1/(a+b)} = ab,$$

giving

$$S^* = \frac{G^2}{S} \tag{13}$$

This is very similar to $H = \dfrac{G^2}{A}$. By replacing $a \to 1/a$, $b \to 1/b$ in definition of $S$ and $S^*$, one gets the surprising fact:

$$S^*(a,b) = \frac{1}{S\left(\dfrac{1}{a}, \dfrac{1}{b}\right)} \quad \text{and} \quad S(a,b) = \frac{1}{S^*\left(\dfrac{1}{a}, \dfrac{1}{b}\right)}. \tag{14}$$

**Definition.** Two means $M, N$ are $T$-similar (related to the mean $T$) if

$$MN = T^2 \tag{15}$$

The mean $M^*$ is named as the conjugate mean of $M$ of

$$M^*(a,b) = \frac{1}{M\left(\dfrac{1}{a}, \dfrac{1}{b}\right)}.$$

For example, $H$ and $A$; as well as $S$ and $S^*$ are $G$-similar (see (13)). We have:

**Theorem 3.** *If $M, N$ are $T$-similar, then $M^*$ and $N^*$ are $T^*$-similar, and reciprocally.*

**Proof.** In $M(a,b)N(a,b) = T^2(a,b)$ apply $a \to 1/a$, $b \to 1/b$ and use the above definition.

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# 5.6 On certain weighted means

## 1. Introduction

For $a, b > 0$ let

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab},$$

$$Q = Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad S(a,b) = (a^a \cdot b^b)^{1/(a+b)}$$

denote some classical means, and let

$$L = L(a,b) = \frac{a-b}{\ln b - \ln a} \ (a \neq b), \quad L(a,a) = a$$

and

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \ (a \neq b), \quad I(a,a) = a$$

be the well-known logarithmic and identric means. For these means many interesting relations, including various identities and inequalities have been proved. For a survey of results, see e.g. [2], [3], [5], [6], [7]. Particularly, the following chain of inequalities holds true:

$$G < L < I < A < Q < S, \tag{1}$$

where in all cases $a \neq b$.

Recently, in paper [1] the following weighted means have been introduced

$$G(a,b;p,q) = \frac{G(ap,bq)}{G(p,q)}, \text{ where } a,b,p,q > 0, \text{ etc.}$$

and more generally

$$M(a,b;p,q,) = \frac{M(ap,bq)}{M(p,q)}, \tag{2}$$

where $M$ is a given homogeneous mean.

In [1] the authors have essentially proved the following:

**Theorem 1.** *For all $0 < a < b$ and $0 < p < q$ one has*

$$G(a,b;p,q) < L(a,b;p,q) < I(a,b;p,q)$$
$$< A(a,b;p,q) < Q(a,b;p,q) < S(a,b;p,q). \tag{3}$$

We note that in the author's paper [7] (see Corollary 4, p. 117) the following is proved:

**Theorem A.** *If $a,b,c,d > 0$ and $c > d$, $ad - bc > 0$ then*

$$\frac{L(a,b)}{L(c,d)} > \frac{G(a,b)}{G(c,d)}. \tag{4}$$

Now put $a := b \cdot q$, $b := a \cdot p$, $c := q$, $d := p$. We get from Theorem A that if $q > p > 0$ and $b > a > 0$, then the first inequality of (3) holds true.

**Remark.** Therefore, the idea of considering the means $G(a,b;p,q)$ and $L(a,b;p,q)$ could be reduced to Theorem A.

The other aim of this paper is to offer new proofs for the remaining inequalities, and even to (4). We point out that the second inequality of (3) has been proved essentially in our paper [4].

## 2. Main results

The following auxiliary result will be used:

**Lemma A.** *Let $M$ and $N$ be two homogeneous means and suppose that the application*

$$f(x) = \frac{M(1,x)}{N(1,x)} \text{ is strictly increasing for } x > 1.$$

*Then for any $b > a > 0$ and $q > p > 0$ one has*

$$M(a,b;p,q) > N(a,b;p,q). \tag{5}$$

*Proof.* Inequality (5) can be written also as

$$\frac{M(ap,bq)}{N(ap,bq)} > \frac{M(p,q)}{N(p,q)},$$

and as $M$, $N$ are homogeneous (of order one),

$$\frac{M\left(1, \frac{b}{a} \cdot \frac{q}{p}\right)}{N\left(1, \frac{b}{a} \cdot \frac{q}{p}\right)} > \frac{M\left(1, \frac{q}{p}\right)}{N\left(1, \frac{q}{p}\right)}, \quad \text{or} \quad f\left(\frac{bq}{ap}\right) > f\left(\frac{q}{p}\right).$$

Now, if $bq > ap$ and $b > a$, then as $f$ is strictly increasing, the result follows.

*Proof of* $I(a, b; p, q) > L(a, b; p, q)$. In our paper [4] it is proved that $\dfrac{I(1, x)}{L(1, x)}$ is strictly increasing function of $x > 1$. By the Lemma, the proof is completed.

*Proof of* $L(a, b; p, q) > G(a, b; p, q)$. In our paper [4] it is proved that

$$\frac{L'}{L} = \frac{1}{x-1}\left(1 - \frac{L}{x}\right),$$

where $L = L(1, x)$ and $L' = \dfrac{d}{dx}L(1, x)$. Now, as $\dfrac{G'}{G} = \dfrac{1}{2x}$, and as it is well known that $L < \dfrac{x+1}{2}$, (particular case of (1): $L < aA$), we easily get $\dfrac{L'}{L} > \dfrac{G'}{G}$. Since $\left(\dfrac{L}{G}\right)' = \dfrac{L'G - G'L}{G^2}$, we get that $\dfrac{L(1, x)}{G(1, x)}$ is strictly increasing for $x > 1$. By the Lemma, the result follows.

*Proof of* $A(a, b; p, q) > I(a, b; p, q)$. As $\dfrac{I'}{I} = \dfrac{1}{x-1}\left(1 - \dfrac{1}{L}\right)$ (see [4]), and by $\dfrac{A'}{A} = \dfrac{1}{x+1}$, where $I = I(1, x)$, etc., we get $\dfrac{A'}{A} > \dfrac{I'}{I}$, as by $L < \dfrac{x+1}{2}$ this holds true. Now, the Lemma implies the result.

*Proof of* $S(a, b; p, q) > Q(a, b; p, q)$. As $\ln S(a, b) = \dfrac{a \ln a + b \ln b}{a+b}$, an easy computation gives

$$\frac{S'}{S} = \left(x + 1 + \frac{x-1}{L}\right) / (x+1)^2.$$

Since $\dfrac{Q'}{Q} = \dfrac{x}{x^2+1}$, we can prove that $\dfrac{S'}{S} > \dfrac{Q'}{Q}$, as by $L < \dfrac{x+1}{2}$ this reduces to $(x^2 + 4x - 1)(x^2 + 1) > x(x + 1)^3$, or after some elementary computations, to $(x - 1)^3 > 0$. The Lemma applies.

## 3. Corrected result and new proofs

In [1] Theorem 8 incorrectly states that $I(a, b; p, q) < I(a, b)$ for $0 < a < b$ and $0 < p < q$, while Theorem 7 states that $L(a, b; p, q) > L(a, b)$. In what follows, the following corrected result, with new proofs, will be offered:

**Theorem 2.** *One has, for any $b > a > 0$ and $q > p > 0$*

$$I(p, q) \cdot I(a, b) > I(pa, qb) \tag{6}$$

*and*

$$L(p, q) \cdot L(a, b) < L(pa, qb). \tag{7}$$

**Proof.** The following integral representations are well-known:

$$\ln I(a, b) = \int_0^1 \ln(ua + (1 - u)b)du \tag{8}$$

and

$$L(a, b) = \int_0^1 a^u \cdot b^{1-u} du. \tag{9}$$

By taking into account of (8), it will be sufficient to show that

$$upa + (1 - u)qb \geq [up + (1 - u)q][ua + (1 - u)b], \ u \in [0, 1]. \tag{10}$$

After some elementary computations, (10) becomes:

$$u(1 - u)(a - b)(p - q) \geq 0$$

which is true. Since in (10) there is equality only for $u = 0$ or $u = 1$, by integration we get the strict inequality (6).

Now, by (9) inequality becomes

$$\int_0^1 a^t b^{1-t} dt \int_0^1 p^t q^{1-t} dt < \int_0^1 (pa)^t (bq)^{1-t} dt,$$

and this is consequence of the Chebysheff integral inequality

$$\frac{1}{y-x}\int_x^y f(t)dt \cdot \frac{1}{y-x}\int_x^y g(t)dt < \frac{1}{y-x}\int_x^y f(t)g(t)dt, \tag{11}$$

where $x < y$ and $f, g : [x, y] \to \mathbb{R}$ are strictly monotonic functions of the same type. In our case $[x, y] = [0, 1]$;

$$f(t) = a^t \cdot b^{1-t} = b\left(\frac{a}{b}\right)^t, \quad g(t) = p^t q^{1-t} = q\left(\frac{p}{q}\right)^t$$

which are both strictly decreasing functions for $b > a > 0$ and $q > p > 0$.

**Remarks.** 1) Another proof of (7) can be given by the formula

$$\ln L(a, b) = \int_0^1 \frac{\ln I(a^t, b^t)}{t} dt,$$

and by taking into account of the relation (6) (proved before via (8) and (10)).

2) A generalization of (6) and (7), for the so-called "Stolarsky means" has been given by E. Neuman and the author in 2003 (see [2], Theorem 3.8).

## 4. Refinements

First we will use the method of proof of Theorem A from [7] to deduce a refinement of $I(a, b; p, q) < A(a, b; p, q)$.

*Refinement of* $I(a, b; p, q) < A(a, b; p, q)$. Let us introduce the mean

$$R = \sqrt{\frac{2A^2 + G^2}{3}}.$$

The following series representations may be found in [7]:

$$\ln \frac{R}{I} = \sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{1}{2k+1} - \frac{1}{3^k}\right) z^{2k} \tag{12}$$

and

$$\ln \frac{R}{G} = \sum_{k=1}^{\infty} \frac{1}{2k} \left(1 - \frac{1}{3^k}\right) z^{2k}, \tag{13}$$

$$\ln \frac{A}{G} = \sum_{k=1}^{\infty} \frac{1}{2k} \cdot z^{2k}, \tag{14}$$

where $z = \dfrac{b-a}{b+a}$. By a subtraction, from (13) and (14) we get

$$\ln \frac{A}{R} = \sum_{k=1}^{\infty} \frac{1}{2^k \cdot 3^k} \cdot z^{2k}. \tag{15}$$

Assume now that $a, b, c, d > 0$ are given in such a way that $b > a$, $d > c$ and $\dfrac{b-a}{b+a} > \dfrac{d-c}{d+c}$. This latest inequality may be written also as $bc > ad$. Then from (12) we get $\dfrac{R}{I}(a,b) > \dfrac{R}{I}(c,d)$, or equivalently

$$\frac{R(a,b)}{R(c,d)} > \frac{I(a,b)}{I(c,d)}.$$

By putting here $a := pa$, $b := qb$, $c := p$, $d := q$ the above conditions become $b > a$ and $q > p$ and we have obtained $I(a,b;p,q) < R(a,b;p,q)$. By using the same method for the representation (15), we get $R(a,b;p,q) < A(a,b;p,q)$. Thus we have obtained the refinement:

**Theorem 3.**

$$I(a,b;p,q) < R(a,b;p,q) < A(a,b;p,q), \ \textit{if } b > a > 0, \ q > p > 0. \tag{16}$$

*Refinements of* $L(a,b;p,q) < I(a,b;p,q)$. Let us introduce the following means:

$$U = U(a,b) = \frac{2G+A}{3}, \quad V = V(a,b) = \frac{2A+G}{3},$$

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)} \ (a \neq b), \ P(a,a) = a.$$

Here the mean $P$ is called also the first Seiffert mean (see [?], [?]).

The following chain of inequalities is known:

$$L < U < P < V < I. \tag{17}$$

The first inequality of (17) is due to B.C. Carlson ([3]), for the second and third inequality see [5], while for the last inequality, see [3], [5] and the references therein (the last inequality has been proved by the author in 1991).

Now, from Section 2 we know that

$$\frac{I'}{I} = \frac{1}{x-1}\left(1 - \frac{1}{L}\right), \quad \frac{L'}{L} = \frac{1}{x-1}\left(1 - \frac{L}{x}\right),$$

where $L = L(1, x)$, etc. and $x > 1$. One can deduce also the following formulae

$$\frac{U'}{U} = \frac{2+\sqrt{x}}{\sqrt{x}(4\sqrt{x}+x+1)}, \quad \frac{V'}{V} = \frac{2\sqrt{x}+1}{2\sqrt{x}(x+\sqrt{x}+1)},$$

and

$$\frac{P'}{P} = \frac{1}{x-1}\left[1 - \frac{2P}{(x+1)\sqrt{x}}\right].$$

Now, the inequality $\dfrac{U'}{U} > \dfrac{L'}{L}$ becomes, after some elementary transformations:

$$L > \frac{2\sqrt{x}(x+1+\sqrt{x})}{4\sqrt{x}+x+1}.$$

Since $\sqrt{x} = G$, $x+1 = 2A$, this inequality is in fact the following:

**Lemma 1.** *One has*

$$L > \frac{G(2A+G)}{2G+A}. \tag{18}$$

*Proof.* By the Leach-Sholander inequality ([3]) $L > \sqrt[3]{G^2A}$, it is sufficient to prove that

$$\sqrt[3]{G^2A} > G(2A+G)/(2G+A).$$

By putting $A/G = t > 1$, and by letting logarithms, this becomes

$$u(t) = \ln t + 3\ln(t+2) - 3\ln(2t+1) > 0.$$

After elementary computations, we can deduce:

$$t(t+2)(2t+1)U'(t) = 2(t-1)^2 > 0.$$

Since $u(t)$ is strictly increasing, we can write $U(t) > U(1) = 0$, so (18) follows. This proves $U'/U < L'/L$.

Inequality $\dfrac{U'}{U} < \dfrac{P'}{P}$, after some elementary transformations becomes:

**Lemma 2.**

$$P < \frac{(2A+G)A}{2G+A}. \tag{19}$$

*Proof.* By $P < V = \dfrac{2A+G}{3}$ and $\dfrac{2A+G}{3} < \dfrac{(2A+G)A}{2G+A} \Leftrightarrow$

$$2G + A < 3A, \text{ i.e. } G < A,$$

which is true.

The inequality $\dfrac{P'}{P} < \dfrac{V'}{V}$ becomes:

**Lemma 3.**

$$P > \frac{A(2G+A)}{2A+G}. \tag{20}$$

*Proof.* In our paper [5] it is proved that

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2}.$$

Therefore, it is sufficient to prove that

$$\sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \frac{A(2G+A)}{2A+G}.$$

Put $\dfrac{A}{G} = t > 1$, and by logarithmation, we have to prove

$$v(t) = 3\ln(2t+1) + 2\ln\left(\frac{t+1}{2}\right) - 2\ln t - 3\ln(t+2) > 0.$$

After elementary computations we get

$$t(t+1)(t+2)(2t+1)v'(t) = (t-1)(5t+4) > 0.$$

This implies $v(t) > v(1) = 0$, so (20) follows.

Finally, the inequality $\frac{V'}{V} < \frac{I'}{I}$ becomes again inequality (18).

By the proved inequalities

$$\frac{L'}{L} < \frac{U'}{U} < \frac{P'}{P} < \frac{V'}{V} < \frac{I'}{I},$$

and by taking into account Lemma A, we get:

**Theorem 4.** *For $b > a > 0$, $q > 0 > 0$ one has*

$$L(a,b;p,q) < U(a,b;p,q) < P(a,b;p,q)$$

$$< V(a,b;p,q) < I(a,b;p,q). \tag{21}$$

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## 5.7 Stolarsky and Gini means

### 1

First class of means studied here was introduced by K. Stolarsky [20]. For $a, b \in \mathbb{R}$ they are denoted by $D_{a,b}(\cdot,\cdot)$ and defined as

$$D_{a,b}(x,y) = \begin{cases} \left[\dfrac{b(x^a - y^a)}{a(x^b - y^b)}\right]^{1/(a-b)}, & ab(a-b) \neq 0 \\ \exp\left(-\dfrac{1}{a} + \dfrac{x^a \ln x - y^a \ln y}{x^a - y^a}\right), & a = b \neq 0 \\ \left[\dfrac{x^a - y^a}{a(\ln x - \ln y)}\right], & a \neq 0,\ b = 0 \\ \sqrt{xy}, & a = b = 0. \end{cases} \tag{1.1}$$

Stolarsky means are sometimes called the difference means (see, e.g., [10], [8]). Here $x, y > 0$, $x \neq y$.

The identric, logarithmic, and power means of order $a$ ($a \neq 0$) will be denoted by $I_a$, $L_a$, and $A_a$, respectively. They are all contained in the family of means under discussion. We have $I_a = D_{a,a}$, $L_a = D_{a,0}$, and $A_a = D_{2a,a}$. When $a = 1$ we will write $I$, $L$, and $A$ instead of $I_1$, $L_1$, and $A_1$. There is a simple relationship between means of order $a$ ($a \neq 1$) and those of order one. We have

$$I_a(x,y) = [I(x^a, y^a)]^{1/a}$$

with similar formulas for the remaining means mentioned above. Finally, the geometric mean of $x$ and $y$ is $G(x,y) = D_{0,0}(x,y)$.

Second family of bivariate means studied was introduced by C. Gini [4]. Throughout the sequel they will denoted by $S_{a,b}(\cdot,\cdot)$ and they are defined as follows

$$S_{a,b}(x,y) = \begin{cases} \left[\dfrac{x^a + y^a}{x^b + y^b}\right]^{1/(a-b)}, & a \neq b \\ \exp\left(\dfrac{x^a \ln x + y^a \ln y}{x^a + y^a}\right), & a = b \neq 0 \\ \sqrt{xy}, & a = b = 0. \end{cases} \tag{1.2}$$

Gini means are also called the sum means. It follows from (1.2) that $S_{0,-1} = H$ – the harmonic mean, $S_{0,0} = G$, and $S_{1,0} = A$. The following mean $S = S_{1,1}$ will play an important role in the discussion that follows.

Alzer and Ruscheweyh [1] have proven that the joint members in the families of the Stokarsky and Gini means are exactly the power means.

## 2

For the reader's convenience we give here a list o basic properties of the Stolarsky and Gini means. They follow directly from the defining formulas (1.1) and (1.2) and most of them can be found in [5] and [20]. Although they are formulated for the Stolarsky means they remain valid for the Gini means, too. In what follows we assume that $a, b, c \in \mathbb{R}$.

(P1) $D_{a,b}(\cdot,\cdot)$ is symmetric in its parameters, i.e. $D_{a,b}(\cdot,\cdot) = D_{b,a}(\cdot,\cdot)$.

(P2) $D_{\cdot,\cdot}(x,y)$ is symmetric in the variables $x$ and $y$, i.e., $D_{\cdot,\cdot}(x,y) = D_{\cdot,\cdot}(y,x)$.

(P3) $D_{a,b}(x,y)$ is homogeneous function of order one in its variables, i.e., $D_{a,b}(\lambda x, \lambda y) = \lambda D_{a,b}(x,y)$, $\lambda > 0$.

(P4) $D_{a,b}(x^c, y^c) = [D_{ac,bc}(x,y)]^c$.

(P5) $D_{a,b}(x,y)D_{-a,-b}(x,y) = xy$.

(P6) $D_{a,b}(x^c, y^c) = (xy)^c D_{a,b}(x^{-c}, y^{-c})$.

(P7) $D_{a,b}(x,y)S_{a,b}(x,y) = D_{a,b}(x^2,y^2) = D^2_{2a,2b}(x,y)$.

(P8) $D_{a,b}$ increases with increase in either $a$ or $b$.

(P9) If $a > 0$ and $b > 0$, then $D_{a,b}$ is log-concave in both $a$ and $b$.
If $a < 0$ and $b < 0$, then $D_{a,b}$ is log-convex in both $a$ and $b$.

Property (P8) for the Stolarsky means in established in [5] and [20]. F. Qi [13] has established (P8) for the Gini means. Logarithmic concavity (convexity) property for the Stolarsky means is established in [12].

(P10) If $a \neq b$, then

$$\ln D_{a,b} = \frac{1}{b-a}\int_a^b \ln I_t dt \quad \text{and} \quad \ln S_{a,b} = \frac{1}{b-a}\int_a^b \ln J_t dt.$$

First formula in (P10) is derived in [20] while the proof of the second one is an elementary exercise in calculus.

We shall prove now the property (P9) for the Gini means. The following result will be utilized.

**Lemma 2.1.** [14] *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function. If $f$ is increasing (decreasing) and/or convex (concave), then the function*

$$g(a,b) = \frac{1}{b-a}\int_a^b f(t)dt$$

*is increasing (decreasing) and/or convex (concave) function in both variables $a$ and $b$.*

Let $r = (x/y)^t$ and let $\mu(t) = \ln S_t$ $(t \in \mathbb{R})$. It follows from (1.2) that

$$t\mu(t) = t\ln x - \frac{\ln r}{r+1}.$$

Hence

$$t^2\mu'(t) = r\left(\frac{\ln r}{r+1}\right)^2 > 0$$

and

$$t^3\mu''(t) = r(1-r)\left(\frac{\ln r}{r+1}\right)^3 < 0$$

for all $t \neq 0$. Thus the function $\mu(t)$ is strictly concave for $t > 0$ and strictly convex or $t < 0$. This in conjunction with Lemma 2.1 and the second formula of (P10) gives the desired result.

We close this section with three comparison theorems for the means under discussion. The following functions will be used throughout the sequel. Let

$$k(x,y) = \begin{cases} \dfrac{|x|-|y|}{x-y}, & x \neq y \\ \operatorname{sign}(x), & x = y \end{cases}$$

and let

$$l(x,y) = \begin{cases} L(x,y), & x > 0,\ y > 0 \\ 0, & x \cdot y = 0 \end{cases}$$

where

$$L(x,y) = \frac{x-y}{\ln x - \ln y},\ x \neq y, \quad L(x,x) = x$$

is the logarithmic mean of $x$ and $y$. Function $l(x,y)$ is defined for non-negative values of $x$ and $y$ only.

The comparison theorem for the Stolarsky mean reads as follows.

**Theorem A.** ([10], [6]) *Let $a,b,c,d \in \mathbb{R}$. Then the comparison inequality*

$$D_{a,b}(x,y) \le D_{c,d}(x,y)$$

*holds true if and only if $a+b \le c+d$ and*

$$\begin{array}{lll} l(a,b) \le l(c,d) & \textit{if} & 0 \le \min(a,b,c,d), \\ k(a,b) \le k(c,d) & \textit{if} & \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ -l(-a,-b) \le -l(-c,-d) & \textit{if} & \max(a,b,c,d) \le 0. \end{array}$$

A comparison result for the Gini means is contained in the following.

**Theorem B.** ([9]) *Let $a, b, c, d \in \mathbb{R}$. Then the comparison inequality*

$$S_{a,b}(x, y) \le S_{c,d}(x, y)$$

*is valid if and only if $a + b \le c + d$ and*

$$\begin{aligned} &\min(a, b) \le \min(c, d) && \text{if} \quad 0 \le \min(a, b, c, d), \\ &k(a, b) \le k(c, d) && \text{if} \quad \min(a, b, c, d) < 0 < \max(a, b, c, d), \\ &\max(a, b) \le \max(c, d) && \text{if} \quad \max(a, b, c, d) \le 0. \end{aligned}$$

A comparison result for the Stolarsky and Gini means is obtained in [8].

**Theorem C.** *Let $a, b \in \mathbb{R}$. If $a + b > 0$, then*

$$D_{a,b}(x, y) < S_{a,b}(x, y)$$

*with the inequality reversed if $a+b < 0$. Moreover, $D_{a,b}(x, y) = S_{a,b}(x, y)$ if and only if $a + b = 0$.*

The new proof of Theorem C is included below.

**Proof.** There is nothing to prove when $a + b = 0$ because

$$d_{a,-a} = S_{a,-a} = G.$$

Define $r = (x/y)^t$ and $\phi(t) = \ln I_t - \ln J_t$. One can verify easily that

$$t\phi(t) = \frac{2r \ln r}{(r + 1)(r - 1)} - 1 = \frac{H(r, 1)}{L(r, 1)} - 1 < 0,$$

where the last inequality follows from the harmonic-logarithmic mean inequality. Also, $\phi(-t) = -\phi(t)$ for $t \in \mathbb{R}$. Hence $\phi(t) < 0$ if $t > 0$ and $\phi(t) > 0$ for $t < 0$. Let $a \ne b$. Making use of (P10) we obtain

$$\ln D_{a,b} - \ln S_{a,b} = \frac{1}{b - a} \int_a^b \phi(t)dt < 0,$$

where the last inequality holds true provided $a + b > 0$. The same argument can be employed to show that $D_{a,b} > S_{a,b}$ if $a + b < 0$. Assume now that $a = b \ne 0$. Sándor and Raşa [17] have proven that $D_{a,a} < S_{a,}$ for $a > 0$ with the inequality reversed if $a < 0$. This completes the proof. $\square$

## 3

Proofs of some results in this section utilize a refinement of the classical inequality which is due to Hermite and Hadamard.

Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{2}[f(a) + f(b)] \tag{3.1}$$

with the inequalities reversed if $f$ is concave on $[a, b]$. Equalities hold in (3.1) if and only if $f$ is a polynomial of degree one or less (see, e.g., [11]).

To obtain a refinement of (3.1) we introduce a uniform partition of $[a, b]$ with the breakpoints $\alpha_k$, i.e., $a = \alpha_0 < \alpha_1 < \ldots < \alpha_n = b$ with $\alpha_k - \alpha_{k-1} = h > 0$. Also, let $\beta_1 < \beta_2 < \ldots < \beta_n$ be the midpoints of the consecutive subintervals. Thus

$$\alpha_k = \frac{(n-k)a + kb}{n},\ 0 \le k \le n$$

and

$$\beta_k = \frac{(2n-2k+1)a + (2k-1)b}{2n},\ 1 \le k \le n.$$

Let $n$ be a positive integer. We define

$$M_n = \frac{1}{n}\sum_{k=1}^{n} f(\beta_k)$$

and

$$T_n = \frac{1}{n}\left\{\frac{1}{2}[f(a) + f(b)] + \sum_{k=1}^{n-1} f(\alpha_k)\right\}.$$

**Lemma 3.1.** *Let $f$ be a convex function on $[a, b]$. Then*

$$M_n \le \frac{1}{b-a}\int_a^b f(t)dt \le T_n. \tag{3.2}$$

*Inequalities (3.2) are reversed if $f$ is concave on $[a, b]$.*

**Proof.** Applying (3.1) to each of the integrals

$$\frac{1}{h}\int_{\alpha_{k-1}}^{\alpha_k} f(t)dt$$

$(h = (b-a)/n)$ and next summing the resulting expression, for $k = 1, 2, \ldots, n$, we obtain the assertion. $\square$

It is easy to verify that if $f$ is a convex function on $[a, b]$, then

$$M_n \geq f\left[\frac{1}{n}\sum_{k=1}^{n}\beta_k\right] = f\left(\frac{a+b}{2}\right)$$

and

$$T_n \leq \frac{1}{2n}[f(a)+f(b)] + \frac{1}{n}\sum_{k=1}^{n-1}[(n-k)f(a)+kf(b)] = \frac{1}{2}[f(a)+f(b)].$$

Thus (3.2) gives the refinement of the Hermite-Hadamard inequality (3.1). Inequality (3.2), when $n = 2$, appears in [3]. See also [2].

We shall use Lemma 3.1 in the proof of the following.

**Theorem 3.2.** *Let $a$ and $b$, $a \neq b$, be nonnegative numbers. Then*

$$\left(\sqrt{I_a I_b}\prod_{k=1}^{n-1} I_{\alpha_k}\right)^{1/n} \leq D_{a,b} \leq \left(\prod_{k=1}^{n} I_{\beta_k}\right)^{1/n}, \tag{3.3}$$

$$\frac{n}{\sum\limits_{k=1}^{n} I_{\beta_k}} \leq \frac{1}{D_{a,b}} \leq \frac{1}{2n}\left(\frac{1}{I_a}+\frac{1}{I_b}+2\sum_{k=1}^{n-1}\frac{1}{I_{\alpha_k}}\right), \tag{3.4}$$

*and*

$$\left(\sqrt{I_{2a}I_{2ab}}\prod_{k=1}^{n-1} I_{2\alpha_k}\right)^{2/n} \leq D_{a,b}S_{a,b} \leq \left(\prod_{k=1}^{n} I_{2\beta_k}\right)^{2/n}. \tag{3.5}$$

*Inequalities (3.3) and (3.5) are reversed if $a \leq 0$ and $b \leq 0$, $a \neq b$.*

**Proof.** Assume that $a \geq 0$ and $b \geq 0$, $a \neq b$. For the proof of (3.3) we use Lemma 3.1 with $f(t) = \ln I_t$ and property (P9) to obtain

$$\frac{1}{n}\left(\ln\sqrt{I_a I_b} + \sum_{k=1}^{n-1}\ln I_{\alpha_k}\right) \leq \frac{1}{b-a}\int_a^b \ln I_t dt \leq \frac{1}{n}\sum_{k=1}^{n}\ln I_{\beta_k}.$$

Application of (P10) to the middle term gives the assertion. In order to establish inequality (3.4) we use inequality (3.3) to obtain

$$\prod_{k=1}^{n}\left(\frac{1}{I_{\beta_k}}\right)^{1/n} \le \frac{1}{D_{a,b}} \le \left(\frac{1}{I_a I_b}\right)^{1/2n} \prod_{k=1}^{n-1}\left(\frac{1}{I_{\alpha_k}}\right)^{1/n}.$$

Application of the geometric mean-harmonic mean inequality together with the use of the arithmetic mean-geometric mean inequality completes the proof of (3.4). Inequality (3.5) follows from (3.3). Replacing $a$ by $2a$ and $b$ by $2b$ and next using the duplication formula $D_{a,b}S_{a,b} = D^2_{2a,2b}$ (see (P7)) we obtain the desired result. Case $a \le 0$ and $b \le 0$ is treated in an analogous manner, hence it is not included here. $\square$

Inequalities (3.3) and (3.4) are valid for the Gini means with the identric means being replaced by $S$-means of the appropriate order.

Bounds on the product $D_{a,b}S_{a,b}$ are obtained below.

**Theorem 3.3.** *Let $a, b \in \mathbb{R}$. Assume that $a + b \ge 0$. Then*

$$D_{a,b}S_{a,b} \le A_q^2 \tag{3.6}$$

*if and only if $q = \max(r_1, r_2)$, where $r_1 = \frac{2}{3}(a+b)$ and*

$$r_3 = \begin{cases} (\ln 4)l(a,b) & \text{if} \quad a \ge 0 \quad \text{and} \quad b \ge 0, \\ 0 & \text{if} \quad a < 0 \quad \text{and} \quad b > 0. \end{cases}$$

*If $a + b \le 0$, then the inequality (3.6) is reversed if and only if $q = \min(r_1, r_2)$, where $r_1$ is the same as above and*

$$r_2 = \begin{cases} -(\ln 4)l(-a,-b) & \text{if} \quad a \le 0 \quad \text{and} \quad b \le 0, \\ 0 & \text{if} \quad a > 0 \quad \text{and} \quad b > 0. \end{cases}$$

**Proof.** We shall use again the duplication formula

$$\sqrt{D_{a,b}S_{a,b}} = D_{2a,2b}.$$

Assume that $a \geq 0$ and $b \geq 0$. Using Theorem A we see that $D_{2a,2b} \leq D_{2q,q}$ if and only if $2(a+b) \leq 3q$ and $l(2a,2b) \leq l(2q,q)$. Solving these inequalities for $q$ we obtain $q \geq r_1$ and $q \geq r_2$. Assume now that $a \geq 0$, $b \leq 0$ with $a+b \geq 0$. Invoking Theorem A we obtain

$$q \geq r_1 \quad \text{and} \quad k(2a,2b) \leq k(2q,q).$$

The last inequality can be written as

$$(a+b)/(a-b) \leq 1.$$

Clearly it is satisfied for all values of $a$ and $b$ in the stated domain because $0 \leq a+b \leq a-b$. Case when $a \leq 0$, $b \geq 0$ with $a+b \geq 0$ is treated in the same way. We omit the proof of theorem when $a+n \leq 0$ because it goes along the lines introduced above. $\square$

Numerous inequalities for the particular means are contained in those of Theorems 3.2 and 3.3.

**Corollary 3.4.** *We have*

$$A_{2/3} < \sqrt{I_{5/6}I_{7/6}} < I, \tag{3.7}$$

$$\sqrt{AL} < \sqrt{I_{1/2}I_{3/2}} < I, \tag{3.8}$$

$$\sqrt{AL} < A_{2/3}, \tag{3.9}$$

$$\sqrt{IS} < A_{\ln 4}. \tag{3.10}$$

**Proof.** First inequalities in (3.7)-(3.8) follows from the second inequalities in (3.3) and (3.5) by letting $n = 2$ and putting $(a,b) = \left(\frac{4}{3}, \frac{2}{3}\right)$ and $(a,b) = (1,0)$, respectively while the second inequalities are an obvious consequence of the logarithmic concavity of the identric mean. Inequalities (3.9)-(3.10) follow from (3.6) by letting $(a,b) = (1,0)$ and $(a,b) = (1,1)$, respectively. $\square$

Combining (3.7) and (3.9) we obtain $\sqrt{AL} < A_{2/3} < I$ (see [15]). The second inequality in the last result is also established in [21].

The following result

$$A^2_{4a/3} \le I_a S_a \le A^2_{(\ln 4)a}, \;\; a \ge 0 \tag{3.11}$$

is also worth mentioning. Inequalities (3.11) are reversed if $a \le 0$. Let $a \ge 0$. Then the second inequality in (3.11) follows immediately from (3.6). For the proof of the first inequality in (3.11) we use (3.7) and the duplication formula (P7) to obtain

$$A^2_{4/3}(x, y) = A_{2/3}(x^2, y^2) < I(x^2, y^2) = I(x, y)S(x, y).$$

This completes the proof when $a = 1$. A standard argument is now used to complete the proof when $a \ge 0$. □

Our next result reads as follows.

**Theorem 3.5.** *Let $a \le 0$ and $b \le 0$. Then*

$$D_{a,b} \le L(I_a, I_b). \tag{3.12}$$

*If $a \ge 0$ and $b \ge 0$, then*

$$D_{a,b} \ge \frac{I_a I_b}{L(I_a, I_b)}. \tag{3.13}$$

**Proof.** There is nothing to prove then $a = b$. Assume that $a \le 0$, $b \le 0$, $a \ne b$. For the proof of (3.12) we use (P10), Jensen's inequality for integrals, logarithmic convexity of $I_t$ and the formula

$$L(x, y) = \int_0^1 x^t y^{1-t} dt$$

(see [7]) to obtain

$$\begin{aligned}\ln D_{a,b} &= \frac{1}{b-a}\int_a^b \ln I_t dt = \int_0^1 \ln I_{ta+(1-t)b} dt \\ &\le \ln\left(\int_0^1 I_{ta+(1-t)b} dt\right) \le \ln\left(\int_0^1 I_a^t I_b^{1-t} dt\right) \\ &= \ln L(I_a, I_b).\end{aligned}$$

Let now $a \geq 0$ and $b \geq 0$. For the proof of (3.13) we use (P5) and (3.12) to obtain

$$D_{a,b}(x,y) = \frac{xy}{D_{-a,-b}(x,y)} \geq \frac{xy}{L(I_{-a}, I_{-b})} = \frac{xy}{L\left(\dfrac{xy}{I_a}, \dfrac{xy}{I_b}\right)}$$

$$= \frac{1}{L\left(\dfrac{1}{I_a}, \dfrac{1}{I_b}\right)} = \frac{I_a I_b}{L(I_a, I_b)}. \qquad \square$$

**Corollary 3.6.** *The following inequality*

$$\frac{IG}{L(I,G)} < L \tag{3.14}$$

*is valid.*

**Proof.** In (3.13) put $(a, b) = (1, 0)$. $\square$

Inequalities similar to those in (3.12)-(3.13) hold true for the Gini means. We have

$$S_{a,b} \leq L(S_a, S_b), \ a \leq 0, \ b \leq 0$$

and

$$S_{a,b} \geq \frac{S_a S_b}{L(S_a, S_b)}, \ a \geq 0, \ b \geq 0.$$

**Theorem 3.7.** *Let $a, b, c \in \mathbb{R}$, $c \neq 0$. Then*

$$[D_{a,b}(x^c, y^c)]^{1/c} \geq D_{a,b}(x,y) \tag{3.15}$$

*if and only if $(a+b)(c-1) \geq 0$. A similar result is valid for the Gini means.*

**Proof.** We shall use (P4) in the form

$$[D_{a,b}(x^c, y^c)]^{1/c} = D_{ac,bc}(x,y). \tag{3.16}$$

It follows from Theorem A that $D_{a,b} \leq D_{ac,bc}$ if and only if $a+b \leq c(a+b)$ and if one of the remaining three inequalities of the above mentioned theorem is valid. Assume that $a + b \geq 0$ and consider the case when

$c > 1$. If $a \geq 0$ and $b \geq 0$, then $\min(a, b, ac, bc) \geq 0$ and $l(ac, bc) = cl(a, b) \geq l(a, b)$. Making use of (3.16) we obtain the desired inequality (3.15). Now let $a \geq 0$, $b \leq 0$ with $a + b \geq 0$. Then $\min(a, b, ac, bc) \leq 0 \leq \max(a, b, ac, bc)$ and $k(ac, bc) = k(a, b)$ which completes the proof of (3.15) in the case under discussion. Cases $0 < c < 1$ and $c < 0$ are treated in a similar way, hence they are not discussed here in detail. For the proof of the counterpart of (3.15) for the Gini means one uses the comparison inequality of Theorem B. $\square$

We close this section with the result which can be regarded as the Chebyshev type inequality for the Stolarsky and Gini means.

**Theorem 3.8.** *Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be positive vectors. Assume that $0 < p_1 < p_2$ and $0 < q_1 < q_2$ or that $0 < p_2 < p_1$ and $0 < q_2 < q_1$. Let $s = (s_1, s_2)$, where $s_1 = p_1q_1$ and $s_1 = p_2q_2$. If $a + b \geq 0$, then*

$$D_{a,b}(p)D_{a,b}(q) \leq D_{a,b}(s). \tag{3.17}$$

*If $a + b \leq 0$, then the inequality (3.17) is reversed. A similar result is valid for the Gini means.*

**Proof.** The following function

$$\psi(t) = \ln I_t(s) - \ln[I_t(p)I_t(q)],\ t \in \mathbb{R} \tag{3.18}$$

plays an important role in the proof of (3.17). We shall prove that $\psi(-t) = -\psi(t)$ and also that $\psi(t) \geq 0$ for $t \geq 0$. We have

$$\psi(t)+\psi(-t) = \ln I_t(s)+\ln_{-t}(s)-[\ln I_t(p)+\ln I_{-t}(p)]-[\ln I_t(q)+\ln I_{-t}(q)]$$

$$= 2[\ln G(s) - \ln G(p) - \ln G(q)] = 0.$$

Here we have used the identity $\ln I_t + \ln I_{-t} = 2 \ln G$ which is a special case of (P5) when $a = b = t$. Nonnegativity of the function $\psi(t)$ $(t \geq 0)$ can be established as follows. Let $0 \leq u \leq 1$ and let $v = (u, 1 - u)$. The dot product of $v$ and $p$, denoted by $v \cdot p$, is defined in the usual way

$$v \cdot p = up_1 + (1 - u)p_2.$$

Using the integral representation for the identric mean of order one

$$\ln I(p) = \int_0^1 \ln(v \cdot p)du$$

we obtain

$$\ln[I(p)I(q)] = \int_0^1 \ln[(v \cdot p)(v \cdot q)]du.$$

Application of the Chebyshev inequality

$$(v \cdot p)(v \cdot q) \leq v \cdot s$$

gives

$$\ln[I(p)I(q)] \leq \int_0^1 \ln(v \cdot s)du = \ln I(s).$$

This implies the inequality

$$I_t(p)I_t(q) \leq I_y(s), \ t \geq 0$$

with the inequality reversed if $t \leq 0$. This completes the proof of (3.17) when $a = b = t$ and shows that $\psi(t) \geq 0$ for $t \geq 0$. Assume now that $a \neq b$. Let $a+b \geq 0$. Using (P10) together with the two properties of the function $\psi$ we obtain

$$\begin{aligned} \ln[D_{a,b}(p)D_{a,b}(q)] &= \frac{1}{b-a}\int_a^b \ln[I_t(p)I_t(q)]dt \\ &\leq \frac{1}{b-a}\int_a^b \ln I_t(s)dt = \ln D_{a,b}(s). \end{aligned}$$

If $a+b \leq 0$, then the last inequality is reversed. Proof of the corresponding inequality for the Gini means goes the lines introduced above. We omit further details. □

## 4

The goal of this section is to obtain the Ky Fan type inequalities for the means discussed in this paper.

To this end we will assume that $0 < x,\, y \le \frac{1}{2}$ with $x \neq y$. We define $x' = 1 - x$, $y' = 1 - y$ and write $G'$ for the geometric mean of $x'$ and $y'$, i.e., $G' = G(x', y')$. The same convention will be used for the remaining means which appear in this section.

We need the following.

**Lemma 4.1.** *Let $a \neq 0$. Then*

$$\left|\frac{x^a - y^a}{x^a + y^a}\right| > \left|\frac{(1-x)^a - (1-y)^a}{(1-x)^a + (1-y)^a}\right|. \tag{4.1}$$

**Proof.** For the proof of (4.1) we define a function

$$\phi_a(t) = \frac{t^a - 1}{t^a + 1},\ t > 0.$$

Clearly $\phi_a$ is an odd function in $a$, i.e., $\phi_{-a} = -\phi_a$. In what follows we will assume that $a > 0$. Also, $\phi_a(t) > 0$ for $t > 1$ and $\phi_a(t) < 0$ for $0 < t < 1$. Since both sides of the inequality (4.1) are symmetric, we may assume, without loss a generality, that $x > y > 0$. Let $z = x/y$ and $w = (1-x)/(1-y)$. It is easy to verify that $z > 1 > w > 0$ and $zw > 1$. In order to prove (4.1) it suffices to show that $|\phi_a(z)| > |\phi_a(w)|$ for $a > 0$. Using the inequalities which connect $z$ and $w$ we obtain

$$z^a - w^a > 0 \quad \text{and} \quad (zw)^a > 1.$$

Hence

$$z^a - w^a + (zw)^a - 1 > z^a - w^a - (zw)^a + 1$$

or what is the same that

$$(z^a - 1)(1 + w^a) > (z^a + 1)(1 - w^a).$$

This implies that $\phi_a(z) > -\phi_a(w) > 0$. The proof is complete. $\square$

**Proposition 4.2.** *Let $a \ge 0$. Then*

$$\frac{G}{G'} \le \frac{I_a}{I'_a} \le \frac{A_a}{A'_a} \le \frac{J_a}{J'_a}. \tag{4.2}$$

*Inequalities* (4.2) *are reversed if* $a \leq 0$ *and they become equalities if and only if* $a = 0$.

**Proof.** There is nothing to prove when $a = 0$. Assume that $a \neq 0$. We need the following series expansions

$$A = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{x-y}{x+y}\right)^{2k}\right], \tag{4.3}$$

$$I = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{x-y}{x+y}\right)^{2k}\right], \tag{4.4}$$

$$J = G \exp\left[\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x-y}{x+y}\right)^{2k}\right] \tag{4.5}$$

(see [18], [19], [16]). It follows from (4.3)-(4.5) that for any $a \neq 0$

$$A_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right], \tag{4.6}$$

$$I_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k+1}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right], \tag{4.7}$$

$$J_a = G \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right]. \tag{4.8}$$

Eliminating $G$ between equations (4.6) and (4.7) and next between (4.6) and (4.8), we obtain

$$I_a = A_a \exp\left[-\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right] \tag{4.9}$$

and

$$J_a = A_a \exp\left[\frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}\left(\frac{x^a-y^a}{x^a+y^a}\right)^{2k}\right]. \tag{4.10}$$

Assume that $a > 0$. For the proof of the first inequality in (4.2) we use (4.7) to obtain

$$\frac{I_a}{I'_a} = \frac{G}{G'} \exp\left[\frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{2k+1}(u^{2k} - v^{2k})\right], \tag{4.11}$$

where

$$u = \frac{x^a - y^a}{x^a + y^a} \quad \text{and} \quad v = \frac{(1-x)^a - (1-y)^a}{(1-x)^a + (1-y)^a}.$$

Making use of Lemma 4.1 we obtain $u^{2k} - v^{2k} > 0$ for $k = 1, 2, \ldots$ This in conjunction with (4.11) gives the desired result. Second and third inequalities in (4.2) can be established in an analogous manner using (4.9) and (4.10), respectively. The case $a < 0$ is treated in the same way. □

The main result of this section reads as follows.

**Theorem 4.3.** *Let $a, b \in \mathbb{R}$. If $a + b \geq 0$, then*

$$\frac{G}{G'} \leq \frac{D_{a,b}}{D'_{a,b}} \leq \frac{S_{a,b}}{S'_{a,b}}. \tag{4.12}$$

*Inequalities (4.12) are reversed if $a + b \leq 0$ and they become equalities if and only if $a + b = 0$.*

**Proof.** There is nothing to prove when $a + b = 0$. Assume that $a + b > 0$. For the proof of the first inequality in (4.12) we use (P5) twice with $a = b = t$ to obtain

$$\ln\frac{I_t}{I'_t} + \ln\frac{I_{-t}}{I'_{-t}} = 2\ln\frac{G}{G'}. \tag{4.13}$$

Let us define

$$h(t) = \ln\frac{G}{G'} - \ln\frac{I_t}{I'_t}.$$

It follows from (4.13) that $h(t) = -h(-t)$. Also $h(t) \geq 0$ for $t \geq 0$ and $h(t) \leq 0$ for $t \leq 0$. This is an immediate consequence of the first inequality in (4.2). Making use of (P10) we obtain

$$0 \geq \frac{1}{b-a}\int_a^b h(t)dt = \ln\frac{G}{G'} - \ln\frac{D_{a,b}}{D'_{a,b}}.$$

For the proof of the second inequality in (4.12) we define not $h(t)$ as

$$h(t) = \ln \frac{I_t}{I'_t} - \ln \frac{S_t}{S'_t}.$$

It follows that

$$h(t) = (\ln I_t - \ln S_t) - (\ln I'_t - \ln S'_t).$$

Since both terms on the right side are odd functions in $t$ (see proof of Theorem C) it follows that function $h(t)$ is also odd as a function of variable $t$. Using (4.2) we see that $h(t) \le 0$ for $t > 0$ with the inequality reversed if $t > 0$. This in conjunction with (P10) gives

$$0 \ge \frac{1}{b-a} \int_a^b h(t)dt = \ln \frac{D_{a,b}}{D'_{a,b}} - \ln \frac{S_{a,b}}{S'_{a,b}}.$$

This completes the proof in the case when $a+b \ge 0$. Case when $a+b \le 0$ is treated in an analogous manner. $\square$

**Corollary 4.4.** *The following inequalities are valid*

$$\frac{H}{H'} \le \frac{G}{G'} \le \frac{L}{L'} \le \frac{A}{A'}.$$

**Proof.** To obtain the inequalities in question we use Theorem 4.3 twice letting $(a,b) = (-1,0)$ and $(a,b) = (1,0)$. $\square$

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## 5.8 A note on the Gini means

### 1

In paper [1], the following two means are compared to each others: Let $0 < a < b$. The power mean of two arguments is defined by

$$M_p = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0 \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1}$$

while the Gini mean is defined as

$$S_p = \begin{cases} \left(\dfrac{a^{p-1} + b^{p-1}}{a+b}\right)^{1/(p-2)}, & p \neq 2, \\ S(a,b), & p = 2, \end{cases} \tag{2}$$

where $S(a,b) = (a^b \cdot b^b)^{1/(a+b)}$. The properties of the special mean $S$ have been extensively studied by us e.g. in [7], [8], [9], [10]. In paper [6] it is conjectured that

$$\frac{S_p}{M_p} = \begin{cases} < 1, & \text{if} \quad p \in (0,1) \\ = 1, & \text{if} \quad p \in \{0,1\} \\ > 1, & \text{if} \quad p \in (-\infty, 0) \cup (1, \infty), \end{cases} \tag{3}$$

while in [1], (3) is corrected to the following:

$$\frac{S_p}{M_p} = \begin{cases} < 1, & \text{if} \quad p \in (0,1) \cup (1,2) \\ = 1, & \text{if} \quad p \in \{0,1\} \\ > 1, & \text{if} \quad p \in (-\infty, 0) \cup [2, \infty) \end{cases} \tag{4}$$

For the proof of (4), for $p \notin \{0,1,2\}$, the author denotes $t = b/a > 1$, when

$$\log \frac{S_p}{M_p} = \frac{1}{p} f(t),$$

where

$$f(t) = \frac{p}{p-2} \cdot \log \frac{1+t^{p-1}}{1+t} - \log \frac{1+t^p}{2}, \ t > 1.$$

Then

$$f'(t) = \frac{p}{p-2} \cdot \frac{g(t)}{(1+t)(1+t^{p-1})(1+t^p)},$$

where

$$g(t) = t^{2p-2} - (p-1)t^p + (p-1)t^{p-2} - 1, \ t > 0.$$

It is immediate that

$$g'(t) = (p-1)t^{p-3}h(t), \quad \text{where} \quad h(t) = 2t^p - pt^2 + p - 2.$$

Then the author **wrongly** writes $h'(t) = 2p(t^{p-1} - 1)$. In fact one has $h'(t) = 2pt(t^{p-2} - 1)$, and by analyzing the monotonicity properties, it follows easily that relations (3) are true (and not the corrected version (4)!).

## 2

However, we want to show, that relations (3) are consequences of more general results, which are known in the literature.

In fact, Gini [2] introduced the two-parameter family of means

$$S_{u,v}(a,b) = \begin{cases} \left(\dfrac{a^u + b^u}{a^v + b^v}\right)^{1/(u-v)}, & u \neq v \\ \exp\left(\dfrac{a^u \log a + b^u \log b}{a^u + b^u}\right), & u = v \neq 0 \\ \sqrt{ab}, & u = v = 0 \end{cases} \tag{5}$$

for any real numbers $u, v \in \mathbb{R}$. Clearly,

$$S_{0,-1} = H \quad \text{(harmonic mean)},$$

$$S_{0,0} = G \quad \text{(geometric mean)},$$

$$S_{1,0} = A \quad \text{(arithmetic mean)},$$

$$S_{1,1} = S \quad \text{(denoted also by } J \text{ in [4], [10])},$$

$$S_{p-1,1} = S_p,$$

where $S_p$ is introduced by (2). In 1988 Zs. Páles [5] proved the following result on the comparison of the Gini means (5).

**Theorem 2.1.** *Let $u, v, t, w \in \mathbb{R}$. Then the comparison inequality*

$$S_{u,v}(a, b) \leq S_{t,w}(a, b) \tag{6}$$

*is valid if and only if $u + v \leq t + w$ and*

*i)* $\min\{u, v\} \leq \min\{t, w\}$, *if* $0 \leq \min\{u, v, t, w\}$,

*ii)* $k(u, v) \leq k(t, w)$, *if* $\min\{u, v, t, w\} < 0 < \max\{u, v, t, w\}$,

*iii)* $\max\{u, v\} \leq \max\{t, w\}$, *if* $\max\{u, v, t, w\} \leq 0$.

*Here* $k(x, y) = \begin{cases} \dfrac{|x| - |y|}{x - y}, & x \neq y \\ \operatorname{sign}(x), & x = y. \end{cases}$

The cases of equality are trivial.

Now, remarking that $S_p = S_{p-1,1}$ and $M_p = S_{p,0}$, results (3) will be a consequence of this Theorem. In our case $u = p - 1$, $v = 1$, $t = p$, $w = 0$; so $u + v \leq t + w = p$, i.e. (6) is satisfied.

Now, it is easy to see that denoting

$$\min\{p - 1, 0, 1, p\} = a_p, \quad \max\{p - 1, 0, 1, p\} = A_p,$$

the following cases are evident:

1) $p \leq 0 \Rightarrow p - 1 < p \leq 0 < 1$, so $a_p = p - 1$, $A_p = 1$;

2) $p \in (0, 1] \Rightarrow p - 1 < 0 < p \leq 1$, so $a_p = p - 1$, $A_p = 1$;

3) $p \in (1, 2] \Rightarrow 0 < p - 1 \leq 1 < p$, so $a_p = 0$, $A_p = p$;

4) $p > 2 \Rightarrow 0 < 1 < p - 1 < p$, so $a_p = 0$, $A_p = p$.

In case 2) one has

$$\frac{|p - 1| - 1}{p - 2} \leq \frac{|p|}{p} \quad \text{if} \quad p - 1 < 0 < p$$

only if

$$\frac{1-p-1}{p-2} \leq 1, \quad \text{i.e.} \quad \frac{2(1-p)}{p-2} \leq 0,$$

which is satisfied. The other cases are not possible.

Now, in case $p \notin (0,1)$ write $S_{p,0} < S_{p-1,1}$, and apply the same procedure.

For another two-parameter family of mean values, i.e. the Stolarsky means $D_{u,v}(a,b)$, and its comparison theorems, as well as inequalities involving these means see e.g. [11], [3], [4], [10], and the references.

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## 5.9 Inequalities for the ratios of certain bivariate means

### 1. Introduction

In recent years a problem of comparison of ratios of certain bivariate homogeneous means has attracted attention of researchers (see, e.g., [17], [6]).

In order to formulate this problem let us introduce a notation which will be used throughout the sequel. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ stand for vectors whose components are positive numbers. To this end we will always assume that $a$ and $b$ satisfy the monotonicity conditions

$$\frac{a_1}{a_2} \geq \frac{b_1}{b_2} \geq 1. \tag{1.1}$$

Further, let $\phi$ and $\psi$ be bivariate means. We will always assume that $\phi$ and $\psi$ are homogeneous of degree 1 (or simply homogeneous) in their variables. The central problem discussed in this paper is formulated as follows. Assume that the variables $a_i$ and $b_i$ $(i = 1, 2)$ satisfy monotonicity conditions (1.1). For what means $\phi$ and $\psi$ does the following inequality

$$\frac{\phi(a)}{\phi(b)} \leq \frac{\psi(a)}{\psi(b)} \tag{1.2}$$

hold true? In [6] the authors have proven that the inequality (1.2) is valid for power means of certain order, logarithmic, identric and the Heronian mean of order $\omega$. For the definition of the latter mean see [7] and formula (2.6).

In this paper we shall obtain inequalities of the form (1.2) for the Stolarsky, Gini, Schwab-Borchardt, and the lemniscatic means. Definitions and basic properties of these means are presented in Section 2. The main results are derived in Section 3. We close this paper with a result which deals with the relationship of the Ky Fan inequality and the inequality (1.2).

## 2. Definitions and basic properties of certain bivariate means

We begin with the definition of the Stolarsky means which have been introduced in [18] and studied extensively by numerous researchers (see, e.g., [4], [8], [10], [11], [15]). For $x > 0$, $y > 0$ and $p, q \in \mathbb{R}$, they are denoted by $D_{p,q}(x, y)$, and defined for $x \neq y$ as

$$D_{p,q}(x,y) = \begin{cases} \left[\dfrac{q(x^p - y^p)}{p(x^q - y^q)}\right]^{\frac{1}{p-q}}, & pq(p-q) \neq 0 \\ \exp\left(-\dfrac{1}{p} + \dfrac{x^p \ln x - y^p \ln y}{x^p - y^p}\right), & p = q \neq 0 \\ \left[\dfrac{x^p - y^p}{p(\ln x - \ln y)}\right]^{\frac{1}{p}}, & p \neq 0,\ q = 0 \\ \sqrt{xy}, & p = q = 0. \end{cases} \tag{2.1}$$

Also, $D_{p,q}(x, x) = x$.

Stolarsky means are sometimes called the extended means or the difference means (see [8], [10], [15]).

A second family of bivariate means employed in this paper was introduced by C. Gini [5]. Throughout the sequel they will be denoted by $S_{p,q}(x, y)$. Following [5]

$$S_{p,q}(x,y) = \begin{cases} \left[\dfrac{x^p + y^p}{x^q + y^q}\right]^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\dfrac{x^p \ln x + y^p \ln y}{x^p + y^p}\right), & p = q \neq 0 \\ \sqrt{xy}, & p = q = 0. \end{cases} \tag{2.2}$$

Gini means are also called the sum means (see, e.g., [10]).

For the reader's convenience we recall basic properties of these two families of means. Properties (P1)-(P3) follow directly from (2.1) and (2.3). Properties (P4)-(P6) are established in [8], [18] and [11]. For the sake of presentation, let $\phi_{p,q}$ stand either for the Stolarsky or Gini mean of order $(p, q)$. We have

(P1) $\phi_{p,q}(\cdot,\cdot) = \phi_{q,p}(\cdot,\cdot)$.

(P2) $\phi_{\cdot,\cdot}(x, y) = \phi_{\cdot,\cdot}(y, x)$.

(P3) $\phi_{p,q}(x, y)$ is homogeneous of degree 1 in its variables, i.e.,

$$\phi_{p,q}(\lambda x, \lambda y) = \lambda\phi_{p,q}(x, y),\ \ \lambda > 0.$$

(P4) $\phi_{p,q}(\cdot,\cdot)$ increases with increase in either $p$ or $q$.

(P5) $\ln D_{p,q}(x, y) = \begin{cases} \dfrac{1}{p-q}\displaystyle\int_p^q \ln I_t(x, y)dt, & p \neq q \\ \ln I_p(x, y), & p = q, \end{cases}$

where

$$I_p(x, y) = D_{p,p}(x, y) \tag{2.3}$$

is the identric mean of order $p$. Similarly

(P6) $\ln S_{p,q}(x, y) = \begin{cases} \dfrac{1}{q-p}\displaystyle\int_p^q \ln J_t(x, y)dt, & p \neq q \\ \ln S_t(x, y), & p = q, \end{cases}$

where

$$S_p(x, y) = S_{p,p}(x, y). \tag{2.4}$$

Other means used in this paper include the power mean $A_p$ of order $p \in \mathbb{R}$. Recall that

$$A_p(x, y) = \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p}, & p \neq q \\ \sqrt{xy}, & p = 0. \end{cases} \tag{2.5}$$

The Heronian mean $H_\omega$ of order $\omega \geq 0$ is defined as

$$H_\omega(x, y) = \frac{x + y + \omega\sqrt{xy}}{2 + \omega} \tag{2.6}$$

(see [7]). Also we will deal with the harmonic, geometric, logarithmic, identric, arithmetic and centroidal means of order one. They will be denoted by $H$, $G$, $L$, $I$, $A$ and $C$, respectively. They are special cases of the

Stolarsky mean $D_{p,q}$. We have

$$\begin{gathered} H = D_{-2,-1}, \quad G = D_{0,0}, \quad L = D_{0,1}, \quad H_1 = D_{1/2,3/2} \\ I = D_{1,1}, \quad A = D_{1,2}, \quad C = D_{2,3}. \end{gathered} \tag{2.7}$$

The Comparison Theorem for the Stolarsky means (see, eg., [15]) implies the chain of inequalities

$$H < G < L < H_1 < I < A < C \tag{2.8}$$

provided $x \neq y$.

Another mean used in this paper is commonly referred to as the Schwab-Borchardt mean. Now let $x \geq 0$ and $y > 0$. The latter mean, denoted by $SB(x, y) \equiv SB$, is defined as the common limit of two sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$, i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1} y_n}, \tag{2.9}$$

$n \geq 0$ (see [2]). It is known that the mean under discussion can be expressed in terms of the elementary transcendental functions

$$SB(x, y) = \begin{cases} \dfrac{\sqrt{y^2 - x^2}}{\arccos(x/y)}, & 0 \leq x < y \\ \dfrac{\sqrt{x^2 - y^2}}{\operatorname{arccosh}(x/y)}, & y < x \\ x, & x = y \end{cases}$$

(see [1, Theorem 8.4], [2, (2.3)]). The Schwab-Borchardt mean has been studied extensively in recent papers [12] and [14].

The lemniscatic mean of $x > 0$ and $y \geq 0$, denoted by

$$LM(x, y) \equiv LM,$$

is also the iterative mean, i.e.,

$$LM = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1}x_n}, \quad n \geq 0.$$

The explicit formula

$$[LM(x,y)]^{-1/2} = \begin{cases} (x^2-y^2)^{-1/4}\mathrm{arcsl}\left(1-\dfrac{y^2}{x^2}\right)^{1/4}, & y<x \\ (y^2-x^2)^{-1/4}\mathrm{arcslh}\left(\dfrac{y^2}{x^2}-1\right)^{1/4}, & x<y \\ x^{-1/2}, & x=y \end{cases}$$

involves two incomplete symmetric integrals of the first kind

$$\mathrm{arcsl}x = \int_0^x \frac{dt}{\sqrt{1-t^4}}, \quad |x| \leq 1$$

and

$$\mathrm{arcslh}x = \int_0^x \frac{dt}{\sqrt{1+t^4}},$$

which are also called the Gauss lemniscate functions, (see [2, (2.5)-(2.6)], [1, p. 259]). It is known [2, (4.1)] that

$$\mathrm{arcsl}x = xR_B(1, 1-x^4) \tag{2.10}$$

and

$$\mathrm{arcslh}x = xR_B(1, 1+x^4), \tag{2.11}$$

where

$$R_B(x,y) = \frac{1}{4}\int_0^\infty (t+x)^{-3/4}(t+y)^{-1/2}dt \tag{2.12}$$

(see [2, (3.14)]). The lemniscatic mean has been studied extensively in [9].

For later use let us record the fact that both $SM$ and $LM$ are homogeneous of degree 1, however, they are not symmetric in their variables. We shall make use of the inequality which has been established in [9, Theorem 5.2]:

$$SB(x,y) \le LM(y,x) \le A \le LM(x,y) \le SB(y,x) \tag{2.13}$$

provided $0 < y \le x$. Inequalities (2.13) are reversed if $y \ge x > 0$.

## 3. Main results

Before we state and prove one of the main results of this section (Theorem 3.3) we shall investigate a function $u(t)$ which is defined as follows

$$u(t) \equiv u(t;x) = \frac{d}{dx} I_t(x,1)$$

($0 < x < 1$), where $I_t$ is the identric mean defined in (2.3). It follows from (2.1) that

$$u(t) = \begin{cases} \dfrac{x^{2t-1} - x^{t-1} - tx^{t-1}\ln x}{(x^t-1)^2}, & t \ne 0 \\ \dfrac{1}{2x}, & t = 0. \end{cases} \tag{3.1}$$

We need the following:

**Lemma 3.1.** *The function $u(t)$ has the following properties*

$$u(t) \ge 0,\ t \in \mathbb{R}, \tag{3.2}$$

$$u(-t) + u(t) = 2u(0), \tag{3.2}$$

$$u(t) \textit{ is strictly decreasing for every } t \ne 0, \tag{3.4}$$

$$u(t) \textit{ is strictly convex for } t > 0 \textit{ and strictly concave for } t < 0. \tag{3.5}$$

**Proof.** In order to establish the inequality (3.2) it suffices to apply the inequality $\ln x^t < x^t - 1$ to the right side of (3.1). Formula (3.3) follows easily from (3.1). For the proof of monotonicity property (3.4) we differentiate (3.1) to obtain

$$\frac{(x^t-1)^3}{x^{t-1}\ln x}u'(t) = y\ln y + \ln y - 2y + 2, \tag{3.6}$$

where $y = x^t$. Letting $z = x^{-t}$ we can rewrite the right side of (3.6) as

$$\frac{(x^t-1)^3}{x^{t-1}\ln x}u'(t) = \frac{(z-1)(z+1)}{z}\left[\frac{1}{A(z,1)} - \frac{1}{L(z,1)}\right]. \tag{3.7}$$

Let $t > 0$. Then $0 < x^t < 1$. This in turn implies that $z > 1$. Application of the well-known inequality $L(z,1) < A(z,1)$ shows that the right side of (3.7) is negative. Hence $u'(t) < 0$ for $t > 0$. The same argument can be used that $u'(t) < 0$ for positive $t$. This completes the proof of (3.4). For the proof of (3.5) we differentiate (3.6) to obtain

$$\frac{(x^t-1)^4}{x^{t-1}(\ln x)^2}u''(t) = 3(y^2-1) - (\ln y)(y^2+4y+1). \tag{3.8}$$

The right side of (3.8) can also be written as

$$6(\ln y)\left[L(y^2-1) - \frac{A(y^2,1)+2G(y^2,1)}{3}\right] =: R.$$

Let $t > 0$. Then $y < 1$. This in turn implies that $R > 0$ because

$$L < \frac{A+2G}{3} \tag{3.9}$$

(see [3], [12]). This in conjunction with (3.8) shows that $u''(t) > 0$ for $t > 0$. Since the proof of strict concavity of $u(t)$ when $t < 0$ goes along the lines introduced above, it is omitted. □

For later use let us record a generalization of the classical Hermite-Hadamard inequalities.

**Proposition 3.2.** ([4]) *Let $f(t)$ be a real-valued function which is concave for $t < 0$, convex for $t > 0$, and satisfies the symmetry condition*

$$f(-t) + f(t) = 2f(0).$$

*Then for any $r$ and $s$ ($r \neq s$) in the domain of $f(t)$ the following inequalities*

$$f\left(\frac{r+s}{2}\right) \le \frac{1}{s-r}\int_r^s f(t)dt \le \frac{1}{2}[f(r)+f(s)] \tag{3.10}$$

*hold true provided $r+s \ge 0$. Inequalities (3.10) are reversed if $r+s \le 0$.*

We are in a position to prove the following.

**Theorem 3.3.** *Let the vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ of positive numbers be such that the inequalities (1.1) are satisfied. Further, let the numbers $p$, $q$, $r$ and $s$ satisfy the conditions $p \le q$ and $r \le s$. Then the following inequality*

$$\frac{D_{r,s}(a)}{D_{r,s}(b)} \le \frac{D_{p,q}(a)}{D_{p,q}(b)} \tag{3.11}$$

*is satisfied if either*

*(i) $r+s \ge 0$ and $p \ge \dfrac{r+s}{2}$*

*or*

*(ii) $r+s \le 0$ and $p \ge r$*

*or*

*(iii) $p+q \ge 0$ and $s \le p$*

*or*

*(iv) $p+q \le 0$ and $s \le \dfrac{p+q}{2}$.*

**Proof.** The following function

$$\phi(x) = \frac{D_{p,q}(x,1)}{D_{r,s}(x,1)}, \tag{3.12}$$

$0 < x < 1$ plays a crucial role in the proof of the inequality (3.11). Logarithmic differentiation together with the use of (P5) yields

$$\frac{\phi'(x)}{\phi(x)} = \begin{cases} \dfrac{1}{q-p}\displaystyle\int_p^q u(t)dt - \dfrac{1}{s-r}\displaystyle\int_r^s u(t)dt, & p \neq q \text{ and } r \neq s \\ u(p) - \dfrac{1}{s-r}\displaystyle\int_r^s u(t)dt, & p = q \text{ and } r \neq s \\ \dfrac{1}{q-p}\displaystyle\int_p^q u(t)dt - u(r), & p \neq q \text{ and } r = s \\ u(p) - u(r), & p = q \text{ and } r = s, \end{cases} \tag{3.13}$$

where

$$u(t) = \frac{d}{dx} I_t(x, 1).$$

We shall prove that $\phi(x)$ is a decreasing function on its domain. Consider the case when $r + s \geq 0$ and $p \geq (r + s)/2$. Taking into account that the function $u(t)$ is strictly decreasing for $t \neq 0$ (see (3.4)) we have

$$\frac{1}{q-p} \int_p^q u(t)dt \leq u(p). \tag{3.14}$$

This in conjuction with the first inequality in (3.10) and the first line of (3.13) gives

$$\frac{\phi'(x)}{\phi(x)} \leq u(p) - u\left(\frac{r+s}{2}\right) \leq 0,$$

where the last inequality holds true because $p \geq (r + s)/2$.
Hence $\phi'(x) \leq 0$ for $0 < x < 1$. Assume now that $r + s \leq 0$. Making use of (3.14) and the second inequality in (3.10) applied to the expression on the right side in the second line of (3.13) we obtain

$$\frac{\phi'(x)}{\phi(x)} \leq u(p) - \frac{1}{2}[u(r) + u(s)] = \frac{1}{2}[u(p) - u(r)] + \frac{1}{2}[u(p) - u(s)] \leq 0,$$

where the last inequality holds true provided $p \geq r$ and $p \geq s$. Since $r \leq p$, $\phi'(x) \leq 0$ provided $p \geq r$. Assume now that $p + q \geq 0$. Utilizing monotonicity of the function $u(t)$ together with the use of $r \leq s$ gives

$$\frac{1}{s-r} \int_r^s u(t)dt \geq u(s). \tag{3.15}$$

This in conjunction with the third member of (3.13) and the second inequality in (3.10) gives

$$\frac{\phi'(x)}{\phi(x)} \leq \frac{1}{2}[u(p) + u(q)] - u(s) = \frac{1}{2}[u(p) - u(s)] + \frac{1}{2}[u(q) - u(s)] \leq 0,$$

where the last inequality is valid provided $p \geq s$ and $q \geq s$.
Thus $\phi'(x) \leq 0$ if $s \leq p$. Finally, let $p + q \leq 0$. Then

$$\frac{\phi'(x)}{\phi(x)} \leq u\left(\frac{p+q}{2}\right) - u(s), \tag{3.16}$$

where the last inequality follows from the first inequality in (3.10) and from (3.15). Since $u(t)$ is strictly decreasing, the right side of (3.16) is nonpositive if $s \le (p+q)/2$. The desired property of the function $\phi(x)$ now follows. In order to establish the inequality (3.11) we employ the inequality $\phi(x) \le \phi(y)$ with

$$x = \frac{a_2}{a_1} \le \frac{b_2}{b_1} = y < 1.$$

Making use of (3.12) and properties (P2) and (P3) we obtain the assertion. The proof is complete. □

We shall establish now an inequality between the ratios of the Stolarsky and Gini means.

**Theorem 3.4.** *Let the vectors a and b satisfy assumptions of Theorem 3.3. If $p+q \ge 0$, then*

$$\frac{D_{p,q}(a)}{D_{p,q}(b)} \le \frac{S_{p,q}(a)}{S_{p,q}(b)}. \tag{3.17}$$

*Inequality (3.17) is reversed if $p+q \le 0$.*

**Proof.** Let now

$$\phi(x) = \frac{D_{p,q}(x,1)}{S_{p,q}(x,1)}, \tag{3.18}$$

where $0 < x < 1$. Using (P5) and (P6) we obtain

$$\ln \phi(x) = \begin{cases} \dfrac{1}{q-p}\displaystyle\int_p^q [\ln I_t(x,1) - \ln S_t(x,1)]dt, & p \ne q \\ \ln I_p(x,1) - \ln S_p(x,1), & p = q. \end{cases}$$

Differentiation with respect to $x$ gives

$$\frac{\phi'(x)}{\phi(x)} = \begin{cases} \dfrac{1}{q-p}\displaystyle\int_p^q u(t)dt, & p \ne q \\ u(p), & p = q, \end{cases} \tag{3.19}$$

where now

$$u(t) = \frac{d}{dx}[\ln I_t(x,1) - \ln S_t(x,1)].$$

Making use of (2.3), (2.1), (2.4), and (2.2) we obtain

$$\ln I_t(x,1) - \ln S_t(x,1) = -\frac{1}{t} + \frac{2x^t \ln x}{x^{2t}-1}, \ t \neq 0.$$

Hence

$$u(t) = \frac{2x^{t-1}}{(x^{2t}-1)^2}[x^{2t} - 1 - (x^{2t}+1)\ln x^t]. \tag{3.20}$$

We shall prove that the function $u(t)$ has the following properties

$$u(t) \begin{cases} > 0 & \text{if} \quad t > 0, \\ < 0 & \text{if} \quad t < 0 \end{cases} \tag{3.21}$$

and

$$u(-t) = -u(t). \tag{3.22}$$

For the proof of (3.21) we substitute $y = x^t$ into (3.20) to obtain

$$\begin{aligned} u(t) &= \frac{4x^{t-1}\ln x}{(x^{2t}-1)^2}\left(\frac{y^2-1}{\ln y^2} - \frac{y^2+1}{2}\right) \\ &= \frac{4x^{t-1}\ln x}{(x^{2t}-1)^2}[L(y^2-1) - A(y^2,1)]. \end{aligned}$$

Since $0 < x < 1$, $0 < y < 1$ for $t > 0$ and $y > 1$ for $t < 0$, the inequality of the logarithmic and arithmetic means implies (3.21). For the proof of (3.22) we rewrite (3.20) as

$$u(t) = \frac{2}{x} \cdot \frac{y}{(y^2-1)^2}[y^2 - 1 - (y^2+1)\ln y],$$

where $y = x^t$. Easy computations give the assertion. It follows from (3.19), (3.21) and (3.22) that $\phi'(x) \geq 0$ if $p + q \geq 0$ and $\phi'(x) \leq 0$ if $p + q \leq 0$ with equalities if $p + q = 0$. To complete the proof of (3.17) we let

$$x = \frac{a_2}{a_1} \leq \frac{b_2}{b_1} = y < 1$$

in $\phi(x) \leq \phi(y)$ when $p + q \geq 0$. This in conjunction with (3.18) and properties (P2) and (P3) completes the proof. The case when $p + q \leq 0$ can be treated in an analogous manner. This completes the proof. □

Our next result reads as follows.

**Theorem 3.5.** *Let the vectors a and b satisfy monotonicity conditions (1.1). Then*

$$\begin{aligned}\frac{H(a)}{H(b)} &\le \frac{G(a)}{G(b)} \le \left[\frac{G^2(a)A(a)}{G^2(b)A(b)}\right]^{1/3} \le \frac{L(a)}{L(b)} \\ &\le \frac{H_4(a)}{H_4(b)} \le \frac{H_1(a)}{H_1(b)} \le \frac{A_{2/3}(a)}{A_{2/3}(b)} \le \frac{I(a)}{I(b)} \\ &\le \frac{H_{e-2}(a)}{H_{e-2}(b)} \le \frac{H_\omega(a)}{H_\omega(b)} \le \frac{A(a)}{A(b)} \le \frac{C(a)}{C(b)}. \end{aligned} \tag{3.23}$$

**Proof.** The first inequality in (3.23) follows from (3.11) and (2.7) with $r = -2$, $s = -1$, $p = q = 0$ while the second one is an immediate consequence of $G(a)/G(b) \le A(a)/A(b)$ which is a part of (3.23). For the proof of the third inequality in (3.23) we define a function

$$\phi(x) = \frac{L^3(x,1)}{G^2(x,1)A(x,1)}, \tag{3.24}$$

$0 < x < 1$. We shall prove that $\phi(x)$ is a decreasing function on the stated domain. Logarithmic differentiation gives

$$\frac{\phi'(x)}{\phi(x)} = 3\left(\frac{1}{x-1} - \frac{1}{x\ln x}\right) - \frac{2x+1}{x(x+1)}.$$

Letting $x = 1/t$ $(t > 1)$ we see that the last formula can be written as

$$\frac{\phi'(x)}{\phi(x)} = \frac{3t}{t-1}\left[\frac{t-1}{\ln t} - \frac{t^2+4t+1}{3(t+1)}\right]. \tag{3.25}$$

To complete the proof of monotonicity of $\phi(x)$ we apply Carlson's inequality (3.9) to obtain

$$\frac{t-1}{\ln t} \le \frac{t^2+4t+1}{3(t+1)}.$$

This in conjunction with (3.25) gives the desired result. To complete the proof of the inequality in question we follow the lines introduced at the end of the proofs of Theorems 3.3 and 3.4. The fourth, sixth, and eighth inequalities in (3.23) are established in [6]. (See Theorems 3.2, 3.1, and 3.3, respectively.) The fifth, ninth, and the tenth inequalities in (3.23) are a consequence of the monotonicity in $\omega$ of the ratio $H_\omega(a)/H_\omega(b)$. We have

$$\frac{H_\alpha(a)}{H_\alpha(b)} \le \frac{H_\beta(a)}{H_\beta(b)} \tag{3.26}$$

provided $\alpha > \beta \ge 0$ and $0 < x \le$. For, let

$$\phi(x) = \frac{H_\alpha(x,1)}{H_\beta(x,1)}. \tag{3.27}$$

Differentiating we obtain

$$\phi'(x) = \frac{(2+\beta)(\alpha-\beta)}{2+\alpha} \cdot \frac{1-x}{2\sqrt{x}\,(x+1+\beta\sqrt{x})^2}.$$

Thus $\phi(x)$ is increasing for $0 < x \le 1$. Letting in (3.27)

$$x = \frac{a_2}{a_1} \le \frac{b_2}{b_1} = y \le 1$$

we obtain the inequality (3.26). The seventh inequality in (3.23) is a consequence of the fact that $A_{2/3}(x,1)/I(x,1)$ is a decreasing function for $0 < x < 1$ (see [13, p. 104]). The remaining part of the proof of the inequality in question goes along the lines introduced in the proofs of Theorems 3.3 and 3.4. The last inequality in (3.23) is a special case of (3.11) when $r = 1$, $s = 2$, $p = 2$ and $q = 3$. The proof is complete. $\square$

We shall now derive inequalities involving ratios of the Schwab-Borchardt means and the lemniscatic means. The following result, sometimes called the L'Hospital-type rule for monotonicity, will be utilized in the sequel.

**Proposition 3.6.** ([21]) *Let $f$ and $g$ be continuous functions on $[c,d]$. Assume that they are differentiable and $g'(t) \ne 0$ on $(c,d)$. If $f'/g'$ is strictly increasing (decreasing) on $(c,d)$, then so are*

$$\frac{f(t)-f(c)}{g(t)-g(c)} \quad \textit{and} \quad \frac{f(t)-f(d)}{g(t)-g(d)}.$$

(See also [16].)

We are in a position to prove the following.

**Theorem 3.7.** *Let the vectors satisfy the monotonicity conditions (1.1). Then the following inequalities*

$$\frac{SB(a_1,a_2)}{SB(b_1,b_2)} \le \frac{LM(a_2,a_1)}{LM(b_2,b_1)} \le \frac{LM(a_1,a_2)}{LM(b_1,b_2)} \le \frac{SB(a_2,a_1)}{SB(b_2,b_1)} \tag{3.28}$$

*hold true.*

**Proof.** In order to establish the first inequality in (3.28) we introduce a function

$$\phi(x) = \frac{SB(x,1)}{LM(1,x)} \tag{3.29}$$

$(x \ge 1)$. Making use of

$$SB(x,1) = \frac{t^2}{\text{arcsinh } t^2}$$

(see [12, (1.3)]) and

$$LM(1,x) = \frac{t^2}{(\text{arcslh}\, t)^2}$$

(see [9, (6.2]) we obtain

$$\phi(x) = \frac{(\text{arcslh } t)^2}{\text{arcsinh } t^2},$$

where $t = \sqrt[4]{x^2-1}$. To prove that $\phi(x)$ is an increasing function on its domain we write

$$\phi(x) = \frac{f(t)}{g(t)},$$

where $f(t) = (\text{arcslh } t)^2$ and $g(t) = \text{arcsinh } t^2$ $(t \ge 0)$. Differentiation gives

$$\frac{f'(t)}{g'(t)} = \frac{\text{arcslh } t}{t} = R_B(1, 1+t^4),$$

where in the last step we have used (2.11). Since $R_B$ is a decreasing function in each of its variables (see (2.12)) we conclude, using Proposition

3.6 and the fact that $f(0) = g(0) = 0$, that $\phi(x)$ has the desired property, i.e., $\phi(x) \geq \phi(y)$ whenever $x \geq y$. Letting

$$x = \frac{a_1}{a_2} \geq \frac{b_1}{b_2} = y \geq 1$$

and next using (3.29) and the fact that both means $SB$ and $LM$ are homogeneous we obtain the assertion. For the proof of the second inequality in (3.28), we define

$$\phi(x) = \frac{LM(1,x)}{LM(x,1)}$$

$(x \geq 1)$. Using [9, (6.1)-(6.2)] we obtain

$$\phi(x) = \left[\frac{f(t)}{g(t)}\right]^2, \tag{3.30}$$

where

$$f(t) = \operatorname{arcsl}\left(\frac{t}{\sqrt[4]{1+t^4}}\right) = tR_B(1+t^4, 1)$$

and

$$g(t) = \operatorname{arcslh} t = tR_B(1, 1+t^4) \quad \text{and} \quad t = \sqrt[4]{x^2-1}.$$

Taking into account that

$$f'(t) = (1+t^4)^{-3/4} \quad \text{and} \quad g'(t) = (1+t^4)^{-1/2}$$

we see that

$$\frac{f'(t)}{g'(t)} = (1+t^4)^{-1/4}$$

is the decreasing function for $t \geq 0$. Making use of Proposition 3.6 we conclude that the function $f(t)/g(t)$ decreases with an increase in $t$. This together with (3.30) implies that $\phi(x) \leq \phi(y)$ whenever $x > y$. We now follow the lines introduced in the proof of the first inequality in (3.28) to obtain the desired result. In order to establish the third inequality in (3.28) we define

$$\phi(x) = \frac{LM(x,1)}{SB(1,x)} \tag{3.31}$$

$(x \geq 1)$. In order to prove that $\phi(x)$ is a decreasing function on its domain it suffices to show that a function

$$\psi(x) = \phi\left(\frac{1}{x}\right)$$

is the increasing function on $(0, 1]$. Using (3.31) and the fact that $LM$ and $SB$ are homogeneous functions we obtain

$$\psi(x) = \frac{LM(1, x)}{SB(x, 1)}$$

$(0 < x \leq 1)$. Making use of [9, (6.1)] and [12, (1.2)] we obtain

$$\psi(x) = \frac{f(t)}{g(t)}, \tag{3.32}$$

where $f(t) = \arcsin t^2$, $g(t) = (\text{arcsl } t)^2$ and $t = \sqrt[4]{1 - x^2}$. Hence

$$\frac{f'(t)}{g'(t)} = \frac{t}{\text{arcsl } t} = \frac{1}{R_B(1, 1 - t^4)},$$

where the last equality follows from (2.10). We conclude that the ratio $f'(t)/g'(t)$ is a decreasing function of $t$ because $R_B$ is also decreasing in each of its variables. This in conjunction with Proposition 3.6 applied to (3.32) and the fact that $t$ and $x$ satisfy $t = \sqrt[4]{1 - x^2}$ leads to the conclusion that $\psi(x)$ is an increasing function on $(0, 1]$. This in turn implies that $\phi(x)$ defined in (3.31) is decreasing for every $x \geq 1$. We follow the lines introduced earlier in this proof to complete the proof of the last inequality in (3.28). □

Before we state and prove a corollary of Theorem 3.7, let us introduce some special means derived from $SB$ and $LM$. To this end let $x > 0$, $y > 0$ and let $G$, $A$ and

$$Q \equiv Q(x, y) = \sqrt{\frac{x^2 + y^2}{2}}$$

stand for the geometric mean, arithmetic mean and the root-mean – square mean of $x$ and $y$. Following [12, (2.8)] let

$$\begin{aligned} L &= SB(A,G), \quad P = SB(G,A), \\ M &= SB(Q,A), \quad T = SB(A,Q), \end{aligned} \tag{3.33}$$

where $L$ stands for the logarithmic mean and $P$ and $T$ are the Seiffert means (see [19], [20]). Clearly all four means defined above are symmetric and homogeneous of degree 1. The lemniscate counterparts of these means have been introduced in [9, (6.4)]:

$$\begin{aligned} U &= LM(G,A), \quad V = LM(A,G), \\ R &= LM(A,Q), \quad J = LM(Q,A). \end{aligned} \tag{3.34}$$

It is easy to see that these means are symmetric and homogeneous of degree 1. The following inequalities

$$L \le U \le V \le P \le A \le M \le R \le J \le T \tag{3.35}$$

have been established in [9, (6.10)].

We are in a position to establish the following.

**Corollary 3.8.** *The means defined in (3.33) and (3.34) satisfy the following inequalities*

$$\frac{L}{M} \le \frac{U}{R} \le \frac{V}{J} \le \frac{P}{T}. \tag{3.36}$$

**Proof.** Let $a_1 = A$, $a_2 = G$, $b_1 = Q$ and $b_2 = A$. Since $A^2 \ge GQ$, the numbers $a_i$ and $b_i$ satisfy the inequalities (1.1). Utilizing (3.28) and (3.33) and (3.34) one obtains the assertion (3.36). □

Let $a$ and $b$ satisfy (1.1). Then the inequalities (3.35) can be obtained immediately from

$$\begin{aligned} \frac{L(a)}{L(b)} &\le \frac{U(a)}{U(b)} \le \frac{V(a)}{V(b)} \le \frac{P(a)}{P(b)} \le \frac{A(a)}{A(b)} \\ &\le \frac{M(a)}{M(b)} \le \frac{R(a)}{R(b)} \le \frac{J(a)}{J(b)} \le \frac{T(a)}{T(b)} \end{aligned} \tag{3.37}$$

by letting $b_1 = b_2$. Since the proof of (3.37) goes along the lines introduced in [9, Theorem 6.2], it is omitted.

We close this section with a result which shows that the inequality (1.2) implies the Ky Fan inequality for the means $\phi$ and $\psi$:

$$\frac{\phi(a)}{\phi(a')} \leq \frac{\psi(a)}{\psi(a')}, \tag{3.38}$$

where $a = (a_1, a_2)$ with $0 < a_1, a_2 \leq \dfrac{1}{2}$ and

$$a' = 1 - a = (1 - a_1, 1 - a_2). \tag{3.39}$$

**Proposition 3.9.** *Let $\phi$ and $\psi$ be symmetric homogeneous means of two positive variables and assume that the inequality (1.2) holds true for the vectors $a$ and $b$ which satisfy monotonicity conditions (1.1). Then the means $\phi$ and $\psi$ also satisfy the Ky Fan inequalities (3.38).*

**Proof.** Without a loss of generality let us assume that $a = (a_1, a_2)$ is such that $0 < a_2 < a_1 \leq \dfrac{1}{2}$ and $b = (b_1, b_2) = (1 - a_2, 1 - a_1)$. It is easy to verify that $a$ and $b$ satisfy (1.1). Since $\phi$ and $\psi$ are symmetric means, inequality (1.2) holds true with $b$ replaced by $a'$ (see (3.39)). $\square$

Application of Proposition 3.9 to Theorems 3.1-3.3 in [6] gives immediately Theorems 4.1, 4.2 and 4.4 in [6].

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## 5.10 Inequalities involving logarithmic mean of arbitrary order

### 1. Introduction

The history of mean values is long and laden with detail. Among means of two variables the logarithmic mean has attracted attention of several researchers. A two-parameter generalizations of the logarithmic mean have been introduced by K. B. Stolarsky (see [15]). A particular case of Stolarsky mean is called the logarithmic mean of arbitrary order (see 2.1). The goal of this note is to establish new inequalities satisfied by the latter mean. Some known inequalities involving logarithmic mean of order one are special cases of the main results established in this paper. In Section 2 we give definitions of bivariate means used in the sequel. Also, some known inequalities involving hyperbolic functions are included in this section. The main results of this note are established in Section 3.

### 2. Definitions and preliminaries

Throughout the sequel we will assume that $x$ and $y$ are positive and unequal numbers. We begin this section with definitions of certain bivariate means used in the sequel. The logarithmic mean of order $t \in \mathbb{R}$ of $x$ and $y$, denoted by $L_t(x, y) \equiv L_t$, is defined as follows [11]:

$$L_t(x,y) = \begin{cases} L(x^t, y^t)^{\frac{1}{t}} & \text{if } \ t \neq 0, \\ G(x,y) & \text{if } \ t = 0, \end{cases} \tag{2.1}$$

where

$$L(x,y) \equiv L = \frac{x-y}{\ln x - \ln y}$$

is the logarithmic mean of order one and

$$G(x,y) \equiv G = \sqrt{xy}$$

is the geometric mean of $x$ and $y$. Another mean used in this paper is the power mean $A_t(x,y) \equiv A_t$ of order $t \in \mathbb{R}$:

$$A_t(x,y) = \begin{cases} \left(\dfrac{x^t + y^t}{2}\right)^{\frac{1}{t}} & \text{if } \ t \neq 0, \\ G(x,y) & \text{if } \ t = 0. \end{cases} \tag{2.2}$$

It is worth mentioning that all means defined above belong to a two-parameter family of mens introduced by K.B. Stolarsky in [15]. These means have been studied by several researchers. See, e.g., [10], [6] and the references therein.

The key inequality used in this paper is the following one

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} \tag{2.3}$$

($x \neq 0$). First inequality in (2.3) is due to Lazarević [2] while the second one is commonly referred to as the Cusa-Huygens inequality for hyperbolic functions. Inequalities (2.3) are special cases of inequalities established in [9].

For later use let us recall a result which has been established in [5] (see Theorem 3.2).

**Theorem 2.1** *Let $u$, $v$, $\gamma$ and $\delta$ be positive numbers which satisfy the following conditions*

*(i) $min(u,v) < 1 < max(u,v)$,*

*(ii) $1 < u^\gamma v^\delta$,*

*(iii) $\gamma + \delta < \gamma\dfrac{1}{u} + \delta\dfrac{1}{v}$.*

*Then the following inequality*

$$2 < \left(\frac{1}{u}\right)^{\gamma p} + \left(\frac{1}{v}\right)^{\delta p} < u^{\gamma p} + v^{\delta p}. \tag{2.4}$$

*holds true provided $\gamma \geq 1$, $\delta \geq 1$, and $p \geq 1$. Second inequality in (2.4) is valid if $p > 0$.*

We will also utilize the following result (see [5], Theorem 3.1).

**Theorem 2.2.** *Assume that the numbers $u$, $v$, $\gamma$ and $\delta$ satisfy assumptions of Theorem 2.1. Further, let $\alpha$ and $\beta$ be positive numbers and assume that $v < 1 < u$. Then*

$$\alpha + \beta < \alpha u^p + \beta v^q \tag{2.5}$$

*if either*

$$p > 0 \qquad \text{and} \qquad q \le p\frac{\delta\alpha}{\gamma\beta}, \tag{2.6}$$

*or if*

$$q \le p \le -1 \qquad \text{and} \qquad \delta\alpha \le \gamma\beta. \tag{2.7}$$

Conditions of validity of (2.5) when $u < 1 < v$ are also obtained in [5]. We omit further details.

## 3. Main results

In this section we shall establish inequalities involving logarithmic mean $L_t$. For the later use let us introduce a variable $\lambda = (t/2)\ln(x/y)$ ($t \in \mathbb{R}$). One can easily verify, using (2.1)-(2.2), that

$$A\left(e^{\lambda}, e^{-\lambda}\right) = \cosh\lambda = \left(\frac{A_t}{G}\right)^t \tag{3.1}$$

and

$$L\left(e^{\lambda}, e^{-\lambda}\right) = \frac{\sinh\lambda}{\lambda} = \left(\frac{L_t}{G}\right)^t. \tag{3.2}$$

This implies that

$$\frac{\tanh\lambda}{\lambda} = \left(\frac{L_t}{A_t}\right)^t \tag{3.3}$$

Our first result reads as follows.

**Theorem 3.1.** *Let $x$ and $y$ be positive and unequal numbers, let $t \ne 0$, and let $p \ge 1$. Then*

$$2 < \left(\frac{G}{L_t}\right)^{2pt} + \left(\frac{A_t}{L_t}\right)^{pt} < \left(\frac{L_t}{G}\right)^{2pt} + \left(\frac{L_t}{A_t}\right)^{pt}. \tag{3.4}$$

*Second inequality in (3.4) holds true for $p > 0$.*

**Proof.** We shall prove the assertion using Theorem 2.1 with

$$u = \frac{\sinh z}{z}, v = \frac{\tanh z}{z}, \gamma = 2, \delta = 1.$$

It is well known that $v < 1 < u$ holds for all $z \neq 0$. Moreover, the first inequality in (2.3) can be written as $1 < u^2 v$ while the second one is the same as $3 < 2\frac{1}{u} + \frac{1}{v}$. Letting $z = \lambda$, where $\lambda$ is the same as above, we obtain

$$2 < \left(\frac{\lambda}{\sinh \lambda}\right)^{2pt} + \left(\frac{\lambda}{\tanh \lambda}\right)^{pt} < \left(\frac{\sinh \lambda}{\lambda}\right)^{2pt} + \left(\frac{\tanh \lambda}{\lambda}\right)^{pt}.$$

Application of (3.2) and (3.3) completes the proof. □

Particular cases of inequality (3.4) have been obtained in [4].

**Corollary 3.2.** *The following inequalities*

$$\frac{2L}{L+G} < \frac{A_{1/2}}{L} < \frac{L^2}{GA_{1/2}} < \frac{L+G}{2G} \tag{3.5}$$

*hold true.*

**Proof.** We utilize the first two members of (3.4) with $p = 1$ and $t = \frac{1}{2}$ and next apply $L_{1/2} = L^2/A_{1/2}$, to obtain

$$2 < \frac{A_{1/2}G}{L^2} + \frac{A_{1/2}}{L}. \tag{3.6}$$

Multiplying both sides of (3.6) by $L/(L+G)$ we obtain the first inequality in (3.5). The second inequality in (3.5) is equivalent to $A_{1/2}^2 G < L^3$ (see [13] and [7]), while the third one is equivalent to the first inequality in (3.5). The proof is complete. □

The first inequality in (3.5) has been established in [12].

A generalization of the inequality which connects first and third members of (3.4) reads as follows.

**Theorem 3.3.** *Let $x > 0$, $y > 0$ $(x \neq y)$, and let $t \neq 0$. Further, let $\alpha > 0$ and $\beta > 0$. Then*

$$\alpha + \beta < \alpha \left(\frac{L_t}{G}\right)^{pt} + \beta \left(\frac{L_t}{A_t}\right)^{qt} \tag{3.7}$$

*if either*

$$p > 0 \qquad and \qquad q \le p\frac{\alpha}{2\beta}, \tag{3.8}$$

*or if*

$$q \le p \le -1 \qquad and \qquad \alpha \le 2\beta. \tag{3.9}$$

**Proof.** We shall prove this result using Theorem 2.2 with

$$u = \frac{\sinh z}{z}, v = \frac{\tanh z}{z}, \gamma = 2, \delta = 1.$$

As pointed out in the proof of Theorem 3.1 that they satisfy conditions (i) - (iii). Letting $z = \lambda$, where $\lambda$ is the same as in the proof of Theorem 3.1, we conclude, using inequality (2.5), that

$$\alpha + \beta < \alpha \left(\frac{\sinh \lambda}{\lambda}\right)^p + \beta \left(\frac{\tanh \lambda}{\lambda}\right)^q.$$

Making use of (3.2) and (3.3) we obtain the desired result. This completes the proof. □

To this end we will assume that $\alpha > 0$ and $\beta > 0$. Several inequalities can be derived from (3.7). For the sake of presentation we define the weights

$$w_1 = \alpha/(\alpha + \beta) \qquad \text{and} \qquad w_2 = \beta/(\alpha + \beta).$$

Clearly $w_1 + w_2 = 1$.

We shall now prove the following.

**Corollary 3.4.** *Let $t \ne 0$. If $\alpha \le 2\beta$, then*

$$L_t^t < w_1 G^t + w_2 A_t^t. \tag{3.10}$$

*Also, if $\alpha \ge 2\beta$, then*

$$L_t^{-t} < w_1 G^{-t} + w_2 A_t^{-t}. \tag{3.11}$$

**Proof.** In order to establish (3.10) it suffices to use Theorem 3.3 with $p = q = -1$. Similarly, (3.11) can be obtained using Theorem 3.3 with $p = q = 1$. This completes the proof. □

Letting in (3.10) $t = 1$ and $t = 1/2$ we obtain, respectively,

$$L < w_1 G + w_2 A$$

and

$$L < w_1 (A_{1/2} G)^{1/2} + w_2 A_{1/2}$$

provided $\alpha \le 2\beta$. The last two inequalities are known in mathematical literature in the case when $\alpha = 2$ and $\beta = 1$ (see [1], [7], and [14]). Similarly, letting in (3.11) $t = -1$ and $t = -1/2$ we obtain, respectively,

$$L^{-1} < w_1 G^{-1} + w_2 A^{-1}$$

and

$$L^{-1} < w_1 (A_{1/2} G)^{-1/2} + w_2 A_{1/2}^{-1}$$

provided $\alpha \ge 2\beta$. For more inequalities involving $L^{-1}$, the interested reader is referred to [7].

Inequalities for the extended logarithmic mean $E_t$, where $E_t^{t-1} = L_t^t / L$ have been derived in [3]. They can be used to obtain more inequalities for the mean discussed in this paper.

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# Chapter 6

# Sequential means

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

(D. Hilbert)

"There is a great power in truth and sincerity. The mathematics community has tremendous reserves of human potential energy. If we are lean and hungry, we are likely to use our energy. If we are honest, it is likely to be effective..."

(W. Thurston)

## 6.1 On some inequalities for means

### 1

Let $x > 0$ and $y > 0$. The logarithmic mean $L(x, y)$ is defined by

$$L(x,y) = \frac{x-y}{\ln x - \ln y} \text{ for } x \neq y; \quad L(x,x) = x.$$

The identric mean of $x$ and $y$ is

$$I(x,y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)} \text{ for } x \neq y; \quad I(x,x) = x,$$

while the arithmetic-geometric mean $M(x,y)$ is defined by

$$x_0 = x,\ y_0 = y,\ x_{n+1} = \frac{x_n + y_n}{2},\ y_{n+1} = (x_n y_n)^{1/2},\ n = 0, 1, 2, \dots,$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = M(x,y).$$

For these means many results, especially inequalities, are known. For historical remarks, applications, and inequalities, see, e.g., [1], [12], [13]. For early results and refinements see also [5], [9], [14], [15], [11], [8], [2], [6], [17]. For an extensive bibliography on the mean $M$ see [3] and [17].

The aim of this note is to obtain new and unitary proofs for certain known inequalities as well as refinements and some new relations.

First we will obtain a simple proof of the double inequality

$$\frac{2}{e} \cdot A < I < A, \tag{1}$$

where $A = A(x,y) = (x+y)/2$ denotes the arithmetic mean of $x$ and $y$. A similar inequality will be

$$L < M < \frac{\pi}{2} \cdot L. \tag{2}$$

We shall deduce a new inequality, namely

$$\frac{2}{\pi}\left(\frac{1}{L} - \frac{1}{A}\right) < \frac{1}{M} - \frac{1}{A} < \frac{12}{5\pi}\left(\frac{1}{L} - \frac{1}{A}\right). \tag{3}$$

In [17] it is proved (by studying certain integrals) that $M < (A+G)/2$, where $G = G(x,y) = (xy)^{1/2}$ denotes the geometric mean of $x$ and $y$.

Here we will prove that

$$M > \sqrt{A \cdot G} \tag{4}$$

so that the mean $M$ separates the geometric and arithmetic mean of $A$ and $G$. (In all inequalities (1)-(4) we suppose $x \neq y$).

## 2

In order to obtain relation (1), apply an inequality of Mitrinović and Djoković (see [10, inequality 3.6.35, p. 280]):

$$\frac{2}{e} < a^{a/(1-a)} + a^{1/(1-a)} < 1 \text{ for all } 0 < a < 1. \tag{5}$$

Put $a = x/y$ in (5), where $0 < x < y$. Then

$$\frac{1}{1-a} = \frac{y}{y-x}, \qquad \frac{a}{1-a} = \frac{x}{y-x}.$$

Remarking that

$$y(y/x)^{x/(y-x)} = e \cdot I(x,y) \quad \text{and} \quad x(y/x)^{y/(y-x)} = e \cdot I(x,y),$$

after some simple computations we get

$$\frac{2}{e} < \frac{x+y}{e} \cdot I < 1,$$

which yields (1). We note that the right side of (1) is due to Stolarski [14].

## 3

Write $t = (y-x)/(y+x)$, so that $y/x = (1+t)/(1-t)$ with $0 < t < 1$. Since $M$ and $L$ are homogeneous means (of order 1) it will suffice to show that (2) holds true in the form

$$L(1+t, 1-t) < M(1+t, 1-t) < \frac{\pi}{2} \cdot L(1+t, 1-t). \tag{6}$$

It is easy to see that

$$L(1+t, 1-t) = \frac{2t}{\ln(1+t)/(1-t)}$$

$$= \left(1 + \frac{1}{3}t^2 + \frac{1}{5}t^4 + \ldots + \frac{1}{2n+1} \cdot t^{2n} + \ldots\right)^{-1}. \tag{7}$$

On the other hand, Gauss [7] (see also [4]) has shown that for $|t| < 1$

$$M(1+t, 1-t) = \left(1 + \frac{1}{4}t^2 + \frac{9}{64}t^4 + \ldots + A_n^{-2}t^{2n} + \ldots\right)^{-1}, \quad (8)$$

where

$$A_n = \frac{2 \cdot 4 \ldots (2n)}{1 \cdot 3 \ldots (2n-1)} \text{ for } n \geq 1.$$

The numbers $(A_n)$ satisfy a relation essentially due to Wallis (see, e.g., [10, p. 192]):

$$\pi \cdot n < (A_n)^2 < \pi \cdot \left(n + \frac{1}{2}\right) \text{ for } n \geq 1. \quad (9)$$

Now, since $\pi \cdot n > 2n+1$, it is immediate that by (7) and (8), relation (9) implies (6). The left side of (2) has been discovered by Carlson and Vuorinen [6]. For the right side (with a different proof), see [17, Theorem 1.3(2)].

## 4

A refinement of the left side of (9) is due to Kazarinoff (see [10, p. 192]):

$$\pi \cdot \left(n + \frac{1}{4}\right) < A_n^2 < \pi \cdot \left(n + \frac{1}{2}\right) \text{ for } n \geq 1. \quad (10)$$

We note that the right side of inequality (10) can be improved to $2/7$ in place of $1/2$ [16], but this fact has no importance here.

Remark first that, since $(2n+1)/(4n+1) \leq 3/5$ for $n \geq 1$, from (10) it results that

$$\frac{12}{5\pi} \cdot \frac{1}{2n+1} > A_n^{-2} > \frac{2}{\pi} \cdot \frac{1}{2n+1} \text{ for } n \geq 1. \quad (11)$$

Now, by using the method used in Section 2, by the homogeneity of $M$, $L$ and $A$, (3) is equivalent to

$$\frac{2}{\pi} \cdot \left(\frac{1}{L} - 1\right) < \frac{1}{M} - 1 < \frac{12}{5\pi} \cdot \left(\frac{1}{L} - 1\right), \quad (12)$$

where $L = L(1+t, 1-t)$, etc.

By (7) and (8), this double inequality follows at once from (11).

**5**

Since

$$M(x,y) = A(x,y) \cdot M(1+t, 1-t), \quad G(x,y) = A(x,y) \cdot \sqrt{1-t^2},$$

(4) is equivalent to the inequality

$$(1-t^2)^{-1/4} > 1 + \frac{1}{4}t^2 + \ldots + A_n^{-2}t^{2n} + \ldots \tag{13}$$

By the binomial theorem,

$$\begin{aligned}(1-t^2)^{-1/4} = 1 + \frac{1}{4 \cdot 1!}t^2 + \frac{1 \cdot 5}{4^2 \cdot 2!}t^4 \\ + \ldots + \frac{1 \cdot 5 \cdot 9 \ldots (4n-3)}{4^n \cdot n!}t^{2n} + \ldots\end{aligned}$$

so if we are able to prove that

$$A_n^{-2} < 1 \cdot 5 \cdot 9 \ldots (4n-3)/4^n \cdot n!, \ n > 1, \tag{14}$$

then (13) is valid. From the definition of $A_n$, it is obvious that (14) and

$$(1 \cdot 2 \cdot 3 \ldots n) \cdot (1 \cdot 5 \cdot 9 \ldots (4n-3)) > 1^2 \cdot 3^2 \ldots (2n-1)^2, \ n>1 \tag{15}$$

are the same. This inequality is true for $n = 2$, and accepting it for $n$, the induction step follows by $(n+1)(4n+1) > (2n+1)^2$, so via mathematical induction, (15) follows. This proves (4).

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## 6.2 On inequalities for means by sequential method

### 1

Let $x, y$ be positive real numbers. The arithmetic-geometric mean of Gauss is defined as the common limit of the sequences $(x_n)$, $(y_n)$ defined recurrently by

$$x_0 = x,\ y_0 = y,\ x_{n+1} = \frac{x_n + y_n}{2},\ y_{n+1} = \sqrt{x_n y_n}\ (n \geq 0). \tag{1}$$

Let $M = M(x, y) := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. The mean $M$ was considered firstly by Gauss [8] and Lagrange [9], but its real importance and connections with elliptic integrals are due to Gauss. For historical remarks and an extensive bibliography on $M$, see [4], [2], [16].

The logarithmic mean and identric mean of $x$ and $y$ are defined by

$$L = L(x, y) := \frac{x - y}{\log x - \log y} \text{ for } x \neq y, \quad L(x, x) = x, \tag{2}$$

and

$$I = I(x, y) := \frac{1}{e}(x^x/y^y)^{1/(x-y)} \text{ for } x \neq y, \quad I(x, x) = x, \tag{3}$$

respectively. For a survey of results, refinements, and extensions related to these means, see [5], [10], [1], [11], [12].

Very recently, by using a variant of L'Hospital's rule and representation theorems with elliptic integrals, Vamanamurthy and Vuorinen [16] have proved, among other results, the inequalities

$$M < \sqrt{AL} \tag{4}$$

$$L < M < \frac{\pi}{2}L \tag{5}$$

$$M < I < A \tag{6}$$

$$M < \frac{A + G}{2} \tag{7}$$

$$A < \frac{M(x^2, y^2)}{M(x, y)} < A_2 := \sqrt{\frac{x^2 + y^2}{2}}, \tag{8}$$

where $A = A(x, y) := (x + y)/2$ and $G = G(x, y) := \sqrt{xy}$ denote, as usual, the arithmetic and geometric mean of $x$ and $y$, respectively. Here, in all cases, $x$ and $y$ are distinct.

The left side of (5) has been discovered by Carslon and Vuorinen [7]. In a recent note [13], by using the homogeneity of the above means and a series representation of $M$ due to Gauss [8], we have obtained, among other results, new proofs for (5), (6), and a counterpart of (7),

$$\sqrt{AG} < M, \tag{9}$$

which shows that, $M$ lies between the arithmetic and geometric means of $A$ and $G$.

The aim of this paper is to deduce new proofs for (4), (6), (7), (8), and (9) by using only elementary methods for recurrent sequences and, in fact, to prove much stronger forms of these results.

## 2

The algorithm (1) giving the mean $M$ is known as Gauss' algorithm. Borchardt's algorithm is defined in a similar manner [3] by

$$a_0 = x,\ b_0 = y, b_1 = \sqrt{xy},\ a_{n+1} = \frac{a_n + b_n}{2}\ (n \geq 0),$$
$$b_{n+1} = \sqrt{a_{n+1} b_n}\ (n \geq 1)$$

It can be shown that for $x \neq y$, $(a_n)$ $(n \geq 1)$ is strictly decreasing, while $(b_n)$ $(n \geq 1)$ is strictly increasing, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L. \tag{10}$$

Here $L$ is exactly the logarithm mean, see [5]. For a new proof of this fact, see [15]. Carlson has proved the important inequality

$$L < \frac{A + 2G}{3}. \tag{11}$$

For a new proof of (11) with improvements, see [14].

We now deduce an important counterpart of (11) due to Leach and Sholander [10]:

$$L^3 > G^2 A. \tag{12}$$

This is based on the sequence $(b_n^2 \cdot a_n)$ $(n \geq 1)$, which is strictly increasing. Indeed, for $n \geq 1$ one has

$$b_{n+1}^2 \cdot a_{n+1} = (a_{n+1} b_n) a_{n+1} = a_{n+1}^2 b_n > b_n^2 a_n$$

by $a_{n+1}^2 > a_n b_n$, i.e.

$$((a_n + b_n)/2)^2 > a_n b_n,$$

which is true. Thus

$$b_n^2 a_n > b_{n-1}^2 a_{n-1} > \ldots > b_2^2 a_2 > b_1^2 a_1 = G^2 A \ (n > 2). \tag{13}$$

For $n \to \infty$, via (10) and (11) one gets

$$L^3 > \left(\frac{A+G}{2}\right)^2 G > G^2 A \tag{14}$$

since $a_1 = A$, $b_1 = G$, etc. Inequality (14) refines (12), and, as can be easily seen by (13), other improvements are also valid.

In what follows, it will be convenient also to introduce the algorithm

$$p_0 = x^2, \ q_0 = y^2, \ p_{n+1} = \frac{p_n + q_n}{2}, \ q_{n+1} = \sqrt{p_n q_n} \ (n \geq 0). \tag{15}$$

Clearly,

$$\lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = M(x^2, y^2). \tag{16}$$

## 3

The idea of proving inequalities such as (14), which is an application of monotonicity of certain sequences, appears in [14]. The first theorem, which follows, is well known and can be proved by mathematical induction.

**Theorem 1.** *Let $n \geq 1$ (and $x \neq y$). Then the sequences $(x_n)$ and $(a_n)$ are strictly decreasing, while the sequences $(y_n)$ and $(b_n)$ are strictly increasing. In fact, one has*

$$0 < y_1 < y_2 < \ldots < y_n < x_n < x_{n-1} < \ldots < x_1 \ (n > 1) \tag{17}$$

$$0 < b_1 < b_2 < \ldots < b_n < a_n < a_{n-1} < \ldots < a_1 \ (n > 1) \tag{18}$$

**Corollary of (17).** *One can write*

$$\sqrt{AG} < M < \frac{A+G}{2}, \tag{19}$$

*i.e., relations (7) and (9).*

Indeed, for $n \geq 2$ one has $x_n < x_2 = (A+G)/2$ and $y_n > y_2 = \sqrt{AG}$. By letting $n \to \infty$, we get (19), whit $\leq$ in place of $<$, but note that actually there are strong inequalities because of $M \leq x_3 < x_2$ and $M \geq y_3 > y_2$. We obtain the following sharpening of (19):

$$\sqrt{AG} < \sqrt{\frac{A+G}{2}\sqrt{AG}} < M < \left(\frac{\sqrt{A}+\sqrt{G}}{2}\right)^2 < \frac{A+G}{2}. \tag{20}$$

Let us now introduce the notation $u_n = x_n/y_n$ and $v_n = a_n/b_n$ $(n \geq 0)$. Clearly, $u_n > 1$, $v_n > 1$ and $u_{n+1} < u_n$, $v_{n+1} < v_n$ for $n \geq 1$, by (17) and (18). On the other hand, a simple computation shows that

$$u_{n+1} = \frac{1}{2}\left(\sqrt{u_n} + \frac{1}{\sqrt{u_n}}\right), \ n \geq 0 \tag{21}$$

$$v_{n+1} = \sqrt{\frac{v_n+1}{2}}, \ n \geq 0. \tag{22}$$

We now prove that

$$u_n \leq v_n \text{ for } n \geq 1, \text{ with equality for } n = 1 \text{ only.} \tag{23}$$

For $n = 1$ there is equality, but $u_2 < v_2$. We remark that the function

$$f(x) = (1/2)\left(\sqrt{x} + 1/\sqrt{x}\right) \ (x > 1)$$

is strictly increasing, so if we admit that $u_n < v_n$, then

$$f(u_n) \leq f(v_n) \quad \text{and} \quad u_{n+1} \leq f(v_n) = (v_n{+}1)/2\sqrt{v_n} = v_{n+1}^2/\sqrt{v_m} < v_{n+1}$$

by (22) and $v_n > 1$, which imply $v_{n+1} < \sqrt{v_n}$, $n \geq 1$. By induction, (23) is proved for all $n \geq 1$.

**Theorem 2.** *Let $t_n = a_n y_n^2/b_n^2$. The sequence $(t_n)$ $(n > 1)$ is strictly decreasing. One has*

$$x_n^2 \leq a_n \cdot A \ (n \geq 1). \tag{24}$$

**Proof.** $t_{n+1} < t_n$ is equivalent to $x_n/y_n < a_n/b_n$ (simple computation), the inequality proved at (23). Now, by $x_n^2 \leq a_n^2 y_n^2/b_n^2$, if we can show that $a_n y_n^2/b_n^2 \leq A$, inequality (24) is proved. By $t_n \leq t_1 = A$ $(n \geq 1)$, this holds true.

**Corollary of Theorem 2.** *We have*

$$M^2 < \sqrt{A\left(\frac{A+G}{2}\right)} \cdot L < AL. \tag{25}$$

Indeed, for $n > 4$ we have $t_n < t_4$ so for $n \to \infty$ one has

$$M^2/L \leq t_4 < t_3 = \sqrt{A((A+G)/2)} < A.$$

This yields (25), which in turn sharpens (4). We note here that it is known [11] that

$$I > \frac{A+L}{2} > \sqrt{AL} \tag{26}$$

so by (26), relation (6) is a consequence of (4).

We now obtain a result concerning the sequences $(x_n)$, $(y_n)$ and $(p_n)$, $(q_n)$.

**Theorem 3.** *Let $h_n = q_{n+1}/y_n$, $r_n = p_{n+1}/x_n$, and $n \geq 1$. Then the sequences $(h_n)$ and $(r_n)$ are strictly decreasing and increasing, respectively.*

**Proof.** One has $h_{n+1} = q_{n+2}/y_{n+1} = \sqrt{q_{n+1}p_{n+1}/x_n y_n} < q_{n+1}/y_n$ iff

$$p_{n+1}/q_{n+1} < x_n/y_n \ (n \geq 1). \tag{27}$$

For $n = 1$ it is true that $p_2/q_2 = A^2/A_2G < A/G = x_1/y_1$ by $A < A_2$. Now, since

$$p_{n+1}/q_{n+1} := s_{n+1} = (1/2)\left(\sqrt{s_n} + 1/\sqrt{s_n}\right) = f(s_n)$$

(see the proof of (21) and (23)), if we assume relation (27), by the monotonicity of $f$ one obtains

$$f(p_{n+1}/q_{n+1}) < f(x_n/y_n), \quad \text{i.e.} \quad p_{n+2}/q_{n+2} < x_{n+1}/y_{n+1},$$

proving (27) and the monotonicity of $(h_n)$.

For $(r_n)$ one can write

$$p_{n+2}/x_{n+1} = (p_{n+1} + q_{n+1})/(x_n + y_n) > p_{n+1}/x_n,$$

which is equivalent to $p_{n+1}/q_{n+1} < x_n/y_n$ $(n \geq 1)$ and this is exactly inequality (27).

**Corollary.** *Since $q_2/y_1 = A_2$ and $p_2/x_1 = A$, one can deduce that*

$$p_{n+1}x_n > A \quad \textit{and} \quad q_{n+1}/y_n < A_2 \ (n > 1),$$

*proving with $n \to \infty$ relation (8). Since*

$$q_3/y_2 = (AA_2)^{1/2} \quad \textit{and} \quad p_3/x_2 = (A^2 + A_2G)/(A+G),$$

*one obtains the refinements*

$$A < \frac{A^2 + A_2G}{A+G} < \frac{M(x^2, y^2)}{M(x,y)} < (AA_2)^{1/2} < A_2. \tag{28}$$

*By computing, e.g. $q_4/y_3$ and $p_4/x_3$, a new refinement of (28) can be deduced.*

Certain other properties of the above sequences are collected in

**Theorem 4.** *Let $n > 1$. Then*

*(a) $p_n > x_n^2$ and $q_n > y_n^2$;* (29)

*(b) if $u_n = x_n/y_n$ and $s_n = p_n/q_n$, then $s_n > u_n^2$.* (30)

**Proof.** (a) $p_1 = (x^2 + y^2)/2 > ((x + y)/2)^2 = x_1^2$; $q_1 = xy = y_1^2$.

By assuming (29) for $n$, one has

$$p_{n+1} = (p_n + q_n)/2 \geq (x_n^2 + y_n^2)/2 > x_{n+1}^2 = ((x_n + y_n)/2)^2$$

and

$$q_{n+1} = \sqrt{p_n q_n} > x_n y_n = y_{n+1}^2,$$

i.e., the properties are valid for $n+1$ too, so via induction, (a) is proved.

(b) $s_1 > u_1^2$ is true since $2(x^2 + y^2) > (x + y)^2$. On the other hand, by

$$u_{n+1} = (1/2)\left(\sqrt{u_n} + 1/\sqrt{u_n}\right) \quad \text{and} \quad s_{n+1} = (1/2)\left(\sqrt{s_n} + 1/\sqrt{s_n}\right)$$

and the induction step,

$$\begin{aligned} s_{n+1} &= f(s_n) > f(u_n^2) = (1/2)(u_n + 1/u_n) \\ &> u_{n+1}^2 = (1/4)\left(\sqrt{u_n} + 1/\sqrt{u_n}\right)^2 \end{aligned}$$

by $\left(\sqrt{u_n} - 1/\sqrt{u_n}\right)^2 > 0$. This proves (30).

**Corollary of (29).** *One obtains*

$$M(x^2, y^2) \geq M^2(x, y). \tag{31}$$

This can be slightly sharpened, since by $L(x^2, y^2) = LA$, and relations (4) and (5) one has

$$M^2(x, y) < L(x^2, y^2) < M(x^2, y^2). \tag{32}$$

This result follows also from (28) and inequality $A > M$.

Finally, we prove:

**Theorem 5.** *Let $\alpha = 2A/(A + G)$ $(> 1)$. Then for all $n \geq 1$ one has*

$$x_n \leq \alpha a_{n+1} \quad \text{and} \quad y_n \leq \alpha b_n. \tag{33}$$

**Proof.** Let $n = 1$. Then $x_1 = A \leq \alpha a_2 = \alpha(A + G)/2$, which holds, by assumption. Similarly, $y_1 = G < \alpha b_1 = \alpha G$ by $\alpha > 1$. Assuming now that (33) is valid, one can write

$$x_{n+1} = (x_n + y_n)/2 \leq \alpha(a_{n+1} + b_n)/2 < \alpha(a_{n+1} + b_{n+1})/2 = \alpha a_{n+2}$$

(see (18)). Analogously,

$$y_{n+1} = \sqrt{x_n y_n} \leq \alpha \sqrt{a_{n+1} b_n} = \alpha \cdot b_{n+1}.$$

This finishes the proof of (33).

**Corollary of (33).** *Letting $n \to \infty$,*

$$M \leq \alpha L \quad \textit{for} \quad \alpha = 2A/(A+G). \tag{34}$$

We note that this result cannot be compared with (5) since $2A/(A+G)$ and $\pi/2$ are not comparable.

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## 6.3 On certain inequalities for means, III

### 1. Introduction

Let $x, y$ be positive real numbers. The logarithmic mean and the identric mean of $x$ and $y$ are defined by

$$L = L(x,y) = \frac{x-y}{\log x - \log y} \text{ for } x \neq y, \quad L(x,x) = x, \tag{1}$$

and

$$I = I(x,y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)} \text{ for } x \neq y, \quad I(x,x) = x, \tag{2}$$

respectively. Let

$$A = A(x,y) = \frac{x+y}{2} \quad \text{and} \quad G = G(x,y) = \sqrt{xy}$$

denote the arithmetic, resp. geometric mean of $x$ and $y$.

It is well-known that for $x \neq y$ one has (see e.g. [5])

$$G < L < I < A. \tag{3}$$

In 1993 H.-J. Seiffert [10] has introduced the mean

$$P = P(x,y) = \frac{x-y}{4\arctan\left(\sqrt{x/y}\right) - \pi} \text{ for } x \neq y, \quad P(x,x) = x.$$

Seiffert [10] proved that for $x \neq y$

$$L < P < I \tag{4}$$

and later [11], by using certain series representations:

$$\frac{1}{P} < \frac{1}{3}\left(\frac{1}{G} + \frac{2}{A}\right), \tag{5}$$

$$GA < GP, \tag{6}$$

$$P < A < \frac{\pi}{2}P. \tag{7}$$

In fact, $P$ can be written also in the equivalent form

$$P(x,y) = \frac{x-y}{2\arcsin\dfrac{x-y}{x+y}} \text{ for } x \neq y \tag{8}$$

(see [9]). Clearly, we may suppose $0 < x < y$, and we note that (8) implies

$$\frac{A}{P} = \frac{\arcsin z}{z} = f(z),$$

where $z = \dfrac{x-y}{z+y}$, $0 < z < 1$, and $f$ being a strictly function, clearly

$$1 = \lim_{x \to 0} f(z) < \frac{A}{P} < \lim_{z \to 1} f(z) = \frac{\pi}{2},$$

giving (7).

Another remark is that (8) can be written also as

$$P(x,y) = \frac{2}{x-y} \arccos\left(\frac{2}{x+y}\sqrt{xy}\right) = \frac{2}{x-y} \arccos\frac{x_0}{y_0} \tag{9}$$

where $x_0 = \sqrt{xy}$, $y_0 = \dfrac{x+y}{2}$. Since $x_0 < y_0$, $P$ is the common limit of a pair of sequences given by

$$x_{n+1} = \frac{x_n + y_n}{2},\ y_{n+1} = \sqrt{x_{n+1} y_n},\ n = 0, 1, \ldots \tag{10}$$

(see [1], p. 498). According to B.C. Carlson [1], the algorithm (10) is due to Pfaff (see also [3]), who determined the common limit (9) of the sequences $(x_n)$ and $(y_n)$.

By using the sequential method from part II of this series (see [7], [8]), we will be able in what follows to improve relations (4)-(6), and to obtain other inequalities related to the mean $P$.

## 2. Gauss', Borchardt's and Pfaff's algorithms

Pfaff's algorithm is given by (10), where

$$x_0 = \sqrt{xy},\ y_0 = \frac{x+y}{2}.$$

Let us denote the Borchardt algorithm by

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_{n+1} b_n}\ (n \geq 1), \qquad a_0 = x,\ b_0 = y,\ b_1 = \sqrt{xy} \tag{11}$$

and the Gauss algorithm by

$$f_{n+1} = \frac{f_n + g_n}{2},\ g_{n+1} = \sqrt{g_n f_n}\ (n \geq 0),\ f_0 = x,\ g_0 = y. \tag{12}$$

It is well known that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L(x,y)\ -\ \text{the logarithmic mean of } x \text{ and } y;$$

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = M(x,y)\ -\ \text{the famous arithmetic-geometric mean of Gauss}$$

(see e.g. [1], [2], [3], [8]). M.K. Vamanamurthy and M. Vuorinen ([14]) have proved that

$$L < M < \frac{\pi}{2} L \tag{13}$$

$$M < I < A \tag{14}$$

$$M < \frac{A+G}{2} \tag{15}$$

and the author [8] has obtained refinements, based on the Gauss and Borchardt algorithm.

The aim of this paper is to offer new proof of (4), (5), (6) and in fact to obtain strong refinements of these relations.

## 3. Monotonicity properties and applications

**Theorem 1.** *For all $n \geq 0$ we have*

$$x_n < P < y_n. \tag{16}$$

*Particularly,*

$$\frac{A+G}{2} < P < \sqrt{\left(\frac{A+G}{2}\right)A}. \tag{17}$$

**Proof.** Since $y_0 < x_0$ and $y_{n+1} > x_{n+1}$ iff

$$\frac{x_n + y_n}{2} > \sqrt{\frac{x_n + y_n}{2} y_n} \quad \text{i.e.} \quad y_n > x_n,$$

by induction it follows that $y_n > x_n$ for all $n$. The inequality $x_{n+1} > x_n$ is equivalent to $y_n > x_n$, while $y_{n+1} < y_n$ to $x_{n+1} < y_n$ i.e. $\frac{x_n + y_n}{2} < y_n$, thus $x_n < y_n$, which is proved. We have proved that the sequence $(x_n)_{n\geq 0}$ is strictly increasing, $(y_n)_{n\geq 0}$ strictly decreasing, having the same limit $P$, so (16) follows. From

$$x_1 = \frac{A+G}{2},\ y_1 = \sqrt{x_1 y_0} = \sqrt{\frac{A+G}{2} A}$$

we obtain relation (17).

**Corollary 1.**

$$L < M < \frac{A+G}{2} < P. \tag{18}$$

This follows by (13), (15) and (17).

**Remark 1.** Relation $L < \frac{A+G}{2}$ follows also from the known fact (see e.g. [5], [7]) that

$$L < A_{1/3} < A_{1/2} = \frac{A+G}{2}, \quad \text{where} \quad A_s = A_s(x,y) = \left(\frac{x^s + y^s}{2}\right)^{1/s}.$$

**Theorem 2.** *For all $n \geq 0$ we have*

$$\sqrt[3]{y_n^2 x_n} < P < \frac{x_n + 2y_n}{3}. \tag{19}$$

*Particularly,*

$$\sqrt[3]{A^2 G} < P < \frac{G + 2A}{3}. \tag{20}$$

**Proof.** One has

$$y_{n+1}^2 x_{n+1} = (x_{n+1} y_n) x_{n+1} = x_{n+1}^2 y_n > y_n^2 x_n$$

iff $x_{n+1}^2 > x_n y_n$ i.e. $\left(\frac{x_n+y_n}{2}\right)^2 < x_n y_n$, which is true. Thus, the sequence $(y_n^2 x_n)_{n\geq 0}$ is strictly increasing, laving as limit $P^3$. This gives the first part of (19). Next, from

$$x_{n+1} + 2y_{n+1} = \frac{x_n+y_n}{2} + 2\sqrt{\frac{x_n+y_n}{2}y_n} < \frac{x_n+y_n}{2} + \frac{x_n+y_n}{2} + y_n$$

(by $2\sqrt{uv} < u+v$ for $u \neq v$) we get that the sequence $(x_n+2y_n)_{n\geq 0}$ is strictly decreasing, having the limit $3P$. This implies the second part of (19). For $n=0$ we obtain the double inequality (20).

**Corollary 2.**

$$\frac{AG}{L} < \sqrt[3]{A^2G} < P < \frac{G+2A}{3} < I. \tag{21}$$

The first inequality is a consequence of $L > \sqrt[3]{G^2A}$, due to Leach and Sholander (for refinements see [8]), and the last inequality is due to the author (see [7]). We will see (Remark 4) that this relation, combined with other results improves known inequalities.

**Remark 2.** The left side of (21) improves inequality (6). Similarly, the left side of (20) improves inequality (5). Indeed,

$$\frac{1}{3}\left(\frac{1}{G}+\frac{2}{A}\right) = \frac{1}{3}\left(\frac{1}{G}+\frac{1}{A}+\frac{1}{A}\right) > \sqrt[3]{\frac{1}{G}\cdot\frac{1}{A^2}} > \frac{1}{P}$$

by the arithmetic-geometric inequality

$$\frac{x+y+z}{3} > \sqrt[3]{xyz}.$$

**Remark 3.** A better estimate for the right side of (21) can be obtained by applying (19), e.g. for $n=1$. Since

$$x_1 = \frac{A+G}{2}, \quad y_1 = \sqrt{\frac{A+G}{2}A},$$

we obtain

$$P < \frac{1}{3}\left(\frac{A+G}{2} + 2\sqrt{\frac{A+G}{2}A}\right) < \frac{G+2A}{3} < I. \tag{22}$$

In an analogous way, the left side of (19) gives

$$\left[\left(\frac{A+G}{2}\right)^2 A\right]^{\frac{1}{3}} < P, \tag{23}$$

which is better than the left side of (17).

This follows by the remark that

$$\sqrt[3]{y_1^2 x_1} = \sqrt[3]{\left(\frac{A+G}{2}\right)^2 A}.$$

**Remark 4.** A.A. Jagers (see [4]) proved that $A_{1/2} < P < A_{2/3}$. The left side inequality is exactly the left side of (17). Since, it is known that $\dfrac{G+2A}{3} < A_{2/3}$ (see [12], where it is mentioned that this inequality was proposed at the "16th Austrian-Polish Mathematics Competition 1993"), the right side of (20) is better than the right side of Jager's inequality. By an inequality of Stolarsky (see [13]) we have $A_{2/3} < I$ so the right side of (21) can be written also in an improved form.

**Theorem 3.** *One has*

*a)* $P(x^k, y^k) \geq (P(x,y))^k$ *and*
$M(x^k, y^k) \geq (M(x,y))^k$ *for all* $k \geq 1$, (24)

*b)* $P(x^k, y^k) > A_{k/2}^k \geq A^k > (P(x,y))^k$ *for all* $k \geq 2$, (25)

*c)* $M(x^k, y^k) > \dfrac{1}{k} \cdot \dfrac{x^k - y^k}{x-y} L(x,y) \geq A^{k-1} L > (M(x,y))^k$
*for all* $k \geq 2$. (26)

**Proof.** a) As in [8] (for the mean $M$, with $k = 2$) we consider the sequences $(p_n)$, $(q_n)$ defined by

$$p_0 = \sqrt{x^k y^k}, \quad q_0 = \frac{x^k + y^k}{2}, \quad p_{n+1} = \frac{p_n + q_n}{2}, \quad q_{n+1} = \sqrt{p_{n+1} q_n}.$$

Clearly, $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n = P(x^k, y^k)$.

We prove inductively that

$$p_n \geq x_n^k, \; q_n \geq y_n^k \text{ for all } n \geq 0, \; k \geq 1. \tag{27}$$

We have $p_0 = x_0^k$ and $q_0 \geq y_0^k$ since $\dfrac{x^k + y^k}{2} \geq \left(\dfrac{x+y}{2}\right)^k$, which follows by the convexity of the function $t \mapsto t^k$ $(k \geq 1)$. Then, if (27) is valid for an $n$, we can write

$$p_{n+1} = \frac{p_n + q_n}{2} \geq \frac{x_n^k + y_n^k}{2} \geq \left(\frac{x_n + y_n}{2}\right)^k = x_{n+1}^k$$

and

$$q_{n+1} = \sqrt{p_{n+1} q_n} \geq \sqrt{x_{n+1}^k y_n^k} = y_{n+1}^k,$$

i.e. (27) is valid for $n+1$, too. By taking $n \to \infty$ in (27), we get the first part of (a). The second part can be proved in a completely analogous way.

b) By writing the right side of inequality (18) for $x^k, y^k$ in place of $x, y$, one has

$$P(x^k, y^k) > \left(\frac{\sqrt{x^k} + \sqrt{y^k}}{2}\right)^2 = \left[\left(\frac{x^{\frac{k}{2}} + y^{\frac{k}{2}}}{2}\right)^{\frac{2}{k}}\right]^2 = A_{k/2}^k \geq A_1^k = A^k$$

for $k \geq 2$ (since $A_s$ is increasing in $s$), by $A > P$, we get b).

Finally, for c) remark that $L(x^k, y^k) < M(x^k, y^k)$, but

$$L(x^k, y^k) = \frac{x^k - y^k}{\ln x^k - \ln y^k} = L\frac{x^k - y^k}{k(x-y)}.$$

We shall prove that

$$(M(x,y))^k < \frac{x^k - y^k}{k(x-y)} L(x,y) \text{ for } k \geq 2. \qquad (*)$$

First, we note that the function $t \mapsto t^{k-1}$ is convex for $k \geq 2$, so by Hadamard's inequality

$$\int_y^x f(t)dt \geq (x-y) f\left(\frac{x+y}{2}\right) \quad (y < x)$$

we get

$$\frac{x^k - y^k}{k(x-y)} \geq A^{k-1}.$$

It is sufficient to prove that

$$M < A^{k-1} L \left( \leq \frac{x^k - y^k}{k(x-y)} L \right).$$

It is known that (see [14], [8]) $M^2 < AL$, i.e. $M^k < A^{k/2} L^{k/2} \leq A^{k-1} L$, since this is equivalent to $L^{k/2-1} \leq A^{k/2-1}$, valid by $L < A$ and $k \geq 2$. So $(*)$ holds true, and this finishes the proof of (26).

Finally, we prove

**Theorem 4.** *a) For all* $k > 1$ *we have*

$$P(x^k, y^k) < \frac{x^k + y^k}{x+y} P(x, y). \tag{27}$$

*b) For* $0 < k < 2$ *we have*

$$P(x^k, y^k) > \frac{x^k + y^k}{x^2 + y^2} P(x^2, y^2). \tag{28}$$

*c) For all* $k > 0$,

$$P(x^{k+1}, y^{k+1}) < \frac{x^{k+1} + y^{k+1}}{x^k + y^k} P(x^k, y^k). \tag{29}$$

**Proof.** We have seen in Introduction that

$$\frac{A}{P} = \frac{\arcsin z}{z} = f(z), \quad \text{where} \quad z = \frac{x-y}{x+y} \ (0 < y < x)$$

is an increasing function of $x$. Since

$$\frac{x^k - y^k}{x^k + y^k} > \frac{x-y}{x+y} \text{ for } k > 1 \text{ (and } 0 < y < x),$$

we get

$$\frac{A(x^k, y^k)}{P(x^k, y^k)} > \frac{A(x, y)}{P(x, y)},$$

giving (27).

Relation (28) follows from

$$\frac{x^k - y^k}{x^k + y^k} < \frac{x^2 - y^2}{x^2 + y^2} \text{ for } k < 2$$

in the same manner. Finally, (29) is a consequence of

$$\frac{x^{k+1} - y^{k+1}}{x^{k+1} + y^{k+1}} > \frac{x^k - y^k}{x^k + y^k}.$$

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## 6.4 On two means by Seiffert

### 1

Let $A, G, Q$ be the classical means of two arguments defined by

$$A = A(x,y) = \frac{x+y}{2}, \quad G = G(x,y) = \sqrt{xy},$$

$$Q = Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}, \ x, y > 0.$$

Let $L$ and $I$ denote the logarithmic and identric means. It is well known that $G < L < I < A$ for $x \neq y$.

In 1993 H.-J. Seiffert [4] introduced the mean

$$P = P(x,y) = \frac{x-y}{4 \arctan \sqrt{\frac{x}{y}} - \pi} \ (x \neq y), \quad P(x,x) = x$$

and proved that $L < P < I$ for $x \neq y$. In [5] he obtained other relations, too. The mean $P$ can be written also in the equivalent form

$$P(x,y) = \frac{x-y}{2 \arcsin \dfrac{x-y}{x+y}} \ (x \neq y), \tag{1}$$

see e.g. [3].

Let $x < y$. In the paper [1] we have shown that the mean $P$ is the common limit of the two sequences $(x_n)$, $(y_n)$, defined recurrently by

$$x_0 = G(x,y), \quad y_0 = A(x,y), \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1} y_n}.$$

This algorithm appeared in the works of Pfaff (see [1]). By using simple properties of these sequences, strong inequalities for $P$ can be deduced. For example, in [1] we have proved that

$$x_n < \sqrt[3]{y_n^2 x_n} < P < \frac{x_n + 2y_n}{3} < y_n \ (n \geq 0)$$

and that e.g.

$$P(x^k, y^k) \geq (P(x,y))^k \text{ for all } k \geq 1.$$

As applications, the following inequalities may be deduced:

$$\frac{AG}{L} < \sqrt[3]{A^2G} < P < \frac{G+2A}{3} < I, \tag{2}$$

$$\frac{A+G}{2} < P < \sqrt{\frac{A+G}{2}A},$$

$$P > \sqrt[3]{\left(\frac{A+G}{2}\right)^2 A}, \tag{3}$$

etc.

In 1995 Seiffet [6] considered another mean, namely

$$T = T(x,y) = \frac{x-y}{2\arctan\dfrac{x-y}{x+y}} \; (x \neq y), \quad T(x,x) = x. \tag{4}$$

(Here $T$, as $P$ in [1], is our notation for these means, see [2]). He proved that

$$A < T < Q. \tag{5}$$

## 2

Our aim in what follows is to show that by a transformation of arguments, the mean $T$ can be reduced to the mean $P$. Therefore, by using the known properties of $P$, these will be transformed into properties of $T$.

**Theorem 1.** *Let $y, v > 0$ and put*

$$x = \frac{\sqrt{2(u^2+v^2)} + u - v}{2}, \quad y = \frac{\sqrt{2(u^2+v^2)} + v - u}{2}.$$

*Then $x, y > 0$ and $T(u,v) = P(x,y)$.*

**Proof.** From $\sqrt{2(u^2+v^2)} > |u-v|$ we get that $x > 0$, $y > 0$. Clearly one has

$$x + y = \sqrt{2(u^2+v^2)}, \quad x - y = u - v.$$

From the definitions (1) and (4) we must prove

$$\arctan\frac{u-v}{u+v}=\arcsin\frac{u-v}{\sqrt{2(u^2+v^2)}}.$$

Let $u>v$ and put $\alpha=\arctan\dfrac{u-v}{u+v}$. By

$$\sin\alpha=\cos\alpha\tan\alpha=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$$

and

$$\frac{\dfrac{u-v}{u+v}}{\sqrt{1+\left(\dfrac{u-v}{u+v}\right)^2}}=\frac{u-v}{\sqrt{2(u^2+v^2)}}$$

we get

$$\arcsin\frac{u-v}{\sqrt{2(u^2+v^2)}}=\alpha-\arctan\frac{u-v}{u+v},$$

and the proof of the above relation is finished.

It is interesting to remark that

$$A(x,y)=\frac{x+y}{2}=\sqrt{\frac{u^2+v^2}{2}}=Q(u,v)$$

$$G(x,y)=\sqrt{xy}=\frac{u+v}{2}=A(u,v).$$

Therefore, by using the transformations of Theorem 1, the following transformations of means will be true:

$$G\to A,\quad A\to Q,\quad P\to T.$$

Thus, the inequality $G<P<A$ valid for $P$, will be transformed into $A<T<Q$, i.e. relation (5). By using our inequality (2), we get for $T$ the following results:

$$\sqrt[3]{Q^2A}<T<\frac{A+2Q}{3}, \tag{6}$$

while using (3), we get

$$T^3 > \left(\frac{Q+A}{2}\right)^2 Q. \tag{7}$$

In fact, the following is true:

**Theorem 2.** *Let $0 < u < v$. Then $T = T(u,v)$ is the common limit of the sequences $(u_n)$ and $(v_n)$ defined by*

$$u_0 = A(u,v), \quad v_0 = Q(u,v), \quad u_{n+1} = \frac{u_n + v_n}{2}, \quad v_{n+1} = \sqrt{u_{n+1}v_n}.$$

*For al ln $\geq 0$ one has $u_n < T < v_n$ and $\sqrt[3]{v_n^2 u_n} < T < \dfrac{u_n + 2v_n}{3}$.*

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## 6.5 The Schwab-Borchardt mean

### 1. Introduction

The Schwab-Borchardt mean of two numbers $x \geq 0$ and $y > 0$, denoted by $SB(x, y) \equiv SB$, is defined as

$$SB(x,y) = \begin{cases} SB(x,y) = \dfrac{\sqrt{y^2 - x^2}}{\arccos(x/y)}, & 0 \leq x < y \\ \dfrac{\sqrt{x^2 - y^2}}{\operatorname{arccosh}(x/y)}, & y < x \\ x, & x = y \end{cases} \tag{1.1}$$

(see [1, Th. 8.4], [3, (2.3)]). It follows from (1.1) that $SB(x, y)$ is not symmetric in its arguments and is a homogeneous function of degree 1 in $x$ and $y$. Using elementary identities for the inverse circular function, and the inverse hyperbolic function, one can write the first two parts of formula (1.1) as

$$SB(x,y) = \frac{\sqrt{y^2 - x^2}}{\arcsin\left(\sqrt{1 - (x/y)^2}\right)} = \frac{\sqrt{y^2 - x^2}}{\arctan\left(\sqrt{(y/x)^2 - 1}\right)}, \tag{1.2}$$

$$0 \leq x < y$$

and

$$SB(x,y) = \frac{\sqrt{x^2 - y^2}}{\operatorname{arcsinh}\left(\sqrt{(x/y)^2 - 1}\right)} = \frac{\sqrt{x^2 - y^2}}{\operatorname{arctanh}\left(\sqrt{1 - (y/x)^2}\right)}$$

$$= \frac{\sqrt{x^2 - y^2}}{\ln\left(x + \sqrt{x^2 - y^2}\right) - \ln y}, \; y < x \tag{1.3}$$

respectively.

The Schwab-Borchardt mean is the iterative mean i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n, \tag{1.4}$$

where

$$x_0 = x,\ y_0 = y,\ x_{n+1} = (x_n + y_n)/2,\ y_{n+1} = \sqrt{x_{n+1} y_n}, \tag{1.5}$$

$n = 0, 1, \ldots$ (see [3, (2.3)], [2]). It follows from (1.5) that the member of two infinite sequences $\{x_n\}$ and $\{y_n\}$ satisfy the following inequalities

$$x_0 < x_1 < \ldots < x_n < \ldots < SB < \ldots < y_n < \ldots < y_1 < y_0\ (x < y) \tag{1.6}$$

and

$$y_0 < y_1 < \ldots < y_n < \ldots < SB < \ldots < x_n < \ldots < x_1 < x_0\ (y < x). \tag{1.7}$$

For later use, let us record the invariance formula for the Schwab-Borchardt mean

$$SB(x, y) = SB\left(\frac{x+y}{2}, \sqrt{\frac{x+y}{2} y}\right) \tag{1.8}$$

which follows from (1.5).

This paper deals mostly with the inequalities involving the mean under discussion and is organized as follows. Particular cases of the Schwab-Borchardt mean are studied in Section 2. They include two means introduced recently by H.-J. Seiffert, the logarithmic mean and a possible new mean of two variables. The Ky Fan inequalities for these means are also included. The main results of this paper are contained in Section 3. Lower and upper bounds for $SB$, that are stronger than those in (1.6)-(1.7) are contained in Theorem 3.3. Inequalities involving the Schwab-Borchardt mean and the Gauss arithmetic-geometric mean are also obtained. Additional bounds for the mean under discussion are presented in Appendix 1. Inequalities involving numbers $x_n$ and $y_n$ and those used in Theorem 3.3 are presented in Appendix 2.

## 2. Inequalities for the particular means

Before we state and prove the main results of this section let us introduce more notation. Let $x \geq 0$ and $y > 0$. The following function

$$R_C(x, y) = \frac{1}{2}\int_0^{\infty} (t + x)^{-1/2}(t + y)^{-1}dt \tag{2.1}$$

plays an important role in the theory of special functions (see [5], [7]). B.C. Carlson [3] has shown that

$$SB(x, y) = [R_C(x^2, y^2)]^{-1} \tag{2.2}$$

(see also [2, (3.21)]). It follows from (2.2) and (2.1) that the mean $SB(x, y)$ increases with an increase in either $x$ or $y$.

To this end we will assume that the numbers $x$ and $y$ are positive and distinct. The symbols $A, L, G$ and $H$ will stand for the arithmetic, logarithmic, geometric, and harmonic mean of $x$ and $y$, respectively. Recall that

$$L(x, y) = \frac{x - y}{\ln x - \ln y} = \frac{x - y}{2\text{arctanh}\left(\dfrac{x - y}{x + y}\right)} \tag{2.3}$$

(see, e.g., [4]-[5]). Other means used in the paper include two means introduced recently by H.-J. Seiffert

$$P(x, y) = \frac{x - y}{2\arcsin\left(\dfrac{x - y}{x + y}\right)} \tag{2.4}$$

(see [12]) and

$$T(x, y) = \frac{x - y}{2\arctan\left(\dfrac{x - y}{x + y}\right)} \tag{2.5}$$

(see [13]). For the last two means we have used notation introduced in [10] and [11]. Several inequalities for the Seiffert means are obtained in

[8], [10]-[11]. Also, we define a possibly new mean

$$M(x,y)=\frac{x-y}{2\operatorname{arcsinh}\left(\dfrac{x-y}{x+y}\right)}. \tag{2.6}$$

In what follows we will write $Q(x,y)\equiv Q$ for the power mean of order two of $x$ and $y$

$$Q(x,y)=\sqrt{\frac{x^2+y^2}{2}}. \tag{2.7}$$

It is easy to see that the means, $L, P, T$, and $M$ are the Schwab-Borchardt means. Use of (1.2) and (1.3) gives

$$\begin{aligned} &L=SB(A,B), \quad P=SB(G,A), \\ &T=SB(A,Q), \quad M=SB(Q,A). \end{aligned} \tag{2.8}$$

A comparison result for $SB(\cdot,\cdot)$ is contained in the following:

**Proposition 2.1.** *Let $x>y$. Then*

$$SB(x,y)<SB(y,x). \tag{2.9}$$

**Proof.** Using the invariance formula (1.8) together with the monotonicity property of the mean $SB$ in its arguments, we obtain

$$SB(x,y)=SB\left(A,\sqrt{Ay}\right)<SB\left(A,\sqrt{Ax}\right)=SB(y,x). \qquad \square$$

Inequalities connecting means $L, P, M$, and $T$ with underlying means $G, A$, and $Q$ can be established easily using (2.9). We have

$$G<L<P<A<M<T<Q. \tag{2.10}$$

For the proof of (2.10) we use monotonicity of the Schwab-Borchardt mean in its arguments, inequalities $G<A<Q$, and (2.8) to obtain

$$\begin{aligned} G&=SB(G,G)<SB(A,G)<SB(G,A)<SB(A,A) \\ &=A<SB(Q,A)<SB(A,Q)<SB(Q,Q)=Q. \end{aligned}$$

The first three inequalities in (2.10) are known (see [4]-[5], [12], [14]) and the sixth one appears in [13]. (See also [11] for the proof of the last inequality in (2.10) and its refinements.)

We shall establish now the Ky Fan inequalities involving the first six means that appear in (2.10). For $0 < x,\ y \le \frac{1}{2}$, let $x' = 1 - x$ and $y' = 1 - y$. In what follows we will write $G'$ for $G(x', y')$, $L'$ for $L(x', y')$, etc.

**Proposition 2.2.** *Let $0 < x,\ y \le \frac{1}{2}$. The following inequalities*

$$\frac{G}{G'} < \frac{L}{L'} < \frac{P}{P'} < \frac{A}{A'} < \frac{M}{M'} < \frac{T}{T'} \tag{2.11}$$

*hold true.*

**Proof.** The first inequality in (2.11) is established in [9]. For the proof of the second one we use (2.3) and (2.4) to obtain

$$\frac{L}{P} = \frac{\arcsin z}{\mathrm{arctanh} z}, \tag{2.12}$$

where $z = (x - y)/(x + y)$. Let $z' = (x' - y')/(x' + y')$. One can easily verify that $z$ and $z'$ satisfy the following inequalities

$$0 < |z'| < |z| < 1,\ zz' < 0. \tag{2.13}$$

Let $f(z)$ stand for the function on the right side of (2.12). The following properties of $f(z)$ will be used in the proof of (2.11). We have: $f(z) = f(-z)$, $f(z)$ is strictly increasing on $(-1, 0)$ and strictly decreasing on $(0, 1)$,

$$\max\{f(z) :\ |z| \le 1\} = f(0) = 1.$$

Assume that $y < x \le \frac{1}{2}$. It follows from (2.13) that $0 < -z' < z < 1$. This in turn implied that $f(-z') > f(z)$ or what is the same, $L/P < L'/P'$. One can show that the last inequality is also valid if $x < y \le \frac{1}{2}$. This completes the proof of the second inequality in (2.11). The remaining three inequalities in (2.11) can be established in the analogous manner using the formulas

$$\frac{P}{A} = \frac{z}{\arcsin z}, \quad \frac{A}{M} = \frac{\mathrm{arcsinh} z}{z}, \quad \frac{M}{T} = \frac{\arctan z}{\mathrm{arcsinh} z}. \tag{2.14}$$

They follow from (2.4), (2.6) and (2.5). □

We close this section giving the companion inequalities to the inequalities 3 through 5 in (1.10). We have

$$\frac{\pi}{2}P > A > \mathrm{arcsinh} M > \frac{\pi}{4}T. \tag{2.15}$$

The proof of (2.15) let us note that the functions on the right side of (2.14) share the properties of the function $f(z)$, used above. In particular, they attain the global minima at $z = \pm 1$. This in turn implies that

$$\frac{P}{A} < \frac{2}{\pi}, \quad \frac{A}{M} > \mathrm{arcsinh}(1), \quad \frac{M}{T} > \frac{\pi}{4\mathrm{arcsinh}(1)}.$$

The assertion (2.15) now follows. The first inequality in (2.15) is also established in [14] by use of different means.

## 3. Main results

We are in position to present the main results of this paper. Several inequalities for the mean under discussion are obtained. New inequalities for the particular means discussed in the previous section are also included.

Our first result reads as follows:

**Theorem 3.1.** *Let $x$ and $y$ be positive and distinct numbers. If $x < y$, then*

$$T(x, y) < SB(x, y) \tag{3.1}$$

*and if $x > y$, then*

$$SB(x, y) < L(x, y). \tag{3.2}$$

*The following inequalities*

$$SB(y,G) < SB(x,y) < SB(y,A) \tag{3.3}$$

*and*

$$SB(x,y) > H(SB(y,x),y) \tag{3.4}$$

*are valid.*

**Proof.** Let $x < y$. For the proof of (3.1) we use (1.8), the inequality $xy > x^2$ and (2.8) to obtain

$$SB(x,y) = SB\left(A, \sqrt{\frac{x+y}{2}y}\right) > SB(A,Q) = T(x,y).$$

Assume now that $x > y$. Making use of (1.8) and (2.8) together with the application of the inequality $A < x$ gives

$$SB(x,y) = SB\left(A, \sqrt{Ay}\right) < SB(A,G) = L(x,y).$$

In order to establish the first inequality in (3.3) we need the following one

$$[(t+x^2)(t+y^2)]^{-1/2} \le (t+G^2)^{-1}$$

(see [4]). Multiplying both sides by $(1/2)(t+y^2)^{-1/2}$ and next integrating from 0 to infinity we obtain, using (2.1),

$$R_C(x^2,y^2) < R_C(y^2,G^2).$$

Application of (2.2) to the last inequality gives the desired result. The second inequality in (3.3) follows from the first one. Substitution $y := A$ together with (1.8) give

$$SB\left(A, \sqrt{Ax}\right) = SB(y,x) < SB(x,A).$$

Interchanging $x$ with $y$ in the last inequality we obtain the asserted result. For the proof of (3.4) we apply the arithmetic mean - geometric mean in inequality to $[(t+x^2)(t+y^2)]^{-1/2}$ to obtain

$$[(t+x^2)(t+y^2)]^{-1/2} < (1/2)[(t+x^2)^{-1} + (t+y^2)^{-1}].$$

Multiplying both sides by $(1/2)(t+y^2)^{-1/2}$ and next integrating from 0 to infinity, we obtain

$$R_C(x^2, y^2) < \frac{1}{2}\left[R_C(y^2, x^2) + \frac{1}{y}\right].$$

Here we have used the identity $R_C(y^2, y^2) = 1/y$. Application of (2.2) to the last inequality gives

$$\frac{1}{SB(x,y)} < \frac{1}{2}\left[\frac{1}{SB(y,x)} + \frac{1}{y}\right] = \frac{1}{H(SB(y,x), y)}.$$

This completes the proof. □

**Corollary 3.2.** *The following inequalities*

$$T(A,G) < P, \quad T(A,Q) < T, \tag{3.5}$$

$$L < L(A,G), \quad M < L(A,Q), \tag{3.6}$$

$$L > H(P,G),\ P > H(L,A),\ M > H(T,A),\ T > H(M,Q) \tag{3.7}$$

*hold true.*

**Proof.** Inequalities (3.5) follows from (3.1) and (2.8) by letting $(x,y) := (G,A)$ and $(x,y) := (A,Q)$. Similarly, (3.6) follows from (3.2). Putting $(x,y) := (A,G)$ and $(x,y) := (Q,A)$ we obtain the desired result. Inequalities (3.7) follow from (3.4). The substitutions $(x,y) := (A,G)$, $(x,y) := (G,A)$, $(x,y) := (Q,A)$, and $(x,y) := (A,Q)$ together with application of (2.8) give the desired result. □

The first inequality in (3.6) is also established in [8].

Before we state and prove the next result, let us introduce some notation. In what follows, the symbols $\alpha$ and $\beta$ will stand for positive numbers such that $\alpha + \beta = 1$. The weighted arithmetic mean and the weighted geometric mean of $x_n$ and $y_n$ (see (1.5)) with weights $\alpha$ and $\beta$ are defined as

$$u_n = \alpha x_n + \beta y_n,\ v_n = x_n^{\alpha} y_n^{\beta}, \quad n = 0, 1, \ldots \tag{3.8}$$

**Theorem 3.3.** *In order for the sequence* $\{u_n\}_0^\infty$ ($\{v_n\}_0^\infty$) *to be strictly decreasing (increasing) it suffices that* $\alpha = 1/3$ *and* $\beta = 2/3$. *Moreover,*

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = SB(x,y) \tag{3.9}$$

*and the inequalities*

$$(x_n y_n^2)^{1/3} < SB(x,y) < \frac{x_n + 2y_n}{3} \tag{3.10}$$

*hold true for all* $n \geq 0$.

**Proof.** For the proof of the monotonicity property of the sequence $\{u_n\}_0^\infty$ we use (3.8), (1.5), and the arithmetic mean - geometric mean inequality to obtain

$$\begin{aligned}
u_{n+1} &= \alpha x_{n+1} + \beta y_{n+1} \\
&= \alpha x_{n+1} + \beta (x_{n+1} y_n)^{1/2} \\
&< \alpha x_{n+1} + \beta \frac{x_{n+1} + y_n}{2} \\
&= \left(\frac{\alpha}{2} + \frac{\beta}{4}\right) x_n + \left(\frac{\alpha}{2} + \frac{3\beta}{4}\right) y_n.
\end{aligned}$$

In order for the inequality $u_{n+1} < u_n$ to be satisfied it suffices that

$$\left(\frac{\alpha}{2} + \frac{\beta}{4}\right) x_n + \left(\frac{\alpha}{2} + \frac{3\beta}{4}\right) y_n = \alpha x_n + \beta y_n.$$

This implies that $\alpha = 1/3$ and $\beta = 2/3$. For the proof of the monotonicity result for the sequence $\{v_n\}_0^\infty$ we follow the lines introduced above to obtain

$$\begin{aligned}
v_{n+1} = x_{n+1}^{\alpha} y_{n+1}^{\beta} &= x_{n+1}^{\alpha} (x_n + y_n)^{\beta/2} \\
&= \left(\frac{x_n + y_n}{2}\right)^{\alpha + \beta/2} y_n^{\beta/2} \\
&> x_n^{\alpha/2 + \beta/4} y_n^{\alpha/2 + 3\beta/4} = x_n^{\alpha} y_n^{\beta},
\end{aligned}$$

where the last equality holds provided $\alpha = 1/3$ and $\beta = 2/3$. The assertion (3.9) follows from (3.8) and (1.4). Inequalities (3.10) are the obvious consequence of (3.9) and the first statement of the theorem. $\square$

Inequalities (3.10) for the Seiffert mean $P$ are obtained in [10].

**Corollary 3.4.** *The following inequality*

$$\frac{1}{SB(x,y)} < \frac{1}{3}\left(\frac{2}{A} + \frac{1}{y}\right) \tag{3.11}$$

*holds true.*

**Proof.** Use of the first inequality in (3.10) with $n = 1$ gives

$$(A^2 y)^{1/3} < SB(x,y).$$

Application of the arithmetic mean - harmonic mean inequality with weights leads to

$$\frac{1}{SB(x,y)} < \left(\frac{1}{A}\right)^{2/3}\left(\frac{1}{y}\right)^{1/3} < \frac{2}{3}\cdot\frac{1}{A} + \frac{1}{3}\cdot\frac{1}{y} = \frac{1}{3}\left(\frac{2}{A} + \frac{1}{y}\right). \qquad \square$$

Inequalities connecting the Schwab-Borchardt mean and the celebrated Gauss arithmetic-geometric mean $AGM(x,y) \equiv AGM$ are contained in Theorem 3.5. For the reader's convenience, let us recall that the Gauss mean is the iterative mean, i.e.,

$$AGM = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ are defined as

$$\begin{aligned} a_0 &= \max(x,y), & b_0 &= \min(x,y), \\ a_{n+1} &= (a_n + b_n)/2, & b_{n+1} &= \sqrt{a_n b_n} \end{aligned} \tag{3.12}$$

($n \ge 0$). (See, e.g., [1], [5]). Clearly,

$$b_0 < b_1 < \ldots < b_n < \ldots < AGM < \ldots < a_n < \ldots < a_1 < a_0, \tag{3.13}$$

$AGM(\cdot, \cdot)$ is a symmetric function in its arguments and

$$AGM(x, y) = AGM(a_n, b_n) \tag{3.14}$$

for all $n \geq 0$.

For later use, let us record two inequalities. If $x > y$, then

$$SB(x, y) < AGM(x, y) \tag{3.15}$$

and

$$AGM(x, y) < SB(x, y) \tag{3.16}$$

provided $x < y$. Inequality (3.15) follows from

$$SB(x, y) < L(x, y) < AGM(x, y),$$

where the first inequality is established in Theorem 3.1 and the second one is due to Carlson and Vuorinen [6]. Inequality (3.16) follows from

$$AGM(x, y) < A(x, y) < T(x, y) < SB(x, y).$$

The first inequality is a special case of (3.13) when $n = 1$, the second one appears in [13] and [11], and the last inequality is established earlier (see (3.1)).

We are in position to prove the following

**Theorem 3.5.** *Let $n = 0, 1, \ldots$. The number $SB(a_n, b_n)$ form a strictly increasing sequence while $SB(b_n, a_n)$ form a strictly decreasing sequence. Moreover,*

$$SB(a_n, b_n) < AGM < SB(b_n, a_n). \tag{3.17}$$

**Proof.** Using (1.8), (3.13), (3.15) and (3.14) we obtain

$$\begin{aligned} SB(a_n, b_n) &= SB\left(a_{n+1}, \sqrt{a_{n+1}b_n}\right) < SB\left(a_{n+1}, \sqrt{a_n b_n}\right) \\ &= SB(a_{n+1}, b_{n+1}) < AGM(a_{n+1}, b_{n+1}) \\ &= AGM(x, y). \end{aligned}$$

Similarly, using (1.8), (3.13), (3.16) and (3.14) one obtains

$$\begin{aligned} SB(b_n, a_n) &= SB\left(a_{n+1}, \sqrt{a_{n+1}a_n}\right) > SB(b_{n+1}, a_{n+1}) \\ &> AGM(b_{n+1}, a_{n+1}) = AGM(x, y). \end{aligned}$$

The proof is complete. $\square$

**Corollary 3.6.** *Let the numbers $a_n$ and $b_n$ $(n \geq 1)$ be the same as in (3.12). If $a_0 = A$ and $b_0 = G$, then*

$$L < L(a_n, b_n) < AGM(x, y) < P(a_n, b_n) < P \tag{3.18}$$

*for all $n > 0$. Similarly, if $a_0 = Q$ and $b_0 = A$, then*

$$M < L(a_n, b_n) < AGM(A, Q) < P(a_n, b_n) < T, \ n \geq 0. \tag{3.19}$$

**Proof.** Inequalities (3.18) follow immediately from Theorem 3.5 and from the formulas

$$SB(a_0, b_0) = SB(A, G) = L,$$

$$SB(a_{n+1}, b_{n+1} = L(a_n, b_n),$$

$$SB(b_0, a_0) = SB(G, A) = P$$

and

$$SB(b_{n+1}, a_{n+1}) = P(b_n, a_n) = P(a_n, b_n), \ n \geq 0.$$

Since the proof of (3.19) goes along the lines introduced above, it is omitted. $\square$

## Appendix 1. Bounds for the Schwab-Borchardt mean

We shall prove the following:

**Proposition A1.** *If $x > y$, then*

$$\frac{2x^2 - y^2}{2x\ln(2x/y)} < SB(x, y) < \frac{2x^2 - y^2}{2x\ln(2x/y) - (y^2/x)\ln 2}. \tag{A1.1}$$

*Otherwise, if $y > x \geq 0$, then*

$$\frac{4y^3}{\pi(x^2+2y^2)-4xy} < SB(x,y) \leq \frac{4y^3}{\pi(x^2/2+2y^2)-4xy}. \tag{A1.2}$$

*Equalities hold in* $(A1.2)$ *if and only if* $x = 0$.

**Proof.** Assume that $x > y$. The following asymptotic expansion

$$R_C(x^2,y^2) = \frac{1}{2x}\left(\ln\frac{4x^2}{y^2} + \frac{y^2}{2x^2-y^2}\ln\frac{\theta x^2}{y^2}\right),\ 1 < \theta < 4,$$

in established in [7, Eq. (23)]. Letting above $\theta = 1$ and $\theta = 4$ and next using (2.2) we obtain inequalities ($A1.1$). Assume now that $y > x \geq 0$. Then

$$R_C(x^2,y^2) = \frac{\pi}{2y} - \frac{x}{y^2} + \frac{\pi x^2}{4y^3}\theta,$$

where $y/(x+y) \leq \theta \leq 1$ (see [7, Eq. (22)]). This in conjunction with (2.2) gives (A1.2). □

It is worth mentioning that the bounds (A1.1) are sharp when $x \gg y$ while (A1.2) are sharp if $y \gg x$.

## Appendix 2. Inequalities connecting sequences (1.5) and (3.8)

Let the numbers $x_n$ and $y_n$ ($n \geq 0$) be the same as in the Schwab-Borchardt algorithm (1.5). Further, let $u_n$ and $v_n$ be defined in (3.8) with $\alpha = 1/3$ and $\beta = 2/3$, i.e.,

$$u_n = \frac{x_n + 2y_n}{3}, \quad v_n = (x_n y_n^2)^{1/3}, \quad n \geq 0.$$

These numbers have been used in [10, Ths. 1 and 2] to obtain several inequalities involving the Seiffert mean $P$ and other means.

The following inequalities, which hold true for all $n \geq 0$, show that the numbers $u_n$ and $v_n$ provide sharper bounds for $SB$ than those obtained from $x_n$ and $y_n$. We have

$$y_n < v_n \quad \text{and} \quad u_n < x_n \quad \text{if} \quad y < x$$

and

$$x_n < v_n \quad \text{and} \quad u_n < y_n \quad \text{if} \quad x < y.$$

We shall prove that these inequalities can be improved if $x$ and $y$ belong to certain cones in the plane.

**Proposition A2.** *Let $c = \sqrt{5} - 2 = 0.236\ldots$ and assume that $x > 0$, $y > 0$ with $x \neq y$. If*

$$cx < y < x, \tag{A2.1}$$

*then*

$$y_{n+1} < v_n \quad \text{and} \quad u_n < x_{n+1} \tag{A2.2}$$

*for all $n \geq 0$. Similarly, if*

$$cy < x < y, \tag{A2.3}$$

*then*

$$x_{n+1} < v_n \quad \text{and} \quad u_n < y_{n+1} \tag{A2.4}$$

*for $n = 0, 1, \ldots$.*

Proof of inequalities (A2.2) and (A2.4) is based upon results that are contained in the following lemmas.

**Lemma 1.** *Let $x$ and $y$ be distinct positive numbers. If $cx < y < x$, then*

$$\frac{x+y}{2} < (xy^2)^{1/3}. \tag{A2.5}$$

*If $x < y$, then*

$$\frac{x+2y}{3} < \sqrt{\frac{x+y}{2}y}. \tag{A2.6}$$

**Proof.** For the proof of (A2.5) let us consider a quadratic function

$$p(y) = y^2 + 4xy - x^2 = \left[y - \left(\sqrt{5} - 2\right)x\right]\left[x + \left(\sqrt{5} + 2\right)x\right].$$

It follows that $p(y) > 0$ if $cx < y$. Inequality $p(y) > 0$ can be written as $(x - y)^2 < 2y(x + y)$. Multiplying both sides by $x - y > 0$ we obtain the desired result. In order to establish the inequality (A2.6) let us introduce a quadratic function

$$q(y) = y^2 + xy - 2x^2 = (y - x)(y + 2x).$$

Clearly $q(y) > 0$ if $x < y$. Inequality $q(y) > 0$ is equivalent to

$$x^2 < \frac{1}{2}(xy + y^2).$$

Adding $4xy + 4y^2$ to both sides of the last inequality we obtain

$$\left(\frac{x + 2y}{3}\right)^2 < \frac{x + y}{2} y.$$

Hence, the assertion follows. $\square$

**Lemma 2.** *If $cx < y < x$, then the following inequalities*

$$cx_n < y_n < x_n \tag{A2.7}$$

*hold true for all $n \geq 0$. Similarly, if $cy < x < y$, then*

$$cy_n < x_n < y_n \tag{A2.8}$$

*for all $n \geq 0$.*

**Proof.** The second inequalities in (A2.7) and (A2.8) follows from (1.7) and (1.6), respectively. For the proof of the first inequalities in (A2.7) and (A2.8) we will use the mathematical induction on $n$. There is nothing to prove when $n = 0$. Assume that $cx_n < y_n$, for some $n > 0$. Using (1.5), the inductive assumption and (1.7) we obtain

$$cx_{n+1} = c \cdot \frac{x_n + y_n}{2} < \frac{y_n + cy_n}{2} = \frac{\sqrt{5} - 1}{2} \cdot y_n < y_n < y_{n+1}.$$

Now let $cy < x < y$. Assume that $cy_n < x_n$ for some $n > 0$. Using (1.5), the arithmetic mean - geometric mean inequality and the inductive assumption we obtain

$$
\begin{aligned}
cy_{n+1} = c\sqrt{x_{n+1}y_n} &< c\cdot\frac{x_{n+1}+y_n}{2} < \frac{1}{2}(cx_{n+1}+x_n)\\
&< \frac{1}{2}(cy_n+x_n) < \frac{1}{2}(y_n+x_n)\\
&= x_{n+1}.
\end{aligned}
$$

**Proof of Proposition A2.** For the proof of the first inequality in (A2.2) we use (A2.7) and (A2.5) to obtain

$$
\frac{x_n+y_n}{2} < (x_ny_n^2)^{1/3},\ n\geq 0. \tag{$A2.9$}
$$

Making use of (1.5) and (A2.9) we obtain

$$
y_{n+1} = (x_{n+1}y_n)^{1/2} = \left(\frac{x_n+y_n}{2}\right)^{1/2} y_n^{1/2} < (x_ny_n^2)^{1/3} = v_n.
$$

The second inequality in (A2.2) can be established as follows. We add to both sides of $y_n < x_n$ (see (1.7)) $2x_n + 3y_n$ and next divide the resulting inequality by 6 to obtain the desired result. For the proof of the first inequality in (A2.4) we use (A2.5) with $x$ replaced by $y$ and $y$ replaced by $x$, the inequalities (A2.8) and $x_n < y_n$ (see (1.6)) to obtain

$$
x_{n+1} = \frac{x_n+y_n}{2} < (x_n^2y_n)^{1/3} < (x_ny_n^2)^{1/3} = v_n.
$$

The second inequality in (A2.4) is obtained with the aid of (A2.8), (A2.6) and (1.5). We have

$$
u_n = \frac{x_n+2y_n}{3} < \sqrt{\frac{x_n+y_n}{2}y_n} = y_{n+1}. \qquad \square
$$

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## 6.6 Refinements of the Mitrinović-Adamović inequality with application

### 1. Introduction

The famous inequality due to D.D. Adamović and D.S. Mitrinović (see [2, p. 238]) states that for any $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$\frac{\sin x}{x} > (\cos x)^{\frac{1}{3}} \tag{1.1}$$

The Seiffert mean $P$ of two positive variables, (see [5], [6]) is defined by

$$P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}} \text{ for } x \neq y; P(x,x) = x \tag{1.2}$$

Let

$$A(x,y) = \frac{x+y}{2}, \quad G(x,y) = \sqrt{xy}$$

denote the arithmetic, resp. geometric means of $x$ and $y$.

Let

$$x_0 = \sqrt{xy}, \quad y_0 = \frac{x+y}{2}$$

and

$$x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1}y_n} \quad (n \geq 0)$$

be the Pfaff algorithm (see e.g. [3]).

In 2001 the author [3] has proved that for any $n \geq 0$ one has

$$\sqrt[3]{y_n^2 x_n} < P < \frac{x_n + 2y_n}{3} \tag{1.3}$$

Particularly, for $n = 0$, from (1.3) we get the double inequality

$$\sqrt[3]{A^2 G} < P < \frac{G + 2A}{3}, \tag{1.4}$$

while for $n = 1$ we get

$$\sqrt[3]{A\left(\frac{A+G}{2}\right)^2} < P < \frac{1}{3}\left(\frac{A+G}{2} + 2\sqrt{\frac{A+G}{2} \cdot A}\right) \tag{1.5}$$

In what follows, we will use the above mean inequalities, as well as certain algebraic inequalities, in order to obtain refinements of the Mitrinović-Adamović inequality (1.1). An application to a new and simple proof of a result from [1] will be offered, too. For mean inequalities and trigonometric and hyperbolic applications, see also [4].

## 2. Main results

The main result of this section is contained in the following:

**Theorem 2.1.** *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^{\frac{2}{3}} > \frac{1 + \sqrt{1 + 8\cos x}}{4} > \sqrt[3]{\cos x} \tag{2.1}$$

**Proof.** First remark that

$$P(1 + \sin x, 1 - \sin x) = \frac{\sin x}{x}$$

and

$$A(1 + \sin x, 1 - \sin x) = 1, \quad G(1 + \sin x, 1 - \sin x) = \cos x.$$

Applying the left side of (1.5) we get the first inequality of (2.1).

For the second inequality of (2.1) put $u = \cos x$ and $\frac{u+1}{2} = v^3$ (here $0 < u, v < 1$). Then $u = 2v^3 - 1$ and the inequality becomes $\left(4v^2 - 1\right)^2 > 16v^3 - 7$ or

$$P(v) = 2v^4 - 2v^3 - v^2 + 1 > 0 \tag{2.2}$$

After elementary transformations, $P(v)$ can be written as

$$P(v) = (v - 1)^2(2v^2 + 2v + 1),$$

so (2.2) follows.

For the proof of the last inequality of (2.1) let again $u = \cos x$ and $u = s^3$. Then we have to prove

$$4s < 1 + \sqrt{1 + 8s^3} \quad \text{or} \quad (4s - 1)^2 < 1 + 8s^3.$$

This becomes $16s^2 - 8s < 8s^3$ or $2s - 1 < s^2$, which is $(s - 1)^2 > 0$, so it is true. $\square$

**Remark 1.** Clearly, by (1.3), the first inequality of (2.1) can be further improved (e.g. by selecting $n = 2$, etc.), and in fact infinitively many improvement are obtainable (as the sequence $(y_n^2 x_n)$ is strictly increasing, see [3]).

## 3. An application

In what follows, we will apply the following part of inequality (2.1):

$$\frac{\sin x}{x} > \frac{1 + \sqrt{1 + 8 \cos x}}{4} \tag{3.1}$$

In 2015 (see [1]) B. Bhayo, R. Klén and the author have proved the following result:

**Theorem 3.1.** *The best constants $\alpha$ and $\beta$ such that*

$$\frac{\cos x + \alpha - 1}{\alpha} < \frac{\sin x}{x} < \frac{\cos x + \beta - 1}{\beta} \tag{3.2}$$

*for any $x \in \left(0, \frac{\pi}{2}\right)$ are $\alpha = \frac{\pi}{\pi - 2}$ and $\beta = 3$.*

The proof of this result in [1] is based on certain series expansions with Bernoulli numbers, and applications of more auxiliary results.

Our aim is to show that (3.1) will offer an easy proof to this theorem.

The inequality

$$\frac{\sin x}{x} < \frac{\cos x + \beta - 1}{\beta}$$

may be written also as

$$\beta > f(x) = \frac{x - x \cos x}{x - \sin x}.$$

We will prove that this function $f(x)$ is strictly decreasing. An immediate computation gives

$$(x-\sin x)^2 f'(x) = -\sin x + \sin x\cos x + x\cos x + x^2\sin x - x = g(x).$$

Also,

$$g'(x) = -2\sin^2 x + x\sin x + x^2\cos x = x^2(-2r^2 + r + \cos x),$$

where

$$r = \frac{\sin x}{x}.$$

Now, the polynomial

$$P(r) = -2r^2 + r + \cos x$$

of variable $r$ has roots $\dfrac{1 \pm \sqrt{1+8\cos x}}{4}$. By inequality (3.1) we get $P(r) < 0$ for $x \in \left(0, \dfrac{\pi}{2}\right)$. Therefore, $g'(x) < 0$, implying $g(x) < g(0) = 0$, so $f(x)$ is indeed strictly decreasing. This implies $f(x) > \lim\limits_{x\to 0} f(x) = 3$, and also $f(x) < f\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{\pi - 2}$, which are the best constants in (3.2).

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# 6.7 A note on bounds for the Neuman-Sándor mean using power and identric means

## 1. Introduction

For $k \in \mathbb{R}$ the $k$th power mean $A_p(a,b)$, Neuman-Sándor Mean $M(a,b)$ [1] and the identric mean $I(a,b)$ of two positive real numbers $a$ and $b$ are defined by

$$A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k} (k \neq 0); A_0(a,b) = \sqrt{ab} = G(a,b) \qquad (1)$$

$$M(a,b) = \frac{a-b}{2\text{arcsinh}\big((a-b)(a+b)\big)} (a \neq b); M(a,a) = a \qquad (2)$$

$$I(a,b) = \frac{1}{e}\big(b^b/a^a\big)^{1/(b-a)} (a \neq b); I(a,a) = a \qquad (3)$$

respectively, where $\text{arcsinh}(x) = \log(x + \sqrt{1+x^2})$ denotes the inverse hyperbolic sine function.

While the $k$th power means and the identric mean have been studied extensively in the last 30-40 years (see e.g. [2] or [3] for surveys of results), the Neuman-Sándor mean has been introduced in 2003 [1] and studied also in 2006 [4], as a particular Schwab-Borchardt mean. In the last 10 years, the Neuman-Sándor mean has been studied by many authors, for many references, see e.g. the papers [5] and [6], [7].

In 2012 and 2013, independently Z.-H. Yang [5] and Y.-M. Chu, B.-Y. Long [7] have considered the bounds

$$A_r < M < A_{4/3}, \qquad (4)$$

where $M = M(a,b)$ for $a \neq b$, etc; and $r = \dfrac{\log 2}{\log\log(3+2\sqrt{2})} = 1.244\ldots$ Also, the constants $r$ and $4/3$ are best possible. Though not mentioned explicitly, the upper bound of (4) is due to E. Neuman and J. Sándor. Indeed, they proved the strong inequalities (see also [7]):

$$M(a,b) < \frac{2A+Q}{3} < \left[He(a^2,b^2)\right]^{1/2} < A_{4/3}(a,b), \tag{5}$$

where

$$He(x,y) = \frac{x+\sqrt{xy}+y}{3}$$

denotes the Heronian mean and

$$A = A(a,b) = A_1(a,b), \quad Q = Q(a,b) = \left(\frac{a^2+b^2}{2}\right)^{1/2} = A_2(a,b).$$

The first inequality of (5) appears in [1], while the second one results by remarking that

$$He(a^2,b^2) = \frac{2a^2+b^2}{3} = \frac{Q^2+2A^2}{3}$$

and the fact that

$$\frac{Q^2+2A^2}{3} > \left(\frac{2A+Q}{3}\right)^2.$$

The last inequality of (5) follows by

$$He(a,b) < A_{2/3}(a,b) \tag{6}$$

(see [8], [9]) applied to $a := a^2, b := b^2$.

We note also that for application purposes, we may choose $1.2 = \dfrac{6}{5}$ in place of $r$ in (4), so the following bounds (though, the lower bound slightly weaker) may be stated:

$$A_{6/5} < M < A_{4/3} \tag{7}$$

In the recent paper [7], $M$ is compared also to the identric mean $I$, in the following manner:

$$1 < \frac{M}{I} < c, \tag{8}$$

where $c = \dfrac{e}{2\log(1+\sqrt{2})}$ and $M = M(a,b)$ for $a \neq b$; etc.

Also, the constants 1 and $c$ in (8) follows from earlier known results. Also, the optimality of constants follows from the proofs of these known results.

## 2. Main results

In [1] it is shown that

$$1 < \frac{M}{A} < \frac{1}{\operatorname{arcsinh}(1)} = \frac{1}{\log(1+\sqrt{2})}, \tag{9}$$

where $M = M(a,b)$ for $a \neq b$; etc.

Now, by a result of H. Alzer [10] one has

$$1 < \frac{A}{I} < \frac{e}{2} \tag{10}$$

We note that inequality (10) has been rediscovered many times. See e.g. the author's papers [9], [11].

Now, by a simple multiplication of (9) and (10), we get (8).

For the proof of the fact that 1 and $c$ are best possible, we shall use the proofs of (9) and (10) from [1] resp. [9]. In [1] it is shown that

$$\frac{M}{A} = \frac{z}{\operatorname{arcsinh} z}, \text{ where } z = \frac{b-a}{b+a} \tag{11}$$

Let $b > a$. Then the function

$$f_1(z) = \frac{z}{\operatorname{arcsinh} z}$$

is strictly increasing in $(0,1)$. Put $\dfrac{b}{a} = x$. Then $z = \dfrac{x-1}{x+1}$ is a strictly increasing function of $x > 1$. Therefore $f_1(z)$, as a composite function, will be strictly increasing also on $x \in (1, +\infty)$.

For the proof of (10) in [9] it is shown that

$$f_2(x) = \frac{A(x,1)}{I(x,1)}$$

is strictly increasing of $x > 1$.

Now, remarking that

$$\frac{M(x,1)}{I(x,1)} = f_1\big(z(x)\big) \cdot f_2(x) = g(x),$$

from the above, we get that $g(x)$ is a strictly increasing function, as the product of two functions having the same property. This gives

$$\lim_{x\to 1} g(x) < g(x) < \lim_{x\to\infty} g(x)$$

As

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} f_1(z(x)) \cdot \lim_{x\to 1} f_2(x) = 1$$

and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f_1(z(x)) \cdot \lim_{x\to\infty} f_2(x) = \frac{1}{\log(1+\sqrt{2})} \cdot \frac{e}{2} = c$$

we get the optimality of the constants from (8).

We note that the proof of (8) given in [7] is complicated, and based on subsequent derivatives of functions.

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## 6.8 A note on a bound of the combination of arithmetic and harmonic means for the Seiffert's mean

### 1. Introduction

Let $a, b > 0$ and

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab}, \quad H = H(a,b) = \frac{2ab}{a+b}$$

be the classical means representing the arithmetic, geometric and harmonic means of $a$ and $b$.

Further, let

$$P = P(a,b) = \frac{a-b}{4\mathrm{arctg}\left(\sqrt{\frac{a}{b}}\right) - \pi}, \ a \neq b; \quad P(a,a) = a$$

be the Seiffert mean of $a$ and $b$. For references of this mean, see the Bibliography of papers [4], [2], [6].

In paper [4] the author remarked that $P$ can be written as the common limit of a pair of sequences $(a_n)$ and $(b_n)$, defined recurrently by

$$a_0 = \sqrt{ab},\ b_0 = \frac{a+b}{2},\ a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_{n+1} \cdot b_n} \quad (n \geq 0).$$

Since this algorithm is due to Pfaff (see e.g. [1]), the author suggested the use of letter ,,P" for this mean.

By using these sequences, the author proved in [4] that

$$\sqrt[3]{b_n^2 a_n} < P < \frac{a_n + 2b_n}{3} \text{ for all } \geq 0. \tag{1}$$

Particularly, the left side of (1) for $n = 0$ gives

$$A^{\frac{2}{3}} \cdot G^{\frac{1}{3}} < P, \tag{2}$$

which is the left side of inequality (20) in [4].

For $n = 1$ we get a better lower bound, namely (see (23) in [4])

$$\left(\frac{A+G}{2}\right)^{\frac{2}{3}} \cdot A^{\frac{1}{3}} < P. \tag{3}$$

We note that from (1) we can deduce better-and-better results for increasing values of $n$.

## 2. Main results

In paper [6] it is shown that

$$A^{\frac{5}{6}} \cdot H^{\frac{1}{6}} < P, \tag{4}$$

which in fact may be considered the main result of the paper.

As $H = \dfrac{G^2}{A}$, it is easy to see that $A^{\frac{5}{6}} \cdot H^{\frac{1}{6}} = A^{\frac{2}{3}} \cdot G^{\frac{1}{3}}$, i.e. relation (4) coincides in fact with (2). The complicated method used by the authors in [6] should be compared to the natural sequential method from [4].

The authors prove also that $\dfrac{5}{6}$ is the best value of $k$ in

$$A^k \cdot H^{1-k} < P, \tag{5}$$

but this has been remarked also in [2], where a generalization of the method from [4] to the general Schwab-Borchardt mean was deduced. In fact one may consider instead the Seiffert mean $P$, the more general mean $SB(a,b)$, representing the ,,Schwab-Borchardt" mean (see also [3]).

The Schwab-Borchardt mean of $a, b > 0$, and denoted by $SB = SB(a,b)$ is defined by

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & \text{if } 0 < a < b \\ \dfrac{\sqrt{a^2-b^2}}{\operatorname{arccosh}(a/b)}, & \text{if } b < a \\ a, & \text{if } a = b \end{cases} \tag{6}$$

(see e.g. [1]). It follows that $SB$ is not symmetric in its arguments and is a homogeneous function of degree 1 in $a$ and $b$. Using elementary identities for the inverse circular function, and the inverse hyperbolic function, one can write the first two parts of (6) as

$$SB(a,b)=\frac{\sqrt{b^2-a^2}}{\arcsin\left(1-(a/b)^2\right)}=\frac{\sqrt{b^2-a^2}}{\arctan\left(\sqrt{(b/a)^2-1}\right)},\quad 0<a<b \tag{7}$$

and

$$\begin{aligned}SB(a,b)&=\frac{\sqrt{a^2-b^2}}{\operatorname{arcsinh}\left(\sqrt{(a/b)^2-1}\right)}=\frac{\sqrt{a^2-b^2}}{\operatorname{arctanh}\left(\sqrt{1-(b/a)^2}\right)}\\&=\frac{\sqrt{a^2-b^2}}{\ln\left(a+\sqrt{a^2-b^2}\right)-\ln b},\qquad \text{if } b<a,\end{aligned} \tag{8}$$

respectively.

The Schwab-Borchardt mean is the common limit of the pair of sequences $(a_n)$ and $(b_n)$ defined recurrently by

$$a_0=a,\ b_0=b,\ a_{n+1}=\frac{a_n+b_n}{2},\ b_{n+1}=\sqrt{a_{n+1}\cdot b_n}\quad (n\geq 0)$$

(see [1]), which means that the mean $P$ is a particular $SB$-mean, i.e.

$$P=SB(G,A). \tag{9}$$

Recall that the logarithmic mean $L=L(a,b)$ of $a$ and $b$ is defined by

$$\begin{aligned}&L=L(a,b)=\frac{a-b}{\ln a-\ln b}=\frac{a-b}{2\operatorname{arctanh}\left(\dfrac{a-b}{a+b}\right)},\ a\neq b\\&L(a,a)=a,\end{aligned}$$

so it is immediate that

$$L=SB(A,G). \tag{10}$$

Another Seiffert mean, denoted by $T$ in [5] is

$$T(a,b) = \frac{a-b}{2\arctan\left(\dfrac{a-b}{a+b}\right)}, \quad a \neq b, \tag{11}$$

as well as a new mean of Neuman and Sándor (see [2], [3]) is

$$M(a,b) = \frac{a-b}{2\mathrm{arcsinh}\left(\dfrac{a-b}{a+b}\right)}, \quad a \neq b. \tag{12}$$

They may be represented also as

$$T = SB(A,Q) \tag{13}$$

and

$$M = SB(Q,A), \tag{14}$$

where

$$Q = Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}$$

is the power mean of order two of $a$ and $b$.

In paper [2], among many other properties of $SB(a,b)$, the following inequalities have been proved:

$$\sqrt[3]{b_n^2 \cdot a_n} < SB(a,b) < \frac{a_n + 2b_n}{3} \quad \text{for all } n \geq 0. \tag{15}$$

By relation (9), (15) extends (1) to the case of Schwab-Borchardt means.

Since here we are interested in inequalities of type (2) (or (3)), we note that for $n = 0$, via relations (9), (10), (13), (14), besides (2) we get the following inequalities:

$$G^{\frac{2}{3}} \cdot A^{\frac{1}{3}} < L, \tag{16}$$

$$Q^{\frac{2}{3}} \cdot A^{\frac{1}{3}} < T, \tag{17}$$

and

$$A^{\frac{2}{3}} \cdot Q^{\frac{1}{3}} < M \tag{18}$$

and all these inequalities are optimal, in the sense of (5).

We note that, if we want, the left sides of (16)-(18) can be expressed also in terms of $A$ and $H$.

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## 6.9 The Huygens and Wilker-type inequalities as inequalities for means of two arguments

### 1. Introduction

The famous Wilker inequality for trigonometric functions states that for any $0 < x < \pi/2$ one has

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \tag{1}$$

while the Huygens inequality asserts that

$$\frac{2\sin x}{x} + \frac{\tan x}{x} > 3. \tag{2}$$

Another important inequality, called as the Cusa-Huygens inequality, says that

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \tag{3}$$

in the same interval $(0, \pi/2)$.

The hyperbolic versions of these inequalities are

$$\left(\frac{\sin hx}{x}\right)^2 + \frac{\tanh x}{x} > 2; \tag{4}$$

$$\frac{2\sinh x}{x} + \frac{\tanh x}{x} > 3; \tag{5}$$

$$\frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \tag{6}$$

where in all cases, $x \neq 0$.

For history of these inequalities, for interconnections between them, generalizations, etc., see e.g. papers [22], [23], [10].

Let $a, b > 0$ be two positive real numbers. The arithmetic, geometric, logarithmic and identric means of these numbers are defined by

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab}, \tag{7}$$

$$L = L(a,b) = \frac{a-b}{\ln a - \ln b}\ (a \neq b);$$

$$I = I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}\ (a \neq b) \tag{8}$$

with $L(a,a) = I(a,a) = a$.

Let

$$A_k = A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}\ (k \neq 0), \quad A_0 = G \tag{9}$$

be the power mean of order $k$, and put

$$Q = Q(a,b) = \sqrt{\frac{a^2+b^2}{2}} = A_2(a,b). \tag{10}$$

The Seiffert's means $P$ and $T$ are defined by

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)}, \tag{11}$$

$$T = T(a,b) = \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}, \tag{12}$$

while a new mean $M$ of Neuman and Sándor (see [6], [8]) is

$$M = M(a,b) = \frac{a-b}{2\text{arcsinh}\left(\frac{a-b}{a+b}\right)}. \tag{13}$$

For history and many properties of these means we quote e.g. [11], [13], [6], [8].

The aim of this paper is to show that inequalities of type (1)-(6) are in fact inequalities for the above stated means. More generally, we will point out other trigonometric or hyperbolic inequalities, as consequences of known inequalities for means.

## 2. Main results

**Theorem 1.** *For all* $0 < a \neq b$ *one has the inequalities*

$$\sqrt[3]{G^2 A} < L < \frac{2G + A}{3}; \tag{14}$$

$$L^2 A + LG^2 > 2AG^2 \tag{15}$$

*and*

$$2LA + LG > 3AG. \tag{16}$$

As a corollary, relations (4)-(6) are true.

**Proof.** The left side of (14) is a well-known inequality, due to Leach and Sholander (see e.g. [3]), while the right side of (14) is another famous inequality, due to Pólya-Stegö and Carlson (see [5], [2]).

For the proof of (15), apply the arithmetic mean - geometric mean inequality

$$u + v > 2\sqrt{uv}$$

($u \neq v > 0$) for $u = L^2 A$ and $v = LG^2$. By the left side of (14) we get

$$L^2 A + LG^2 > 2\sqrt{L^3 AG^2} > 2\sqrt{(G^2 A)(AG^2)} = 2AG^2,$$

and (15) follows.

Similarly, apply the inequality

$$u + u + v > 3\sqrt[3]{u^2 v}$$

($u \neq v > 0$) for $u = LA$, $v = LG$. Again, by the left side of (14) one has

$$2LA + LG > 3\sqrt[3]{(L^2 A^2)(LG)} > 3\sqrt[3]{(G^2 A)(A^2 G)} = 3AG,$$

and (16) follows.

Put now $a = e^x$, $b = e^{-x}$ ($x > 0$) in inequalities (14)-(16). As in this case one has

$$A = A(e^x, e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$G = G(e^x, e^{-x}) = 1$$

$$L = L(e^x, e^{-x}) = \frac{e^x - e^{-x}}{2x} = \frac{\sinh x}{x}$$

which are consequences of the definitions (7)-(8), from (14) we get the inequalities

$$\sqrt[3]{\cosh x} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3}. \tag{17}$$

The left side of (17) is called also as "Lazarević's inequality" (see [5]), while the right side is exactly the hyperbolic Cusa-Huygens inequality (6) (18).

By the same method, from (15) and (16) we get the hyperbolic Wilker inequality (4), and the hyperbolic Huygens inequality (5), respectively.

**Remark 1.** For any $a, b > 0$ one can find $x > 0$ and $k > 0$ such that $a = e^x k$, $b = e^{-x} k$. Indeed, for $k = \sqrt{ab}$ and $x = \frac{1}{2}\ln(a/b)$ this is satisfied. Since all the means $A, G, L$ are homogeneous of order one (i.e. e.g. $L(kt, kp) = kL(t, p)$) the set of inequalities (14)-(16) is in fact equivalent with the set of (4)-(6).

Thus, we could call (15) as the "Wilker inequality for the means $A, G, L$", while (16) as the "Huygens inequality for the means $A, G, L$", etc.

The following generalizations of (15) and (16) can be proved in the same manner:

**Theorem 2.** *For any $t > 0$ one has*

$$L^{2t}A^t + L^t G^{2t} > 2A^t G^{2t} \tag{18}$$

*and*

$$2L^t A^t + L^t G^t > 3A^t G^t. \tag{19}$$

These will imply the following generalizations of (4) and (5) (see [9])

$$\left(\frac{\sinh x}{x}\right)^{2t} + \left(\frac{\tanh x}{x}\right)^t > 2; \tag{20}$$

$$2\left(\frac{\sinh x}{x}\right)^t+\left(\frac{\tanh x}{x}\right)^t>3. \tag{21}$$

We now state the following Cusa-Wilker, Wilker and Huygens type inequalities for the means $P, A, G$:

**Theorem 3.** *For all* $0<a\neq b$ *one has the inequalities*

$$\sqrt[3]{A^2G}<P<\frac{2A+G}{3}; \tag{22}$$

$$P^2G+PA^2>2GA^2, \tag{23}$$

*and*

$$2PG+PA>3AG. \tag{24}$$

As a corollary, relations (1)-(3) are true.

**Proof.** Inequalities (22) are due to Sándor [11].

For the proof of (23) and (24) apply the same method as in the proof of Theorem 1, but using, instead of the left side of (14), the left side of (22). Now, for the proof of the second part of this theorem, put

$$a=1+\sin x,\quad b=1-\sin x\quad \left(x\in\left(0,\frac{\pi}{2}\right)\right)$$

in inequalities (22)-(24). As one has by (7) and (11)

$$A=A(1+\sin x,1-\sin x)=1$$

$$G=G(1+\sin x,1-\sin x)=\cos x$$

$$P=P(1+\sin x,1-\sin x)=\frac{\sin x}{x},$$

from (22) we can deduce relation (3), while (23), resp. (24) imply the classical Wilker resp. Huygens inequalities (1), (2).

**Remark 2.** Since for any $a,b>0$ one can find $x\in\left(0,\frac{\pi}{2}\right)$ and $k>0$ such that

$$a=(1+\sin x)k,\quad b=(1-\sin x)k.$$

[Indeed, $k = \dfrac{a+b}{2}$, $x = \arcsin \dfrac{a-b}{a+b}$], by the homogeneity of the means $P, A, G$ one can state that the set of inequalities (22)-(24) is equivalent with the set (1)-(3).

Thus, e.g. inequality (23) could be called as the "classical Wilker inequality for means".

**Remark 4.** Inequalities (14) and (22) can be improved "infinitely many times" by the sequential method discovered by Sándor in [16] and [11]. For generalization, see [6].

The extensions of type (18)-(19) and (20), (21) can be made here, too, but we omit further details.

We now state the corresponding inequalities for the means $T$, $A$ and $Q$ ((25), along with infinitely many improvements appear in [13]).

**Theorem 4.** *For all* $0 < a \neq b$ *one has the inequalities*

$$\sqrt[3]{Q^2 A} < T < \frac{2Q+A}{3}, \tag{25}$$

$$T^2 A + TQ^2 > 2AQ^2 \tag{26}$$

*and*

$$2TA + TQ > 3AQ. \tag{27}$$

The corresponding inequalities for the means $M$, $A$ and $Q$ will be the following:

**Theorem 5.** *For all* $0 < a \neq b$ *one has the following inequalities:*

$$\sqrt[3]{A^2 Q} < M < \frac{2A+Q}{3}; \tag{28}$$

$$M^2 Q + MA^2 > 2AQ^2; \tag{29}$$

$$2MQ + MA > 3QA. \tag{30}$$

**Proof.** Inequalities (28) can be found essentially in [6]. Relations (29) and (30) can be proved in the same manner as in the preceding theorems, by using in fact the left side of (28).

**Remark 5.** For an application, put $a = e^x$, $b = e^{-x}$ in (28). Since

$$A = \frac{e^x + e^{-x}}{2} = \cosh x, \quad Q = \sqrt{\frac{e^{2x} + e^{-2x}}{2}} = \sqrt{\cosh(2x)}$$

and

$$a - b = 2\sinh x, \quad \frac{a-b}{a+b} = \tanh x$$

and by

$$\operatorname{arcsinh}(t) = \ln(t + \sqrt{t^2 + 1}),$$

we get

$$M(a,b) = \frac{\sinh x}{\ln(\tanh x + \sqrt{1 + \tanh^2 x})}.$$

For example, the Wilker inequality (29) will become:

$$\frac{(\tanh x)^2}{\ln^2(\tanh x + \sqrt{1 + \tanh^2 x})} + \frac{\sinh x}{\sqrt{\cosh 2x}} \cdot \frac{1}{\ln(\tanh x + \sqrt{1 + \tanh^2 x})} > 2. \tag{31}$$

Since $\sqrt{\cosh 2x} = \sqrt{\cosh^2 x - \sinh^2 x} < \cosh x$, here we have

$$\frac{\sinh x}{\sqrt{\cosh x}} > \tanh x,$$

so formula (31) is a little "stronger" than e.g. the classical form (1).

Finally, we point out e.g. certain hyperbolic inequalities, which will be the consequences of the various existing inequalities between means of two arguments.

**Theorem 6.**

$$1 < \frac{\sinh t}{t} < e^{t \coth t - 1} < \cosh t \tag{32}$$

$$e^{(t \coth t - 1)/2} < \frac{\sinh t}{t} < \frac{\cosh t + 3\cosh t/3}{4} \tag{33}$$

$$\sqrt[3]{\cosh t} < e^{(t \coth t - 1)/2} < \frac{\sinh t}{t} < L(\cosh t, 1) < \left(\frac{\sqrt[3]{\cosh t} + 1}{2}\right)^3 \tag{34}$$

$$\sqrt[3]{\cosh t} < \frac{2\cosh t + 1}{3} < e^{(t\coth t - 1)/2} < \frac{\sinh t}{t}$$

$$< \frac{\cosh t + 2}{3} < \frac{2\cosh t + 1}{3} < e^{t\coth t - 1} \tag{35}$$

$$\frac{\cosh 2t + 3\cosh 2t/3}{4} < e^{2t\coth t - 2} < \frac{2\cosh^2 t + 1}{3} \tag{36}$$

$$\sqrt[3]{\cosh^2 t} < P(e^t, e^{-t}) < \frac{2\cosh t + 1}{3} \tag{37}$$

$$\frac{2}{e}\cosh t < e^{t\coth t - 1} < \frac{2}{e}(\cosh t + 1) \tag{38}$$

$$2\cosh^2 t - 1 < e^{2t\tanh t} \tag{39}$$

$$4\ln(\cos ht) > t\tanh t + 3t\coth t - 3. \tag{40}$$

**Proof.** For the identric mean of (8) one has $I(e^t, e^{-t}) = e^{t\coth t - 1}$, so for the proof of (32) apply the known inequalities (see the references in [10])

$$1 < L < I < A. \tag{41}$$

For the proof of (33) apply

$$\sqrt{G \cdot I} < L < A_{1/3}. \tag{42}$$

The left side of (42) is due to Alzer ([?]), while the right side to T.P. Lin ([4]). As

$$A_{1/3}(e^t, e^{-t}) = \left(\frac{e^{t/3} + e^{-t/3}}{2}\right)^3 = \frac{e^t + e^{-t} + 3(e^{2t/3} + e^{-2t/3})}{8},$$

(33) follows. For the proof of (34) apply

$$\sqrt[3]{A \cdot G^2} < \sqrt{I \cdot G} < L < L(A, G) \tag{43}$$

and $L(t, 1) < A_{1/3}(t, 1)$.

The first two inequalities of (43) are due to Sándor [15] and Alzer [1] respectively, while the last one to Neuman and Sándor [7]. Inequality (35) follows by

$$\sqrt[3]{A^2 \cdot G} < \sqrt{I \cdot G} < L < \frac{A+2G}{3} < I \tag{44}$$

and these can be found in [11].

For inequality (3) apply

$$A_{2/3}^2 < I^2 < \frac{2A^2 + G^2}{3}. \tag{45}$$

The left side of (45) is due to Stolarsky [21], while the right side of Sándor-Trif [12].

Inequality (37) follows by (22), while (38) by

$$\frac{2}{e}A < I < \frac{2}{e}(A+G), \tag{46}$$

see Neuman-Sándor [7].

Finally, for inequalities (39) and (40) we will use the mean $S$ defined by $S(a,b) = (a^a \cdot b^b)^{1/(a+b)}$, and remarking that $S(e^t, e^{-t}) = e^{\tanh t}$, apply the following inequalities:

$$2A^2 - G^2 < S^2 \tag{47}$$

(see Sándor-Raşa [17]), while for the proof of (40) apply the inequality

$$S < A^4/I^3 \tag{48}$$

due to Sándor [14].

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## 6.10 On Huygens' inequalities

### 1. Introduction

The famous Huygens' trigonometric inequality (see e.g. [3], [14], [7]) states that for all $x \in (0, \pi/2)$ one has

$$2\sin x + \tan x > 3x. \tag{1.1}$$

The hyperbolic version of inequality (1.1) has been established recently by E. Neuman and J. Sándor [7]:

$$2\sinh x + \tanh x > 3x, \text{ for } x > 0. \tag{1.2}$$

Let $a, b > 0$ be two positive real numbers. The logarithmic and identric means of $a$ and $b$ are defined by

$$\begin{aligned} L = L(a,b) &:= \frac{b-a}{\ln b - \ln a} fora \neq b;\ L(a,a) = a, \\ I = I(a,b) &:= \frac{1}{e}(b^b/a^a)^{1/(b-a)}\ (a \neq b);\ I(a,a) = a, \end{aligned} \tag{1.3}$$

respectively. Seiffert's mean $P$ is defined by

$$P = P(a,b) := \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)}\ (a \neq b), \quad P(a,a) = a. \tag{1.4}$$

Let

$$A = A(a,b) := \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab},$$
$$H = H(a,b) = 2/(1/a + 1/b)$$

denote the arithmetic, geometric and harmonic means of $a$ and $b$, respectively. These means have been also in the focus of many research papers in the last decades. For a survey of results, see e.g. [8], [10], [12]. In what follows, we shall assume $a \neq b$.

Now, by remarking that letting $a = 1 + \sin x$, $b = 1 - \sin x$, where $x \in (0, \pi/2)$, in $P, G, A$ we find that

$$P = \frac{\sin x}{x}, \quad G = \cos x, \quad A = 1, \tag{1.5}$$

so Huygens' inequality (1.1) may be written also as

$$P > \frac{3AG}{2G + A} = 3/\left(\frac{2}{A} + \frac{1}{G}\right) = H(A, A, G). \tag{1.6}$$

Here $H(a, b, c)$ denotes the harmonic mean of the numbers $a, b, c$:

$$H(a, b, c) = 3/\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

On the other hand, by letting $a = e^x$, $b = e^{-x}$ in $L, G, A$ we find that

$$L = \frac{\sinh x}{x}, \quad G = 1, \quad A = \cosh x, \tag{1.7}$$

so Huygens' hyperbolic inequality (1.2) may be written also as

$$L > \frac{3AG}{2A + G} = 3/\left(\frac{2}{G} + \frac{1}{A}\right) = H(G, G, A). \tag{1.8}$$

## 2. First improvements

Suppose $a, b > 0$, $a \neq b$.

**Theorem 2.1.** *One has*

$$P > H(L, A) > \frac{3AG}{2G + A} = H(A, A, G) \tag{2.1}$$

*and*

$$L > H(P, G) > \frac{3AG}{2A + G} = H(G, G, A). \tag{2.2}$$

**Proof.** The inequalities $P > H(L, A)$ and $L > H(P, G)$ have been proved in paper [5] (see Corollary 3.2). In fact, stronger relations are valid, as we will see in what follows.

Now, the interesting fact is that the second inequality of (2.1), i.e.

$$\frac{2LA}{L+A} > \frac{3AG}{2G+A}$$

becomes, after elementary transformations, exactly inequality (1.8), while the second inequality of (2.2), i.e.

$$\frac{2PG}{P+G} > \frac{3AG}{2A+G}$$

becomes inequality (1.6).

Another improvements of (1.6), resp. (1.8) are provided by

**Theorem 2.2.** *One has the inequalities*

$$P > \sqrt[3]{A^2G} > \frac{3AG}{2G+A} \tag{2.3}$$

*and*

$$L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{2.4}$$

**Proof.** The first inequality of (2.3) is proved in [12], while the first inequality of (2.8) is a well known inequality due to Leach and Sholander [4] (see [8] for many related references). The second inequalities of (2.3) and (2.4) are immediate consequences of the arithmetic-geometric inequality applied for A, A, G and A, G, G, respectively.

**Remark 2.1.** By (2.3) and (1.5) we can deduce the following improvement of the Huygens' inequality (1.1):

$$\frac{\sin x}{x} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1},\ x \in (0, \pi/2). \tag{2.5}$$

From (2.1) and (1.5) we get

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \frac{3\cos x}{2\cos x + 1},\ x \in (0, \pi/2) \tag*{(2.5)$'$}$$

Similarly, by (2.4) and (1.7) we get

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x + 1},\ x > 0. \tag{2.6}$$

From (2.2) and (1.7) we get

$$\frac{\sinh x}{x} > \frac{2P^*}{P^*+1} > \frac{3\cosh x}{2\cosh x + 1}, \; x > 0. \tag{2.6$'$}$$

Here $L^* = L(1+\sin x, 1-\sin x)$, $P^* = P(e^x, e^{-x})$.

We note that the first inequality of (2.5) has been discovered by Adamović and Mitrinović (see [7]), while the first inequality of (2.6) by Lazarević (see [7]).

Now, we will prove that inequalities (2.2) of Theorem 2.1 and (2.4) of Theorem 2.2 may be compared in the following way:

**Theorem 2.3.** *One has*

$$L > \sqrt[3]{G^2 A} > H(P,G) > \frac{3AG}{2A+G}. \tag{2.7}$$

**Proof.** We must prove the second inequality of (2.7). For this purpose, we will use the inequality (see [12])

$$P < \frac{2A+G}{3}. \tag{2.8}$$

This implies

$$\frac{G}{P} > \frac{3G}{G+2A}, \quad \text{so} \quad \frac{1}{2}\left(1 + \frac{G}{P}\right) > \frac{2G+A}{G+2A}.$$

Now, we shall prove that

$$\frac{2G+A}{G+2A} > \sqrt[3]{\frac{G}{A}}. \tag{2.9}$$

By letting $x = \dfrac{G}{A} \in (0,1)$ inequality (2.9) becomes

$$\frac{2x+1}{x+2} > \sqrt[3]{x}. \tag{2.10}$$

Put $x = a^3$, where $a \in (0,1)$. After elementary transformations, inequality (2.10) becomes $(a+1)(a-1)^3 < 0$, which is true.

**Note.** The Referee suggested the following alternative proof: Since $P < (2A+G)/3$ and the harmonic mean increases in both variables, it suffices to prove stronger inequality $\sqrt[3]{A^2G} > H((2A+G)/3, G)$ which can be written as (2.9).

**Remark 2.2.** The following refinement of inequalities $(2.6)'$ is true:

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{2P^*}{P^*+1} > \frac{3\cosh x}{2\cosh x + 1}, \ x > 0. \tag{2.11}$$

Unfortunately, a similar refinement to (2.7) for the mean $P$ is not possible, as by numerical examples one can deduce that generally $H(L,A)$ and $\sqrt[3]{A^2G}$ are not comparable. However, in a particular case, the following result holds true:

**Theorem 2.4.** *Assume that $A/G \geq 4$. Then one has*

$$P > H(L,A) > \sqrt[3]{A^2G} > \frac{3AG}{2G+A}. \tag{2.12}$$

First we prove the following auxiliary result:

**Lemma 2.1.** *For any $x \geq 4$ one has*

$$\sqrt[3]{(x+1)^2}(2\sqrt[3]{x}-1) > x\sqrt[3]{4}. \tag{2.13}$$

**Proof.** A computer computation shows that (2.13) is true for $x = 4$. Now put $x = a^3$ in (2.13). By taking logarithms, the inequality becomes

$$f(a) = 2\ln\left(\frac{a^3+1}{2}\right) - 9\ln a + 3\ln(2a-1) > 0.$$

An easy computation implies

$$a(2a-1)(a^3+1)f'(a) = 3(a-1)(a^2+a-3).$$

As

$$\sqrt[3]{4^2} + \sqrt[3]{4} - 3 = 2\sqrt[3]{2} + (\sqrt[3]{2})^2 - 3 = (\sqrt[3]{2}-1)(\sqrt[3]{2}+3) > 0,$$

we get that $f'(a) > 0$ for $a \geq \sqrt[3]{4}$. This means that $f(a) > f(\sqrt[3]{4}) > 0$, as the inequality is true for $a = \sqrt[3]{4}$.

**Proof of the theorem.** We shall apply the inequality

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2}, \tag{2.14}$$

due to the author [11]. This implies

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \frac{1}{2}\left(1+\sqrt[3]{\frac{4A^3}{G(A+G)^2}}\right) = N.$$

By letting $x = \dfrac{A}{G}$ in (2.13) we can deduce

$$N < \sqrt[3]{\frac{A}{G}}$$

so

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \sqrt[3]{\frac{A}{G}}.$$

This immediately gives $H(L, A) > \sqrt[3]{A^2G}$.

**Remark 2.3.** If $\cos x \le \dfrac{1}{4}$, $x \in \left(0, \dfrac{\pi}{2}\right)$ then

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1}, \tag{2.15}$$

which is a refinement, in this case, of inequality $(2.5)'$.

## 3. Further improvements

**Theorem 3.1.** *One has*

$$P > \sqrt{LA} > \sqrt[3]{A^2G} > \frac{AG}{L} > \frac{3AG}{2G+A} \tag{3.1}$$

*and*

$$L > \sqrt{GP} > \sqrt[3]{G^2A} > \frac{AG}{P} > \frac{3AG}{2A+G}. \tag{3.2}$$

**Proof.** The inequalities $P > \sqrt{LA}$ and $L > \sqrt{GP}$ are proved in [6]. We shall see, that further refinements of these inequalities are true. Now, the second inequality of (3.1) follows by the first inequality of (2.3), while the second inequality of (3.2) follows by the first inequality of (2.4). The last inequality is in fact an inequality by Carlson [2]. For the inequalities on $\frac{AG}{P}$ we use (2.3) and (2.8).

**Remark 3.1.** One has

$$\frac{\sin x}{x} > \sqrt{L^*} > \sqrt[3]{\cos x} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x + 1},\ x \in (0, \pi/2) \qquad (3.3)$$

and

$$\frac{\sinh x}{x} > \sqrt{P^*} > \sqrt[3]{\cosh x} > \frac{\cosh x}{P^*} > \frac{3\cosh x}{2\cosh x + 1},\ x > 0 \qquad (3.4)$$

where $L^*$ and $P^*$ are the same as in $(2.5)'$ and $(2.6)'$.

**Theorem 3.2.** *One has*

$$P > \sqrt{LA} > H(A, L) > \frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G + A} \qquad (3.5)$$

*and*

$$L > L \cdot \frac{I - G}{A - L} > \sqrt{IG} > \sqrt{PG} > \sqrt[3]{G^2 A} > \frac{3AG}{2A + G}. \qquad (3.6)$$

**Proof.** The first two inequalities of (3.5) follow by the first inequality of (3.1) and the fact that $G(x, y) > H(x, y)$ with $x = L$, $y = A$.

Now the inequality $H(A, L) > \frac{AL}{I}$ may be written also as

$$I > \frac{A + L}{2},$$

which has been proved in [8] (see also [9]).

Further, by Alzer's inequality $L^2 > GI$ (see [1]) one has

$$\frac{L}{I} > \frac{G}{L}$$

and by the inequality $L < \frac{2G + A}{3}$ (see [2]) we get

$$\frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G + A},$$

so (3.5) is proved.

The first two inequalities of (3.6) have been proved by the author in [10]. Since $I > P$ (see [16]) and by (3.2), inequalities (3.6) are completely proved.

**Remark 3.2.**

$$\frac{\sin x}{x} > \sqrt{L^*} > \frac{2L^*}{L^*+1} > \frac{L^*}{I^*} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x+1},\ x \in \left(0, \frac{\pi}{2}\right), \quad (3.7)$$

where $I^* = I(1+\sin x, 1-\sin x)$;

$$\frac{\sinh x}{x} > \frac{\sinh x}{x}\left(\frac{e^{x\coth x-1}-1}{\cosh x - \sinh x/x}\right) > e^{(x\coth x-1)/2} > \sqrt{P^*}$$

$$> \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x+1}. \quad (3.8)$$

**Theorem 3.3.** *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \sqrt{A\left(\frac{A+2G}{3}\right)} > \sqrt{AL}$$

$$> H(A,L) > \frac{AL}{I} > \frac{3AG}{2G+A} \quad (3.9)$$

*and*

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \sqrt{IG} > \sqrt{G\left(\frac{2A+G}{3}\right)}$$

$$> \sqrt{PG} > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \quad (3.10)$$

**Proof.** In (3.9) we have to prove the first three inequalities, the rest are contained in (3.5).

The first inequality of (3.9) is proved in [12]. For the second inequality, put $A/G = t > 1$. By taking logarithms, we have to prove that

$$g(t) = 4\ln\left(\frac{t+1}{2}\right) - 3\ln\left(\frac{t+2}{3}\right) - \ln t > 0.$$

As

$$g'(t)t(t+1)(t+2) = 2(t-1) > 0,$$

$g(t)$ is strictly increasing, so

$$g(t) > g(1) = 0.$$

The third inequality of (3.9) follows by Carlson's relation

$$L < \frac{2G+A}{3}.$$

The first inequality of (3.10) is proved in [11], while the second one in [15]. The third inequality follows by $I > \dfrac{2A+G}{3}$ (see [9]), while the fourth one by relation (2.9). The fifth one follows by (2.3).

**Remark 3.3.** The first three inequalities of (3.9) offer a strong improvement of the first inequality of (3.1); and the same is true for (3.10) and (3.2).

## 4. New Huygens type inequalities

The main result of this section is contained in

**Theorem 4.1.** *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \frac{3A(A+G)}{5A+G} > \frac{A(2G+A)}{2A+G} > \frac{3AG}{2G+A} \tag{4.1}$$

*and*

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \frac{3G(A+G)}{5G+A} > \frac{G(2A+G)}{2G+A} > \frac{3AG}{2A+G}. \tag{4.2}$$

**Proof.** The first inequalities of (4.1), resp. (4.2) are the first ones in relations (3.9) resp. (3.10).

Now apply the geometric mean - harmonic mean inequality

$$\sqrt[3]{xy^2} = \sqrt[3]{x \cdot y \cdot y} > 3/\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}\right) = 3/\left(\frac{1}{x}+\frac{2}{y}\right)$$

for $x = A$, $y = \dfrac{A+G}{2}$ in order to deduce the second inequality of (4.1). The last two inequalities become, after certain transformation,

$$(A-G)^2 > 0.$$

The proof of (4.2) follows on the same lines, and we omit the details.

**Theorem 4.2.** *For all $x \in (0, \pi/2)$ one has*

$$\sin x + 4\tan\frac{x}{2} > 3x. \tag{4.3}$$

*For all $x > 0$ one has*

$$\sinh x + 4\tanh\frac{x}{2} > 3x. \tag{4.4}$$

**Proof.** Apply (1.5) for $P > \dfrac{3A(A+G)}{5A+G}$ of (4.1).

As

$$\cos x + 1 = 2\cos^2\frac{x}{2} \quad \text{and} \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},$$

we get inequality (4.3). A similar argument applied to (4.4), by an application of (4.2) and the formulae

$$\cosh x + 1 = 2\cosh^2\frac{x}{2} \quad \text{and} \quad \sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2}.$$

**Remarks 4.1.** By (4.1), inequality (4.3) is a refinement of the classical Huygens inequality (1.1):

$$2\sin x + \tan x > \sin x + 4\tan\frac{x}{2} > 3x. \tag*{(4.3)$'$}$$

Similarly, (4.4) is a refinement of the hyperbolic Huygens inequality (1.2):

$$2\sinh x + \tanh x > \sinh x + 4\tanh\frac{x}{2} > 3x. \tag*{(4.4)$'$}$$

We will call (4.3) as the **second Huygens inequality**, while (4.4) as the **second hyperbolic Huygens inequality**.

In fact, by (4.1) and (4.2) refinements of these inequalities may be stated, too.

The inequality $P > \dfrac{A(2G+A)}{2A+G}$ gives

$$\frac{\sin x}{x} > \frac{2\cos x + 1}{\cos x + 2},$$

or written equivalently:

$$\frac{\sin x}{x} + \frac{3}{\cos x + 2} > 2, \ x \in \left(0, \frac{\pi}{2}\right). \tag{4.5}$$

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## 6.11 New refinements of the Huygens and Wilker hyperbolic and trigonometric inequalities

### 1. Introduction

The famous Huygens, resp. Wilker trigonometric inequalities can be stated as follows: For any $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$2\sin x + \tan x > 3x, \tag{1.1}$$

resp.

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.2}$$

Their hyperbolic variants are: For any $x > 0$ hold

$$2\sinh x + \tanh x > 3x, \tag{1.3}$$

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{1.4}$$

Clearly, (1.4) hold also for $x > 0$, so it holds for any $x \neq 0$. In what follows, we shall assume in all inequalities $x > 0$ (or $0 < x < \frac{\pi}{2}$ in trigonometric inequalities).

For references, connections, extensions and history of these inequalities we quote the recent paper [5].

In what follows, by using the theory of means of two arguments, we will offer refinements of (1.1) or (1.3), as well as (1.2) or (1.4).

### 2. Means of two arguments

Let $a$, $b$ be positive real numbers. The logarithmic mean and the identric mean of a and b are defined by

$$L = L(a,b) = \frac{a-b}{\ln a - \ln b}, \text{ for } a \neq b, \quad L(a,a) = a \tag{2.1}$$

and

$$I = I(a,b) = \frac{1}{e}(a^a/b^b)^{1/(a-b)}, \text{ for } a \neq b, \quad I(a,a) = a. \tag{2.2}$$

The Seiffert mean $P$ is defined by

$$P = P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)}, \text{ for } a \neq b, \quad P(a,a) = a. \tag{2.3}$$

Let

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab}, \quad H = H(a,b) = \frac{2ab}{a+b}$$

denote the classical arithmetic, geometric, and harmonic means of $a$ and $b$.

There exist many inequalities related to these means. We quote e.g. [1] for $L$ and $I$, while [2] for the mean $P$. Recently, E. Neuman and the author [3] have shown that all these means are the particular cases of the so-called "Schwab-Borchardt mean" (see also [4]).

In what follows, we will use two inequalities which appear in [3] (see Corollary 3.2 of that paper), namely:

$$L > H(P,G), \tag{2.4}$$

and

$$P > H(L,A). \tag{2.5}$$

Our method will be based on the remark that

$$\frac{\sinh x}{x} = L(e^x, e^{-x}), \quad x \neq 0 \tag{2.6}$$

which follows by (2.1), as well as

$$\frac{\sin x}{x} = P(1+\cos x, 1-\cos x), \quad x \in \left(0, \frac{\pi}{2}\right) \tag{2.7}$$

which may be obtained by definition (2.3).

Note also that

$$G(e^x, e^{-x}) = 1, \quad A(e^x, e^{-x}) = \cosh x \tag{2.8}$$

and

$$G(1+\sin x, 1-\sin x) = \cos x, \quad A(1+\sin x, 1-\sin x) = 1. \tag{2.9}$$

## 3. Main result

**Theorem 3.1.** *Define*

$$P^* = P(e^x, e^{-x}), \quad X^* = \frac{2\sinh x}{P^*}.$$

*Then for any $x > 0$ one has*

$$\tanh x + x > X^* > 4x - 2\sinh x \tag{3.1}$$

*and*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > k^2 - 2k + 3, \tag{3.2}$$

*where $k = \dfrac{X^*}{2x}$.*

**Proof.** Writing inequality (2.4) for $L = L(e^x, e^{-x})$, etc., and using (2.6) and (2.8), we get:

$$\frac{\sinh x}{x} > \frac{2P^*}{P^* + 1}. \tag{3.3}$$

Similarly, for (2.5), we get:

$$P^* > \frac{2\sinh x \cosh x}{\sinh x + x\cosh x}. \tag{3.4}$$

By (3.4) and (3.3) we can write that

$$\tanh x + x = \frac{\sinh x + x\cosh x}{\cosh x} > \frac{2\sinh x}{P^*} > 4x - 2\sinh x,$$

so (3.1) follows.

Now,

$$\frac{X^*}{x} = 2 \cdot \frac{\sinh x}{x \cdot P^*} = 2 \cdot \frac{L}{P}(e^x, e^{-x}) < 2$$

by the known inequality (see e.g. [2])

$$L < P. \tag{3.5}$$

By the right side of (3.1) one has

$$\frac{\sinh x}{x} > 2 - \frac{X^*}{2x} = 2 - k > 0. \tag{3.6}$$

From the left side of (3.1) we get

$$\frac{\tanh x}{x} > \frac{X^*}{x} - 1 = 2k - 1. \tag{3.7}$$

Thus, by (3.6) and (3.7) we can write

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > (2-k)^2 + 2k - 1 = k^2 - 2k + 3 > 2$$

by $(k-1)^2 > 0$. In fact, $k = \dfrac{L}{P}(e^x, e^{-x}) < 1$, so there is strict inequality also in the last inequality.

In case of trigonometric functions, the Huygens inequality is refined in the same manner, but in case of Wilker's inequality the things are slightly distinct. □

**Theorem 3.2.** *Define*

$$L^* = L(1+\sin x, 1-\sin x) \quad \textit{and} \quad X^{**} = \frac{2\sin x}{L^*}$$

*for* $x \in \left(0, \dfrac{\pi}{2}\right)$.

*Then one has the inequalities*

$$\tan x + x > X^{**} > 4x - 2\sin x \tag{3.8}$$

*and*

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>\begin{cases}(k^*)^2-2k^*+3, & \textit{if} \quad k^*\le 2\\ 2k^*-1, & \textit{if} \quad k^*>2\end{cases}>2 \tag{3.9}$$

*where* $k^*=\dfrac{X^{**}}{2x}$.

**Proof.** Applying inequality (2.4) for $L=L(1+\sin x,1-\sin x)=L^*$, by (2.7) and (2.9) we get

$$L^*>\frac{2\sin x\cos x}{\sin x+x\cos x}. \tag{3.10}$$

From (2.5) we get

$$\frac{\sin x}{x}>\frac{2L^*}{L^*+1}. \tag{3.11}$$

Thus,

$$\tan x+x=\frac{\sin x+\cos x}{\cos x}>\frac{2\sin x}{L^*}>4x-2\sin x$$

by (3.10) and (3.11) respectively. This gives relation (3.8), which refines (1.1).

Now, by the right side of (3.8) we can write

$$\frac{\sin x}{x}>2-\frac{\sin x}{x\cdot L^*}=2-k^*.$$

Since

$$k^*=\frac{P}{L}(1+\sin x,1-\sin x)>1,$$

and the upper bound of $k^*$ is $+\infty$ as $x\to\dfrac{\pi}{2}$, clearly $2-k^*>0$ is not true.

i) If $2-k^*\ge 0$, then as in the proof of Theorem 3.1, we get from (3.8)

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>(2-k^*)^2+2k^*-1=(k^*)^2-2k^*+3>2.$$

ii) If $2-k^*<0$, then we use only $\left(\dfrac{\sin x}{x}\right)^2>0$. Thus

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>\frac{\tan x}{x}>2k^*-1.$$

In this case $2k^* - 1 > 3 > 2$.

Thus, in any case, inequality (3.9) holds true, so a refinement of the trigonometric Wilker inequality (1.2) is valid. $\square$

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## 6.12 On two new means of two variables

### 1. Introduction

Let $a$, $b$ be two positive numbers. The logarithmic and identric means of $a$ and $b$ b are defined by

$$\begin{aligned} L = L(a,b) &= \frac{a-b}{\ln a - \ln b}\ (a \neq b), \quad L(a,a) = a; \\ I = I(a,b) &= \frac{1}{e}(b^b/a^a)^{1/(b-a)}\ (a \neq b), \quad I(a,a) = a. \end{aligned} \tag{1.1}$$

The Seiffert mean $P$ is defined by

$$P = P(a,b) = \frac{b-a}{2\arcsin\dfrac{b-a}{a+b}}\ (a \neq b), \quad P(a,a) = a. \tag{1.2}$$

Let

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab} \quad \text{and} \quad H = H(a,b) = \frac{2ab}{a+b}$$

denote the classical arithmetic, geometric, resp. harmonic means of $a$ and $b$. There exist many papers which study properties of these means. We quote e.g. [1], [2] for the identric and logarithmic means, and [3] for the mean $P$.

The means $L$, $I$ and $P$ are particular cases of the "Schwab-Borchardt mean", see [4], [5] for details. The means of two arguments have important applications also in number- theoretical problems. For example, the solution of certain conjectures on prime numbers in paper [11] is based on the logarithmic mean $L$.

The aim of this paper is the study of two new means, which we shall denote by $X = X(a,b)$ and $Y = Y(a,b)$, defined as follows:

$$X = A \cdot e^{\frac{G}{P}-1}, \tag{1.3}$$

resp.

$$Y = G \cdot e^{\frac{L}{A}-1}. \tag{1.4}$$

Clearly $X(a,a) = Y(a,a) = a$, but we will be mainly interested for properties of these means for $a \neq b$.

## 2. Main results

**Lemma 2.1.** *The function $f(t) = te^{\frac{1}{t}-1}$, $t > 1$ is strictly increasing. For all $t > 0$, $t \neq 0$ one has $f(t) > 1$. For $0 < t < 1$, $f$ is strictly decreasing. As a corollary, for all $t > 0$, $t \neq 1$ one has*

$$1 - \frac{1}{t} < \ln t < t - 1. \tag{2.1}$$

**Proof.** As $\ln f(t) = \ln t + \frac{1}{t} - 1$, we get

$$\frac{f'(t)}{f(t)} = \frac{t-1}{t^2},$$

so $t_0 = 1$ will be a minimum point of $f(t)$, implying $f(t) \geq f(1) = 1$, for any $t > 0$, with equality only for $t = 1$. By taking logarithm, the left side of (2.1) follows. Putting $1/t$ in place of $t$, the left side of (2.1) implies the right side inequality. $\square$

**Theorem 2.1.** *For $a \neq b$ one has*

$$G < \frac{A \cdot G}{P} < X < \frac{A \cdot P}{2P - G} < P. \tag{2.2}$$

**Proof.** Applying (2.1) for $t = \dfrac{X}{A}$ ($\neq 1$, as $G \neq P$ for $a \neq b$), and by taking into account of (1.3) we get the middle inequalities of (2.2). As it is well known that (see [3])

$$\frac{A+G}{2} < P < A, \tag{2.3}$$

the first inequality of (2.3) implies the last one of (2.2), while the second inequality of (2.3) implies the first one of (2.2).

In a similar manner, the following is true:

**Theorem 2.2.** *For $a \neq b$ one has*

$$H < \frac{L \cdot G}{A} < Y < \frac{G \cdot A}{2A - L} < G. \tag{2.4}$$

**Proof.** Applying (2.1) for $t = \dfrac{Y}{G}$ by (1.4) we can deduce the second and third inequalities of (2.4). Since $H = \dfrac{G^2}{A}$, the first and last inequality of (2.4) follows by the known inequalities (see e.g. [1] for references)

$$G < L < A. \tag{2.5}$$

□

The second inequality of (2.2) can be strongly improved, as follows:

**Theorem 2.3.** *For $a \neq b$ one has*

$$1 < \frac{L^2}{G \cdot I} < \frac{L}{G} \cdot e^{\frac{G}{L}-1} < \frac{X \cdot P}{A \cdot G}. \tag{2.6}$$

**Proof.** As $L < P$ (due to Seiffert; see [3] for references) and

$$f\left(\frac{P}{G}\right) = X \cdot \frac{P}{A \cdot G}, \tag{2.7}$$

where $f$ is defined in Lemma 2.1, and by taking into account of the inequality (see [1])

$$\frac{L}{I} < e^{\frac{G}{L}-1}, \tag{2.8}$$

by the monotonicity of $f$ one has

$$f\left(\frac{P}{G}\right) = \frac{L}{G} \cdot e^{\frac{G}{L}-1} > \frac{L}{G} \cdot \frac{L}{I} = \frac{L^2}{G \cdot I}. \tag{2.9}$$

By an inequality of Alzer (see [1] for references) one has

$$L^2 > G \cdot I, \tag{2.10}$$

thus all inequalities of (2.6) are established.

The following estimates improve the left side of (2.4):

**Theorem 2.4.** *For $a \neq b$,*

$$H < \frac{G^2}{I} < \frac{L \cdot G}{A} < \frac{G \cdot (A + L)}{3A - L} < Y. \tag{2.11}$$

**Proof.** Since $H = \dfrac{G^2}{A}$, the first inequality of (2.11) follows by the known inequality $I < A$ (see [1] for references). The second inequality of (2.11) follows by another known result of Alzer (see [1] for references, and [3] for improvements)

$$A \cdot G < L \cdot I. \tag{2.12}$$

Finally, to prove the last inequality of $Y$, remark that the logarithmic mean of $Y$ and $G$ is

$$L(Y, G) = \frac{Y - G}{\ln Y/G} = \frac{(G - Y)A}{A - L}. \tag{2.13}$$

Now, by the right side of (2.5) applied to $a = Y$, $b = G$ we have

$$L(Y, G) < (Y + G)/2,$$

so

$$2A(G - Y) < (A - L)(Y + G),$$

which after some transformations gives the desired inequality. □

Similarly to (2.11) we can state:

**Theorem 2.5.**

$$\frac{A \cdot G}{P} < \frac{A(P + G)}{3P - G} < X. \tag{2.14}$$

**Proof.** $L(X, A) = (X - A) \log X/A = \dfrac{(A - X)P}{P - G} < \dfrac{X + A}{2}$, so after simple computations we get the second inequality of (2.14). The first inequality becomes

$$(P - G)^2 > 0.$$ □

A connection between the two means $X$ and $Y$ is provided by:

**Theorem 2.6.** *For $a \neq b$,*

$$A^2 \cdot Y < P \cdot L \cdot X. \tag{2.15}$$

**Proof.** By using the inequality (see [3])

$$\frac{A}{L} < \frac{P}{G}, \tag{2.16}$$

and remarking that

$$f\left(\frac{A}{L}\right) = \frac{A}{L \cdot G} \cdot Y, \tag{2.17}$$

by the monotonicity of $f$ one has

$$f\left(\frac{A}{L}\right) < f\left(\frac{P}{G}\right),$$

so by (2.7) and (2.17) we can deduce inequality (2.15). □

**Remark 2.1.** By the known identity (see [1], [2])

$$\frac{I}{G} = e^{\frac{A}{L}-1} \tag{2.18}$$

and the above methods one can deduce the following inequalities (for $a \neq b$):

$$1 < \frac{L \cdot I}{A \cdot G} < \frac{G}{P} \cdot e^{\frac{P}{G}-1}. \tag{2.19}$$

Indeed, as

$$f\left(\frac{L}{A}\right) = \frac{L}{A} \cdot e^{\frac{A}{L}-1} = \frac{L \cdot I}{A \cdot G} > 1$$

we reobtain inequality (2.12). On the other hand, by (2.16) we can write, as $1 > \frac{L}{A} > \frac{G}{P}$ that $f\left(\frac{L}{A}\right) < f\left(\frac{G}{P}\right)$ i.e. the complete inequality (2.19) is established.

**Theorem 2.7.** *For $a \neq b$*

$$X < A\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{G}{P}\right] \tag{2.20}$$

*and*

$$Y < G\left[\frac{1}{e} + \left(1 - \frac{1}{e}\right)\frac{L}{A}\right]. \tag{2.21}$$

**Proof.** The following auxiliary result will be used:

**Lemma 2.2.** *For the function $f$ of Lemma 2.1, for any $t > 1$ one has*

$$f(t) < \frac{1}{e}(t + e - 1) \tag{2.22}$$

*and*

$$f(t) < \frac{1}{e}\left(t + \frac{1}{2t} + e - \frac{3}{2}\right) < \frac{1}{e}(t + e - 1). \tag{2.23}$$

**Proof.** By the series expansion of $e^x$ and by $t > 1$, we have

$$\begin{aligned} f(t) &= \frac{1}{e}\left(t + 1 + \frac{1}{2t} + \frac{1}{3!t^2} + \frac{1}{4!t^3} + \dots\right) \\ &= \frac{1}{e}\left(t + \frac{1}{1!} + \frac{1}{2!} + \dots\right) = \frac{1}{e}(t + e - 1), \end{aligned}$$

so (2.22) follows.

Similarly,

$$f(t) = \frac{1}{e}\left(t + 1 + \frac{1}{2t} + \frac{1}{3!t^2} + \frac{1}{4!t^3} + \dots\right)$$

$$< \frac{1}{e}\left[t + 1 + \frac{1}{2t} + e - \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)\right] = \frac{1}{e}\left(t + \frac{1}{2t} + e - \frac{3}{2}\right),$$

so (2.23) follows as well.

Now, (2.20) follows by (2.22) and (2.7), while (2.21) by (2.23) and (2.17). □

**Theorem 2.8.** *For $a \neq b$ one has*

$$P^2 > A \cdot X \tag{2.24}$$

**Proof.** Let $x \in \left(0, \frac{\pi}{2}\right)$. In the recent paper [7] we have proved the following trigonometric inequality:

$$\ln\frac{x}{\sin x} < \frac{\sin x - \cos x}{2\sin x}. \tag{2.25}$$

Remark that by (1.2) one has

$$P(1+\sin x, 1-\sin x) = \frac{\sin x}{x},$$

$$A(1+\sin x, 1-\sin x) = 1, \quad G(1+\sin x, 1-\sin x) = \cos x,$$

so (2.24) may be rewritten also as

$$P^2(1+\sin x, 1-\sin x) > A(1+\sin x, 1-\sin x) \cdot X(1+\sin x, 1-\sin x). \tag{2.26}$$

For any $a, b > 0$ one can find $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 0$ such that

$$a = (1+\sin x)k, \quad b = (1-\sin x)k.$$

Indeed, let $k = \frac{a+b}{2}$ and $x = \arcsin\frac{a-b}{a+b}$.

Since the means $P$, $A$ and $X$ are homogeneous of order one, by multiplying (2.26) by $k$, we get the general inequality (2.24). □

**Corollary 2.1.**

$$P^3 > \frac{A^2L^2}{I} > A^2G. \tag{2.27}$$

**Proof.** By (2.6) of Theorem 2.3 and (2.24) one has

$$\frac{L^2A}{I \cdot P} < X < \frac{P^2}{A}, \tag{2.28}$$

so we get $P^3 > \frac{A^2L^2}{I} > A^2G$ by inequality (2.10). □

**Remark 2.2.** Inequality (2.27) offers an improvement of

$$P^3 > A^2G \tag{2.29}$$

from paper [3]. We note that further improvements, in terms of $A$ and $G$ can be deduced by the "sequential method" of [3]. For any application of (2.29), put $a = 1+\sin x$, $b = 1-\sin x$ in (2.29) to deduce

$$\frac{\sin x}{x} > \sqrt[3]{\cos x}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{2.30}$$

which is called also the Mitrinović-Adamović inequality (see [6]).

Since (see [3])

$$P < \frac{2A+G}{3}, \tag{2.31}$$

by the above method we can deduce

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}, \tag{2.32}$$

called also as the Cusa-Huygens inequality. For details on such trigonometric or related hyperbolic inequalities, see [6].

**Theorem 2.9.** *One has the inequalities*

$$P^2 > \sqrt[3]{A^2\left(\frac{A+G}{2}\right)^4} > A \cdot X, \tag{2.33}$$

**Proof.** The first inequality of (2.33) appeared in our paper [3]. For the second inequality, consider the application

$$f(x) = \ln\frac{\cos x + 1}{2} - \frac{3}{4}(x\cot x - 1).$$

An easy computation gives

$$\begin{aligned} 4(\sin x)^2(\cos x + 1)f'(x) &= 3x + 3x\cos x - (\sin x)^3 \\ &\quad - 3\sin x - 3\sin x\cos x = g(x). \end{aligned}$$

Here

$$g'(x) = 3\sin x \cdot h(x),$$

where

$$h(x) = 2\sin x - x - \sin x\cos x.$$

One has

$$h'(x) = 2(\cos x)(1 - \cos x) > 0,$$

so $h(x) > h(0) = 0$. This in turn gives $g(x) > g(0)$ implying $f'(x) > 0$. Therefore, $f(x) > f(0) = 0$ for $x \in \left(0, \frac{\pi}{2}\right)$. This proves the second inequality of (2.33). $\square$

**Remark 2.3.** From the second inequality of (2.14) it is immediate that the following weaker inequality holds:

$$X > \frac{2G+A}{3}, \tag{2.34}$$

This follows by $P < (2A+G)/3$ (See [3]). Since $L < (2G+A)/2$, we get

$$X > \frac{2G+A}{3} > L, \tag{2.35}$$

so we can deduce by (2.33) a chain of inequalities for P, which improves a result from our paper [8].

**Theorem 2.10.** *One has*

$$X > A+G-P, \tag{2.36}$$

**Proof.** By using the notations from the proof of Theorem 2.8, and by taking logarithms, the inequality becomes

$$f(x) = x\cot x - 1 - \ln(1+\cos x - \sin x/x)) > 0$$

After elementary computations one finds that the sign of derivative of f depends on the sign of the function

$$g(x) = \frac{x}{\sin x} + (\sin x)^2 - (x^2)\cos x - x^2.$$

To prove that $g(x) > 0$ we have to show that

$$F(x) = x\sin x + (\sin x)^2 - (x^2)(\cos x) - x^2 > 0.$$

We get

$$F'(x) = \sin x - x\cos x + 2\sin x\cos x + x^2\cos x - 2x$$

and

$$F''(x) = 3x\sin x + x^2\cos x - 4(\sin x)^2.$$

We will prove that

$$F''(x) > 0, \quad \text{or equivalently} \quad 4t^2 - t - \cos x < 0,$$

where $t = \sin x/x$. By solving the above quadratic inequality, and by taking into account of the Cusa-Huygens inequality $t < (2+u)/3$, where $u = \cos x$, we have to prove the following relation

$$(2+u)/3 < [3 + \sqrt{9+16u}]/8.$$

Or after some computations, with $(2u+1)(u-1) < 0$, which is true. Since $F''(x) > 0$, we get $F'(x) > F'(0) = 0$, so $F(x) > F(0) = 0$. The function $f$ being strictly increasing, the result follows, as $f(0+) = 0$. $\square$

**Remark 2.4.** Inequality (2.36) is slightly stronger than the right side of (2.14). Indeed, after some transformations, this becomes

$$3P^2 - 2P \cdot (A + 2G) + 2AG + G^2 < 0.$$

Resolving this quadratic inequality, this becomes

$$[3P - (2A + G)](P - G) < 0,$$

which is true by $G < P < (2A + G)/3$.

**Theorem 2.11.**

$$P \cdot X < \left(\frac{A+G}{2}\right)^2, \tag{2.37}$$

**Proof.** It is immediate that we have to prove the following inequality

$$f(x) = \ln(x/\sin x) - 2\ln[2/(1+\cos x)] - (x \cot x - 1) > 0.$$

After computations we get that the sign of $f'(x)$ depends on the sign of

$$\begin{aligned} g(x) &= (\sin x)(1+\cos x)(\sin x - x\cos x) - (x \sin x \cos x)(1+\cos x) \\ &= (x^2)(1+\cos x) + 2x(\sin x)^3. \end{aligned}$$

By

$$1 = \cos x = 2[\cos(x/2)]^2$$

and

$$\sin x = 2\sin(x/2)\cos(x/2)$$

we can write

$$g(x) = 2[\cos(x/2)]^2 h(x),$$

where

$$h(x) = (\sin x)^2 - 2x\sin x\cos x + x^2 + 8x\cos(x/2)[\sin(x/2)]^3.$$

We can deduce

$$h'(x)/4[\sin(x/2)]^2 = 7x[\cos(x/2)]^2 + 2\sin(x/2)\cos(x/2) - x[\sin(x/2)]^2.$$

As $\cos(x/2) > \sin(x/2)$, we get $h'(x) > 0$. Thus we have $h(x) > h(0) = 0$, so $g(x) > 0$ and finally, $f'(x) > 0$. The result follows by the remark that $f(x) > f(0+) = 0$. $\square$

**Remark 2.5.** Relation (2.37) combine with the weaker inequality of (2.14) shows that, $\sqrt{(}P \cdot X)$ lies between the geometric and arithmetic means of $A$ and $G$.

**Remark 2.6.** In a recent paper, B. Bhayo and the author [9] have proved the following counterpart of relation (2.24):

**Theorem 2.12.** *The following inequality holds true:*

$$P < (X^c) \cdot (A^{1-c}), \quad \textit{where} \quad c = \ln(\pi/2), \tag{2.38}$$

**Remark 2.7.** An earlier version of this work appeared in the last paragraph of the arXiv paper [10].

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