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AN EXTENSION OF THE CONCEPT OF A COMPLEX NUMBER

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Abstract. In this paper we present extensions of the complex numbers, involving new sets depending on a certain ε, i.e. Hilger's complex plane. Algebric properties and somple examples are provided.

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Key words. Hilger Complex Plane, cylindrical transformation.

DEFINITION 1. For any $\varepsilon > 0$ we define the Hilger's complex plane C_ε, the Hilger's real axis R_ε, the Hilger's alternating axis A_ε and the Hilger's imaginary circle I_ε as:

$$C_\varepsilon = \left\{ z \in \mathbb{C} \middle| z \neq -\frac{1}{\varepsilon} \right\};$$

$$R_\varepsilon = \left\{ z \in \mathbb{R} \middle| z > -\frac{1}{\varepsilon} \right\};$$

$$A_\varepsilon = \left\{ z \in \mathbb{R} \middle| z < -\frac{1}{\varepsilon} \right\};$$

$$I_\varepsilon = \left\{ z \in \mathbb{C} \middle| \left| z + \frac{1}{\varepsilon} \right| = \frac{1}{\varepsilon} \right\};$$

For $\varepsilon = 0$ we define $C_0 = \mathbb{C}, R_0 = \mathbb{R}, A_0 = \phi, I_0 = i\mathbb{R}$.

DEFINITION 2. Let $\varepsilon > 0$ and for any $z \in C_\varepsilon$ we define *the Hilger real (imaginary) part of* z by:

$$\Re_\varepsilon(z) = \frac{|z\varepsilon + 1| - 1}{\varepsilon} = \left| z + \frac{1}{\varepsilon} \right| - \frac{1}{\varepsilon}$$

and *the Hilger imaginary part of* z by

$$\mathrm{Im}_\varepsilon(z) = \frac{\arg(z\varepsilon + 1)}{\varepsilon} = \frac{\arg\left(z + \frac{1}{\varepsilon} \right)}{\varepsilon},$$

where $\arg(z)$ denotes the principal argument of z, i.e. (i.e. $-\pi < \arg(z) \leq \pi$).

Note that $\arg\left(z + \frac{1}{\varepsilon} \right)$ is the argument of the vector that joins $-\frac{1}{\varepsilon}$ with z (i.e. the angle made by this vector with the axis R_ε). The arc of circle made by this angle or I_ε will have the length $\frac{1}{\varepsilon} \arg\left(z + \frac{1}{\varepsilon} \right) = \mathrm{Im}_\varepsilon(z)$.

Remark 1. For every $z = a + bi$, we have:

$$\lim_{\varepsilon \to 0} \Re_\varepsilon(z) = \Re(z); \quad \lim_{\varepsilon \to 0} \mathrm{Im}_\varepsilon(z) = \mathrm{Im}(z).$$

Indeed, $\lim_{\varepsilon \to 0} \Re_\varepsilon(z) = \lim_{\varepsilon \to 0} \frac{|z\varepsilon+1|-1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|(a\varepsilon+1)+b\varepsilon i|-1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\sqrt{(a\varepsilon+1)^2+b^2\varepsilon^2}-1}{\varepsilon} =$
$= \lim_{\varepsilon \to 0} \frac{(a\varepsilon+1)^2+b^2\varepsilon^2-1}{\varepsilon\left(\sqrt{(a\varepsilon+1)^2+b^2\varepsilon^2}-1\right)} = \lim_{\varepsilon \to 0} \frac{a^2\varepsilon^2+2a\varepsilon+b^2\varepsilon^2}{\varepsilon\left(\sqrt{(a\varepsilon+1)^2+b^2\varepsilon^2}-1\right)} = \frac{2a}{\sqrt{1}+1} = a = \Re(z)$.
Also, $\lim_{\varepsilon \to 0} \mathrm{Im}_\varepsilon(z) = \lim_{\varepsilon \to 0} \frac{\arg(z\varepsilon+1)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \arg((1+a\varepsilon)+b\varepsilon i) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \arctan \frac{b\varepsilon}{1+a\varepsilon} =$
$b =$
$= \mathrm{Im}(z)$.

Example 1. For $\varepsilon = \frac{1}{n}$, we have

$$C_\varepsilon = \left\{ z \in \mathbb{C} \middle| z \neq -n \right\};$$

$$R_\varepsilon = \left\{ z \in \mathbb{R} \middle| z > -n \right\};$$

$$A_\varepsilon = \left\{ z \in \mathbb{R} \middle| z < -n \right\};$$

$$I_\varepsilon = \left\{ z \in \mathbb{C} \middle| \left| z + n \right| = n \right\}.$$

When $z = -n + 2ni$, we have:

$$\Re_{\frac{1}{n}}(z) = |z + \varepsilon| - \varepsilon = |2ni| - n = n$$

$$\mathrm{Im}_{\frac{1}{n}}(z) = \frac{\arg(z+\varepsilon)}{\frac{1}{n}} = h \arg(2ni) = n\frac{\pi}{2} = \frac{n\pi}{2}$$

Example 2. For $\varepsilon = \frac{1}{n}$, we have

$$C_\varepsilon = \left\{ z \in \mathbb{C} \middle| z \neq -2 \right\};$$

$$R_\varepsilon = \left\{ z \in \mathbb{R} \middle| z > -2 \right\};$$

$$A_\varepsilon = \left\{ z \in \mathbb{R} \middle| z < -2 \right\};$$

$$I_\varepsilon = \left\{ z \in \mathbb{C} \middle| \left| z + 2 \right| = 2 \right\}.$$

When $z = -n + 2ni$, we have:

$$\Re_{\frac{1}{n}}(z) = |z + \varepsilon| - \varepsilon = |2ni| - n = n$$

$$\mathrm{Im}_{\frac{1}{n}}(z) = \frac{\arg(z+\varepsilon)}{\frac{1}{n}} = h \arg(2ni) = n\frac{\pi}{2} = \frac{n\pi}{2}$$

If $z = -3 + i$, then $z\varepsilon + 1 = -\frac{1}{2} + \frac{i}{2}, |z\varepsilon + 1| = \frac{\sqrt{2}}{2}, \Re_{\frac{1}{2}}(z) = \sqrt{2} - 2$.

Definition 3. For any $\omega \in \left(-\frac{\pi}{\varepsilon}\right]$ we define *the purely imaginary Hilger number*, $i_\varepsilon \omega$ such:

$$i_\varepsilon \omega = \frac{e^{i\varepsilon\omega} - 1}{\varepsilon}$$

.

Remark 2. $\lim\limits_{\varepsilon \to 0} i_\varepsilon \omega = i\omega$.

Remark 3. $i_\varepsilon \operatorname{Im}_\varepsilon(z) \in I_\varepsilon, \forall z \in C_\varepsilon$

Indeed, $i_\varepsilon \operatorname{Im}_\varepsilon(z) = \frac{e^{i\varepsilon \operatorname{Im}_\varepsilon(z)} - 1}{\varepsilon}$, $\left|\frac{e^{i\varepsilon \operatorname{Im}_\varepsilon(z)} - 1}{\varepsilon} + \frac{1}{\varepsilon}\right| = \frac{1}{\varepsilon}\left|e^{i\varepsilon \operatorname{Im}_\varepsilon(z)}\right| = \frac{1}{\varepsilon}$.

Definition 4. Addition on C_ε is defined as:

$$z \oplus w = z + w + zw\varepsilon.$$

Proposition 1. $(C_\varepsilon, \oplus)$ is an abelian group.

Proof. We specify that if $z, w \in C_\varepsilon$, then $z \oplus w \in C_\varepsilon$. Indeed, $1 + \varepsilon(z \oplus w) = 1 + \varepsilon(z + w + zw\varepsilon) = (1 + \varepsilon z)(1 + \varepsilon w) \neq 0$. We observe that: $z \oplus 0 = z, \forall z \in C_\varepsilon$.
The symmetric of z is: $\theta z = -\frac{z}{1+z\varepsilon}$.
Indeed: $z \oplus \theta z = z - -\frac{z}{1+z\varepsilon} - \frac{z^2}{1+z\varepsilon}\varepsilon = \frac{1+z^2\varepsilon - z - z^2\varepsilon}{1+z\varepsilon} = 0$.
The commutative is obvious, and the associativity is proven in a canonically mode. □

Remark 4. (1) $\theta(\theta z) = z$;
(2) $\theta(i_\varepsilon \omega) = \overline{i_\varepsilon \omega}$;
Here we give some examples of the symmetrical of some elements:
If $z = 1$ then $\theta z = -\frac{1}{1+\varepsilon}$;
If $z = i$, then $\theta z = \frac{1}{1-\varepsilon}$;
If $z = \frac{1}{\varepsilon}$, then $\theta z = -\frac{1}{2\varepsilon}$.

Theorem 1. For any $z \in C_\varepsilon$, we have:

$$z = \Re_\varepsilon(z) \oplus i_\varepsilon(\operatorname{Im}_\varepsilon z).$$

Proof. $\Re_\varepsilon(z) \oplus i_\varepsilon(\operatorname{Im}_\varepsilon z) = \frac{|z\varepsilon+1|-1}{\varepsilon} + \frac{e^{i\arg(z\varepsilon+1)}-1}{\varepsilon} + \varepsilon\frac{|z\varepsilon+1|-1}{\varepsilon}\frac{e^{i\arg(z\varepsilon+1)}-1}{\varepsilon} =$
$= \frac{z\varepsilon}{\varepsilon} = z$. (We used the fact that $w = |w|e^{i\arg(w)}, \forall w \in \mathbb{C}$) □

Definition 5. For any $n \in \mathbb{N}$ and for any $z \in \mathbb{C}$ we define

$$n \odot z = \underbrace{z \oplus z \oplus \ldots \oplus z}_{n \text{ times}}$$

.

Mathematical induction to verify that: $n \odot z = \frac{(1+\varepsilon z)^n - 1}{\varepsilon}, n \in \mathbb{N}^*$.

Definition 6. Let

$$Z_\varepsilon = \left\{ z \in \mathbb{C} \middle| -\frac{\pi}{\varepsilon} < \operatorname{Im}(z) \le \frac{\pi}{\varepsilon} \right\}.$$

For $\varepsilon > 0$ we define the cylindrical transformation $\zeta_\varepsilon(z) = \begin{cases} \frac{\ln(1+z\varepsilon)}{\varepsilon}, \text{ if } \varepsilon \neq 0 \\ z, \text{ if } \varepsilon = 0 \end{cases}$

where $\ln(1+z\varepsilon) = \ln|1+z\varepsilon| + i\arg(1+z\varepsilon)$. Addition on Z_ε is defined as:

$$z + w = z + w \left(modulo \left(\frac{2\pi i}{\varepsilon} \right) \right).$$

Theorem 2. The cylindrical transformation $\zeta_\varepsilon : (C_\varepsilon, \oplus) \to (Z_\varepsilon, +)$ is a homeomorphism group.

Proof. The case $\varepsilon = 0$ is peddling. For $\varepsilon > 0$ we have:

$$\zeta_\varepsilon(z \oplus w) = \frac{1}{\varepsilon}\ln(1 + (z \oplus w)\varepsilon) = \frac{1}{\varepsilon}\ln\left(1 + z\varepsilon + w\varepsilon + zw\varepsilon^2\right) =$$
$$= \frac{1}{\varepsilon}\ln(1 + z\varepsilon) + \frac{1}{\varepsilon}\ln(1 + w\varepsilon) = \zeta_\varepsilon(z) + \zeta_\varepsilon(w)$$

□

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