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SOME IDENTITIES IN TRIANGLE AND CONSEQUENCES

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Abstract. The purpose of this article is to express in a triangle ABC the sums $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$ and $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$ in terms of R, r and s.

MSC 2000: 51M04.
Key words: Trigonometric identities in triangle.

1. INTRODUCTION

In the following we will consider a given ABC triangle with the sides $AB = c$, $BC = a$, $CA = b$, r, R the inradius and circumradius of triangle, $s = \dfrac{a+b+c}{2}$ the semiperimeter and A, B, C the angles.

In [3] it's proved the following identities.

LEMMA 1.1. *In ABC triangle are true the following identities*

$$\left(\sum \cos\frac{A}{2}\right)^2 - \left(\sum \sin\frac{A}{2} + 1\right)^2 = \frac{r}{R} \tag{1.1}$$

and

$$2R\sum \cos\frac{A}{2}\left(\sum \sin\frac{A}{2} - 1\right) = s. \tag{1.2}$$

We denote $f = \sum \sin\frac{A}{2}$ and $g = \sum \cos\frac{A}{2}$. From (1.1) and (1.2) we obtain

$$\begin{cases} g^2 - (f+1)^2 &= \dfrac{r}{R} \\ g(f-1) &= \dfrac{s}{2R}. \end{cases} \tag{1.3}$$

THEOREM 1.1. *In ABC triangle we have*

$$f^4 + \left(\frac{r}{R} - 2\right)f^2 - \frac{2r}{R}f + 1 + \frac{r}{R} - \frac{s^2}{4R^2} = 0. \tag{1.4}$$

LEMMA 1.2. *In ABC triangle are true the inequalities*

$$1 < f \le \frac{3}{2}. \tag{1.5}$$

In [1] W.J. Blundon proved in 1965 the following theorem.

THEOREM 1.2. *In ABC triangle are true the inequalities*

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\leq s^2\leq \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\,. \tag{1.6}$$

REMARK 1.1. Inequalities from (1.6) represent the necessary and sufficient conditions for existence of a triangle with equality if and only if the triangle is equilateral with elements R, r and s.

THEOREM 1.3 (G. Colombier - T. Doucet, see [2] page 49). *In ABC triangle is true inequality*

$$3s^2\leq (4R+r)^2. \tag{1.7}$$

THEOREM 1.4 (L. Euler's inequality, see [2] page 48). *In ABC triangle is true the inequality*

$$R\geq 2r, \tag{1.8}$$

with equality if and only if ABC is equilateral.

2. MAIN RESULT

The identity (1.4) it suggest us to consider the equation

$$x^4+\left(\frac{r}{R}-2\right)x^2-\frac{2r}{R}x+1+\frac{r}{R}-\frac{s^2}{4R^2}=0 \tag{2.1}$$

We will study the equation from (2.1) using the Rolle's sequence.

Let $h:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$h(x)=x^2+4\left(\frac{r}{R}-2\right)x^2-\frac{2r}{R}x+1+\frac{r}{R}-\frac{s^2}{4R^2}\,,\ x\in\mathbb{R}.$$

We have that

$$h'(x)=4x^3+2\left(\frac{r}{R}-2\right)x-\frac{2r}{R}=(x-1)\left(4x^2+4x+\frac{2r}{R}\right). \tag{2.2}$$

The roots of equation $4x^2+4x+\frac{2r}{R}=0$, taking (1.8) into account are $x_1=\dfrac{-1-\sqrt{\frac{R-2r}{R}}}{2}$, $x_2=\dfrac{-1+\sqrt{\frac{R-2r}{R}}}{2}$, which verify $4x_k^2+4x_k+\frac{2r}{R}=0$, $k\in\{1,2\}$, from where

$$x_k^2=-x_k-\frac{r}{2R},\ k\in\{1,2\}. \tag{2.3}$$

Then taking into account (2.3), we have

$$x_k^3 = -x_k^2 - \frac{r}{2R}\,x_k = \left(1 - \frac{r}{2R}\right)x_k + \frac{r}{2R}$$

and

$$x_k^4 = \left(1 - \frac{r}{2R}\right)x_k^2 + \frac{r}{2R}\,x_k = \left(1 - \frac{r}{2R}\right)\left(-x_k - \frac{r}{2R}\right) + \frac{r}{2R}\,x_k,$$

so

$$x_k^4 = \left(\frac{r}{R} - 1\right)x_k - \frac{r}{2R} + \frac{r^2}{4R^2}\,,\ k \in \{1,2\}. \tag{2.4}$$

From (2.3) and (2.4) for $k \in \{1,2\}$, we have

$$\begin{aligned} h(x_k) &= x_k^4 + \left(\frac{r}{R} - 2\right)x_k^2 - \frac{2r}{R}\,x_k + 1 + \frac{r}{R} - \frac{s^2}{4R^2} = \\ &= \left(1 - \frac{2r}{R}\right)x_k + 1 + \frac{3r}{2R} - \frac{r^2}{4R^2} - \frac{s^2}{4R^2}\,,\ k \in \{1,2\}. \end{aligned}$$

If we replace x_1 in above inequality, we obtain

$$\begin{aligned} h(x_1) &= \left(1 - \frac{2r}{R}\right)\frac{-1 - \sqrt{\frac{R-2r}{R}}}{2} + 1 + \frac{3r}{2R} - \frac{r^2}{4R^2} - \frac{s^2}{4R^2} = \\ &= -\frac{1}{2} + \frac{r}{R} - \frac{R-2r}{2R}\sqrt{\frac{R-2r}{R}} + 1 + \frac{3r}{2R} - \frac{r^2}{4R^2} - \frac{s^2}{4R^2}\,, \end{aligned}$$

from which

$$h(x_1) = \frac{2R^2 + 10Rr - 2(R-2r)\sqrt{R^2 - 2Rr} - r^2 - s^2}{4R^2}\,. \tag{2.5}$$

In the same way

$$h(x_2) = \frac{2R^2 + 10Rr + 2(R-2r)\sqrt{R^2 - 2Rr} - r^2 - s^2}{4R^2}\,. \tag{2.6}$$

We have that

$$h(1) = -\frac{s^2}{4R^2} \tag{2.7}$$

and

$$h\left(\frac{3}{2}\right) = \frac{25R^2 + 4Rr - 4s^2}{16R^2}\,. \tag{2.8}$$

We will prove that $\dfrac{(4R+r)^2}{3} \le \dfrac{25R^2 + 4Rr}{4}$, or $64R^2 + 32Rr + 4r^2 \le 75R^2 + 12Rr$, which is equivalent with $(11R + 2r)(R - 2r) \ge 0$, which is

true since $R \geq 2r$.
From (1.7) and the above inequality it results that

$$25R^2 + 4Rr - 4s^2 \geq 0, \tag{2.9}$$

whith equality if and only if the triangle is equilateral.
We will prove that

$$x_1 \leq x_2 < 1 < \frac{3}{2} \tag{2.10}$$

The first inequality is true, with equality if and only if $R = 2r$, so the ABC triangle is equilateral. The inequality $x_2 < 1$ is equivalent with $\sqrt{\frac{R-2r}{R}} < 3$, or $8R > -2r$ which is true.
If $R = 2r$, equivalent with triangle ABC is equilateral, equation (2.1) is equivalent with

$$x^4 - \frac{3}{2}x^2 - x - \frac{3}{16} = 0, \quad \text{or} \quad \left(x - \frac{3}{2}\right)\left(x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}\right) = 0.$$

This equation has a only positive root $x = \frac{3}{2}$. So

$$f = \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = \frac{3}{2}\,. \tag{2.11}$$

In the following we consider the case $R > 2r$. Then from (2.10) it follows that

$$-\infty < x_1 < x_2 < 1 < \frac{3}{2} < +\infty. \tag{2.12}$$

We have $\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to-\infty} h(x) = +\infty$. From (2.5) and (1.6) it result that $h(x_1) < 0$, from (2.6) and (1.6) that $h(x_2) > 0$ and from (2.7) that $h(1) < 0$.
According Rolle's sequence, the function h has four real roots $\alpha_1 \in (-\infty, x_1)$, $\alpha_2 \in (x_1, x_2)$, $\alpha_3 \in (x_2, 1)$ and $\alpha_4 \in (1, +\infty)$. From (2.8) and (2.9) it follows that $\alpha_4 \in \left(1, \frac{3}{2}\right)$. Since is true (1.4) and (1.5) it follows that in the case $R > 2r$ we have

$$\alpha_4 = f \in \left(1, \frac{3}{2}\right) \tag{2.13}$$

and

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4. \tag{2.14}$$

In the following we remember the formulas of solving the 4^{th} degree equation $ax^4+bx^3+cx^2+dx+e$, $a,b,c,d,e\in\mathbb{R}$, $a\neq 0$, with Ferrari's method

$$\alpha_{1,2}=-\frac{b}{4a}-S\pm\frac{1}{2}\sqrt{-4S^2-2p+\frac{q}{S}},\ \alpha_{3,4}=-\frac{b}{4a}-S\pm\frac{1}{2}\sqrt{-4S^2-2p-\frac{q}{S}},$$

where $p=\dfrac{8ac-3b^2}{8a^2}$, $q=\dfrac{b^3-4abc+8a^2d}{8a^3}$, $S=\dfrac{1}{2}\sqrt{-\dfrac{2}{3}p+\dfrac{1}{3a}\left(Q+\dfrac{\Delta_0}{Q}\right)}$, $Q=\sqrt[3]{\dfrac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}}$, $\Delta_0=c^2-3bd+12ae$, $\Delta_1=2c^3-9bcd+27b^2e+27ad^2-72ace$.

In our case taking into account (2.1), we have that $a=1$, $b=0$, $c=\dfrac{r}{R}-2$, $d=-\dfrac{2r}{R}$ and $e=1+\dfrac{r}{R}-\dfrac{s^2}{4R^2}$. Then $q=d=-\dfrac{2r}{R}<0$ and since $S\geq 0$ and from (2.13), (2.14) it results that

$$f=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=-S+\frac{1}{2}\sqrt{4S^2-2p-\frac{q}{S}}. \tag{2.15}$$

Since $p=c=\dfrac{r}{R}-2$, according Wolphram Alpha, we have that

$$\Delta_0=c^2-12e=\frac{r^2-16rR-8R^2+3s^2}{R^2} \tag{2.16}$$

and

(2.17)

$$\Delta_1=2c^3+27d^2-72ce=\frac{2\left(r^3+12r^2R+48rR^2+9rs^2+64R^3-18Rs^2\right)}{R^3}.$$

Also after perform some calculation we obtain

$$\begin{aligned}\Delta_1^2-4\Delta_0^3=\frac{36}{R^6}\Big[&512R^6+1024R^5r+1088R^4r^2+512R^3r^3-56R^2r^4+\\&+8Rr^5-\left(320R^4+320R^3r+192R^2r^2-52Rr^3-r^4\right)s^2+\\&+\left(60R^2+12Rr+6r^2\right)s^4-3s^6\Big].\end{aligned}$$

Taking into account (1.3) and (2.15) we have

$$g = \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = \frac{s}{2R}\,\frac{1}{f-1} = \tag{2.18}$$

$$= \frac{s}{2R}\,\frac{1}{-S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}}.$$

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