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IDENTITIES AND INEQUALITIES IN A QUADRILATERAL

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Abstract. In this paper we will demonstrate some inequalities that occur in an inscribable and circumscribable quadrilater.

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1. INTRODUCTION

Let ABCD be a convex quadrilateral and denote AB=a, BC=b, CD=c, DA=d, BD=e, AC=f, $s=\dfrac{a+b+c+d}{2}$, AC$\cap$ BD=$\{O\}$, angle measure AOB is φ and F is the area of quadrilateral ABCD.

If ABCD is an inscribable and circumscribable quadrilateral, let R, r be the radius of the circumscribed circle, respectively of the inscribed circle to ABCD.

It is well known that the sides a, b, c, d are solutions of the equation (see [3], page 164)

(1.1)

$$x^4-2sx^3+(s^2+2r^2+2r\sqrt{4R^2+r^2})x^2-2rs(\sqrt{4R^2+r^2}+r)X+r^2+s^2=0.$$

The following inequalities are true (see [3], page 168)

$$2\sqrt{2r\sqrt{4R^2+r^2}-r)}\leq s \tag{1.2}$$

with equality either if and only if ABCD is square when $R=r\sqrt{2}$ or if ABCD is an isosceles trapezoid when $R\neq\sqrt{2}$.

$$s\leq\sqrt{4R^2+r^2}+r, \tag{1.3}$$

with equality if and only if ABCD is an orthodiagonal quadrilateral

$$2\sqrt{2r\sqrt{4R^2+r^2}-r)}\leq s\leq\sqrt{4R^2+r^2}+r. \tag{1.4}$$

The inequalities (1.4) hold simultaneously when $R=r\sqrt{2}$ if and only if ABCD is square. When $R\neq\sqrt{2}$ at least an inequality from (1.4) is strict.

On the other hand, we have the Inequality L. Fejes Tóth

$$R\geq r\sqrt{2}, \tag{1.5}$$

which suits, with equality if and only if ABCD is a square. The following identities are true

$$ef = 2r(\sqrt{4R^2+r^2}+r), \tag{1.6}$$

$$F = sr, \tag{1.7}$$

$$e^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}, \tag{1.8}$$

$$ef = ac+bd, \tag{1.9}$$

$$\frac{e}{f} = \frac{ab+cd}{ad+bc}, \tag{1.10}$$

and

$$16R^2F^2 = (ab+cd)(ac+bd)(ad+bc) \tag{1.11}$$

called the relation of Girard.

2. IDENTITIES AND INEQUALITIES WITH R_1, R_2, R_3, R_4

Let it be ABCD a convex quadrilateral and denote by R_1, R_2, R_3, R_4 the radii of the circumscribed circles by the triangles AOB, BOC, COD și DOA.

Lemma 2.1. *The following identities holds:*

$$R_1+R_2+R_3+R_4 = \frac{sef}{2F} \tag{2.1}$$

and

$$\begin{aligned} a &= \frac{2sR_1}{R_1+R_2+R_3+R_4}, b = \frac{2sR_2}{R_1+R_2+R_3+R_4},\\ c &= \frac{2sR_13}{R_1+R_2+R_3+R_4}, d = \frac{2sR_4}{R_1+R_2+R_3+R_4}. \end{aligned} \tag{2.2}$$

Demonstration. From the theorem it follows that $R_1 = \frac{a}{2\sin\varphi}$, $R_2 = \frac{b}{2\sin\varphi}$, $R_3 = \frac{c}{2\sin\varphi}$, $R_4 = \frac{d}{2\sin\varphi}$, where from $R_1+R_2+R_3+R_4 = \frac{a+b+c+d}{s\sin\varphi} = \frac{s}{2\sin\varphi}$. But $F = \frac{ef\sin\varphi}{2}$, from where results the relation (2.1). On the other hand, we have $\frac{R_1}{a} = \frac{R_2}{b} = \frac{R_3}{c} = \frac{R_4}{d} = \frac{1}{2\sin\varphi}$, from where $b = \frac{R_2}{R_1}a$, $c = \frac{R_3}{R_1}a$, $d = \frac{R_4}{R_1}a$. From the above relations, by addition, we obtain that $2s = a+b+c+d = a+\frac{R_2}{R_1}a+\frac{R_3}{R_1}a+\frac{R_4}{R_1}a = \frac{a}{R_1}(R_1+R_2+R_3+R_4)$ and from there the relations (2.2).

Corollary 2.1. *In a convex quadrilateral the inequality occurs*

$$\frac{sef}{2F} \geq \sqrt[4]{R_1R_2R_3R_4} \tag{2.3}$$

Lemma 2.2. *If quadrilateral ABCD is inscribable and circumscribable, then*

$$a = \frac{2sR_1}{\sqrt{4R^2+r^2}+r} \tag{2.4}$$

Demonstration. From (2.2), considering (2.1), (1.6) and (1.7), the relationship (2.4) is obtained.

Theorem 2.1. *If the quadrilater ABCD is inscribable and circumscribable, then R_1, R_2, R_3, R_4 are solutions of the equation*

(2.5)

$$16s^2x^4 - 16s^2(\sqrt{4R^2+r^2}+r)x^3 + (s^2+2r^2+2r\sqrt{4R^2+r^2})\sqrt{4R^2+r^2}+r)^2x^2$$
$$-4(\sqrt{4R^2+r^2}+r)^4x + r^2(\sqrt{4r^2+r^2}+r)^4 = 0$$

Demonstration. Considering (2.4), we obtain a solution for the equation (1.1) replacing a from (2.4) in (1.1) and after calculations we get that R_1 is a solution of the equation (2.5). Analogously, it can be demonstrated that R_2, R_3, R_4 are solutions of the equation (2.5).

Theorem 2.2. *If the quadrilater ABCD is inscribable and circumscribable, then we have the following identities*

$$\sum R_1 = \sqrt{4R^2+r^2}+r \tag{2.6}$$

$$\sum R_1R_2 = \frac{(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2}{4s^2}, \tag{2.7}$$

$$\sum R_1R_2R_3 = \frac{r(\sqrt{4R^2+r^2}+r)^4}{4s^2}, \tag{2.8}$$

$$R_1R_2R_3R_4 = \frac{r^2(\sqrt{4R^2+r^2}+r)^4}{4s^2}, \tag{2.9}$$

$$\sum R_1^2 = \frac{(s^2-2r^2-2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2}{,}4s^2 \tag{2.10}$$

$$\sum \frac{1}{R_1} = \frac{4}{r}, \tag{2.11}$$

$$\sum \frac{1}{R_1R_2} = \frac{4(s^2+2r^2+2r\sqrt{4R^2+r^2})}{r^2(\sqrt{4R^2+r^2}+r)^2}, \tag{2.12}$$

$$,\sum \frac{1}{R_1R_2R_3} = \frac{16s^2}{r^2(\sqrt{4R^2+r^2}+r)^3}. \tag{2.13}$$

The demonstrations results from the Theorem (2.1).

Theorem 2.3. *If the quadrilater ABCD is inscribable and circumscribable the following identities are true*

$$4r \leq \sum R_1 \leq 2R\sqrt{2}, \tag{2.14}$$

$$\begin{aligned} 6r^2 &\leq R^2 + r^2 + r\sqrt{4R^2+r^2} \leq \sum R_1R_2 \\ &\leq \frac{(5\sqrt{4R^2+r^2}-3r)(\sqrt{4R^2+r^2}+r)^3}{64R^2} \\ &\leq \frac{(5\sqrt{4R^2+r^2}-3r)R\sqrt{2}}{4} \leq \frac{(10R\sqrt{2}-3r)R\sqrt{2}}{4}, \end{aligned} \tag{2.15}$$

$$4r^3 \leq \frac{r\sqrt{4R^2+r^2}+r)^2}{16} \leq \sum R_1R_2R_3 \leq \frac{(\sqrt{4R^2+r^2}+r)^5}{128R^2} \leq R^3\sqrt{2}, \tag{2.16}$$

$$r^4 \leq \frac{r\sqrt{4R^2+r^2}+r)^2}{16} \leq R_1R_2R_3R_4 \leq \frac{(\sqrt{4R^2+r^2}+r)^5}{512R^2} \leq \frac{R^4}{4}, \tag{2.17}$$

$$\frac{8r^4}{R^2} \leq \frac{(\sqrt{4R^2+r^2}-5r)(\sqrt{4R^2+r^2}+r)^3}{32R^2} \leq \sum R_1^2 \leq 2R^2, \tag{2.18}$$

$$\frac{4\sqrt{2}}{R} \leq \sum \frac{1}{R_1}, \tag{2.19}$$

$$\begin{aligned} \frac{12}{R^2} &\leq \frac{8(\sqrt{4R^2+r^2}-3r)}{r(\sqrt{4R^2+r^2}+r)} \leq \sum \frac{1}{R_1R_2} \\ &\leq \frac{(\sqrt{4R^2+r^2}-5r)(\sqrt{4R^2+r^2}+r)}{32R^2} \leq \frac{3R^2}{r^4}, \end{aligned} \tag{2.20}$$

$$\frac{8}{2R^2} \leq \frac{512R^2}{r(\sqrt{4R^2+r^2}+r)^2} \leq \sum \frac{1}{R_1R_2R_3} \leq \frac{16}{r(\sqrt{4R^2+r^2}+r)} \leq \frac{4}{r^2}. \tag{2.21}$$

The demonstrations results from the Theorem (2.2) and from the inequalities (1.2), (1.3) and (1.5).

3. IDENTITIES AND INEQUALITIES WITH r_1, r_2, r_3, r_4

In this section we consider that ABCD is inscribable and circumscribable and we denote r_1, r_2, r_3, r_4 the radii of the circles inscribed in the triangles AOB, BOC, COD and DOA.

Lemma 3.1. *The following identities are true*

$$AM = \frac{eda}{ab+cd}, \tag{3.1}$$

$$BM = \frac{eab}{ab+cd}, \tag{3.2}$$

$$CM = \frac{ebc}{ab+cd}, \tag{3.3}$$

$$DM = \frac{ecd}{ab+cd}. \tag{3.4}$$

Demonstration. From the similarity of triangles ABO și DCO, respectively ADO and BCO, we have that $\frac{AO}{DO} = \frac{BO}{CO} = \frac{AB}{CD} = \frac{a}{c}$ and $\frac{AO}{BO} = \frac{DO}{CO} = \frac{AD}{BC} = \frac{d}{b}$. From the above inequalities, it follows that $BO = \frac{d}{b}AO$, $CO = \frac{c}{a}BO = \frac{bc}{ad}AO$ and $DO = \frac{c}{a}AO$. If we choose BO and DO from equality BO+DO=e we get $\frac{b}{d}AO + \frac{c}{a}AO = e$, where the relationship is obtained from (3.1). The relations (3.2)-(3.4) are proved analogously.

Note that s_1, F_1 are the semiperimeter, respectively the area of the triangle AOB and $\alpha = \frac{sr^2(\sqrt{4R^2+r^2}+r)}{2R(2R+r+\sqrt{4R^2+r^2})}$.

Lemma 3.2. *We have that*

$$r_1 = \frac{\alpha}{c} \tag{3.5}$$

Demonstration. Considering s = a + c = b + d, (3.1), (3.2) and (1.6)-(1.11), calculate

$$\begin{aligned} s_1 &= \frac{1}{2}(AB+OB+OA) = \frac{1}{2}(a + \frac{eab}{ab+cd} + \frac{eda}{ab+cd}) \\ &= \frac{a}{2}(a + \frac{e(b+d)}{ab+cd}) = \frac{a}{2}(1 + \frac{s}{ab+cd}) = (\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}) \\ &= \frac{a}{2}(1 + s\frac{ef}{\sqrt{(ad+bc)(ac+bd)(ab+cd)}}) = \frac{a}{2}(1 + s\frac{ef}{4Rr}) \\ &= \frac{a}{2}(1 + s\frac{ef}{4Rr}) = \frac{a}{2}(1 + \frac{2R\sqrt{4R^2+r^2}+r}{4Rr}), \end{aligned}$$

from where

$$s_1 = a\frac{2R+\sqrt{4R^2+r^2}+r}{4R} \tag{3.6}$$

So we have $F_1 = \frac{OAOB\sin\varphi}{4Rr}$ and considering $F = \frac{ef\sin\varphi}{2}$ and (3.1), (3.2), (1.6), (1.7), it results that

$$\begin{aligned} F_1 &= \frac{1}{2}\frac{e^2a^2bd}{2(ab+cd)}\frac{2F}{ef} = \frac{e}{f}\frac{a^2bdFef}{f(ab+cd)^2} = \frac{a^2bdF}{(ab+cd)(ad+bc)} \\ &= \frac{a^2bdFef}{(ab+cd)(ac+bd)(ad+bc)} = \frac{a^2bdFef}{16R^2F^2}, \end{aligned}$$

from where

$$F_1 = \frac{a^2bd(\sqrt{4R^2+r^2}+r)}{8sR^2} \tag{3.7}$$

Because $abcd = F^2 = S^2r^2$, from (3.6) and (3.7) it results that $r_1 = \frac{F_1}{s_1} = \frac{abcd\sqrt{4r^2+r^2}+r}{8sR^2}\frac{4R}{c(2R+r+\sqrt{4R^2+r^2})}$, from where (3.5) is true.

Remark 3.1. *Similar, we obtain that* $r_2 = \frac{\alpha}{d}$, $r_3 = \frac{\alpha}{a}$, $r_4 = \frac{\alpha}{b}$.

Remark 3.2. *Because* $R_1 = \frac{\alpha OBOA}{4T_1}$, *using (3.1)-(3.4), (3.5) and (3.7) we can calculate* R_1, R_2, R_3, R_4.

Theorem 3.1. *If the quadrilater ABCD is inscribable and circumscribable, then* r_1, r_2, r_3, r_4 *are solutions of the equation*

(3.8)

$$16R^4(2R+r+\sqrt{4R^2+r^2})^4x^4 - 16rR^3(\sqrt{4R^2+r^2}+r)^2(2R+r+\sqrt{4R^2+r^2})^3x^3$$
$$+ 4r^2R^2(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2(2R+r+\sqrt{4R^2+r^2})^2x^2$$
$$- 4s^2r^4R(\sqrt{4R^2+r^2}+r)^3(2R+r+\sqrt{4R^2+r^2})x + s^2r^6(\sqrt{4R^2+r^2}+r)^4 = 0$$

Demonstration: Considering c from (3.5) as a solution of the equation (1.1), substituting c from (3.5) in (1.1), after the calculations (3.8) is true.

Theorem 3.2. *If the quadrilater ABCD is inscribable and circumscribable, then*

$$\sum r_1 = \frac{r(\sqrt{4R^2+r^2}+r)^2}{R(2R+r+\sqrt{4R^2+r^2})}, \tag{3.9}$$

$$\sum r_1r_2 = \frac{r^2(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2}{4R^2(2R+r+\sqrt{4R^2+r^2})^2}, \tag{3.10}$$

$$\sum r_1r_2r_3 = \frac{s^2r^4(\sqrt{4R^2+r^2}+r)^3}{4R^3(2R+r+\sqrt{4R^2+r^2})^3}, \tag{3.11}$$

$$r_1r_2r_3r_4 = \frac{s^2r^6(\sqrt{4R^2+r^2}+r)^4}{16R^4(2R+r+\sqrt{4R^2+r^2})^4}, \tag{3.12}$$

$$\sum r_1^2 = \frac{r^2(\sqrt{4R^2+r^2}+r)^2(8R^2+2r^2+2r\sqrt{4R^2+r^2}-s^2)}{2R^2(2R+r+\sqrt{4R^2+r^2})^2}, \tag{3.13}$$

$$\sum \frac{1}{r_1} = \frac{4R(2R+r+\sqrt{4R^2+r^2})}{r^2(\sqrt{4R^2+r^2}+r)}, \tag{3.14}$$

$$\sum \frac{1}{r_1r_2} = \frac{4R^2(s^2+2r^2+2r\sqrt{4R^2+r^2})(2R+r+\sqrt{4R^2+r^2})^2}{s^2r^4(\sqrt{4R^2+r^2}+r)}, \tag{3.15}$$

$$\sum \frac{1}{r_1r_2r_3} = \frac{16R^3(2R+r+\sqrt{4R^2+r^2})^3}{s^2r^5(\sqrt{4R^2+r^2}+r)^2}. \tag{3.16}$$

Demonstration results from Theorem (3.1).

THEOREM 3.3. *If the quadrilater ABCD is inscribable and circumscribable, then we have the following inequalities*

$$\frac{8r^3(\sqrt{2}-1)}{R^2} \le \sum r_1 \le 2\sqrt{2}(\sqrt{2}-1)R, \tag{3.17}$$

$$\frac{24r^6(\sqrt{2}-1)^2}{R^4} \le \sum r_1r_2 \le \frac{3R^4(\sqrt{2}-1)^2}{2r^2}, \tag{3.18}$$

$$\frac{32r^9(\sqrt{2}-1)^3}{R^6} \le \sum r_1r_2r_3 \le \frac{R^6(\sqrt{2}-1)^3}{2r^3}, \tag{3.19}$$

$$\frac{64r^{12}(\sqrt{2}-1)^4}{R^8} \le r_1r_2r_3r_4 \le \frac{R^4(\sqrt{2}-1)^4}{4}, \tag{3.20}$$

$$\frac{4\sqrt{2}(\sqrt{2}+1)}{R} \le \sum \frac{1}{r_1} \le \frac{R^2(\sqrt{2}+1)^4}{r^3}. \tag{3.21}$$

The demonstration follows from Theorem (3.2) and from inequalities (1.2), (1.3) and (1.5).

4. APPLICATIONS

(1) Let it be ABCD a bicentric quadrilateral. We denote AB=a, BC=b, CD=c, DA=d, BD=e, AC=f, the radius of the inscribed circle r. Show that $ef \ge 6r^2$.

Solution. From (1.6) we have $ef = 2r(\sqrt{4R^2+r^2}+r)$. Using the inequality L. Fejes Tóth $R \ge r\sqrt{2}$ we replace R depending on r so that $ef \ge 2r(\sqrt{8r^2+r^2}+r)$ equivalent to $ef \ge 2r(3r+r)$ from where $ef \ge 6r^2$, what had to be demonstrated.

(2) Let it be ABCD a bicentric quadrilateral. We denote R_1, R_2, R_3, R_4 the radii of the circles circumscribed by the triangles AOB, BOC, COD and DOA. Let r be the radius of the inscribed circle. Show that $\sum aR_1 \ge 2sr$.

Solution. How systems R_1, R_2, R_3, R_4 and a, b, c, d are equally ordered, from Cebisev theorem $4(aR_1+bR_2+cR_3+dR_4) \ge (a+b+c+d)(R_1+R_2+R_3+R_4)$ equivalent to $\sum aR_1 \ge \frac{s^2ef}{4sr}$, and from (1.6) and (1.7) we have $\sum aR_1 \ge \frac{s\sqrt{4R^2+r^2}+r}{2}$. From inequality L. Fejes Tóth $R \ge r\sqrt{2}$ from where $\sum aR_1 \ge 2sr$.

(3) Let it be ABCD a bicentric quadrilateral. We denote AB=a, BC=b, CD=c, DA=d, BD=e, AC=f, $s = \frac{a+b+c+d}{2}$, R_1, R_2, R_3, R_4 the radii of the circles circumscribed by the triangles AOB, BOC, COD and DOA, r the radius of the circle inscribed in the quadrilateral ABCD. Show that $\sum aR_1 \ge \frac{s^2ef}{4sr}$

Solution. How systems R_1, R_2, R_3, R_4 and a,b,c,d are equally ordered, from Cebisev theorem $4(aR_1 + bR_2 + cR_3 + dR_4) \geq (a + b + c + d)(R_1 + R_2 + R_3 + R_4)$ so we have $\sum aR_1 \geq \frac{s^2ef}{4sr}$.

(4) Let it be ABCD a bicentric quadrilateral. We denote AB=a, BC=b, CD=c, DA=d, BD=e, AC=f, the radius of the circle inscribed r, the radius of the circumscribed circle R, R_1, R_2, R_3, R_4 the radii of the circles circumscribed by the triangles AOB, BOC, COD and DOA. Show that $\sum aR_1 \geq \frac{s(\sqrt{4R^2+r^2}+r)}{2}$.

Solution. How systems R_1, R_2, R_3, R_4 and a,b,c,d are equally ordered, from Cebisev theorem $4(aR_1 + bR_2 + cR_3 + dR_4) \geq (a + b + c + d)(R_1 + R_2 + R_3 + R_4)$ equivalent to $\sum aR_1 \geq \frac{s^2ef}{4sr}$, form (1.6) and (1.7) we have $\sum aR_1 \geq \frac{s(\sqrt{4R^2+r^2}+r)}{2}$.

(5) Let it be ABCD a bicentric quadrilateral. We denote AB=a, BC=b, CD=c, DA=d, BD=e, AC=f, $s = \frac{a+b+c+d}{2}$, R_1, R_2, R_3, R_4 the radii of the circles circumscribed by the triangles AOB, BOC, COD and DOA, r the radius of the circle inscribed in the quadrilateral ABCD. Show that $r \leq \frac{\sqrt{2}}{4}ef$.

Solution. From (2.11) $\sum \frac{1}{r_1} = \frac{4}{r}$. From the inequality of averages, $m_a \geq m_b \Rightarrow \frac{R_1+R_2+R_3+R_4}{4} \geq \frac{4}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}} = \frac{4}{\sum \frac{1}{R_1}}$. From (2.11) $\Rightarrow \frac{R_1+R_2+R_3+R_4}{4} \geq \frac{4}{\frac{4}{r}}$ equivalent to $r \leq \frac{R_1+R_2+R_3+R_4}{4}$ (a.1). From (2.1) $R_1 + R_2 + R_3 + R_4 = \frac{sef}{2F}$ it follows that $R_1 + R_2 + R_3 + R_4 = \frac{s}{2}\frac{ef}{2}$. From (1.7) $R_1 + R_2 + R_3 + R_4 = \frac{ef}{2r}$. Using here (a.1) we have $r \leq \frac{1}{4}\frac{ef}{2r}$ equivalent to $r^2 \leq \frac{1}{8}ef$ which leads to $r \leq \frac{\sqrt{2}}{4}ef$, what was to be demonstrated.

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