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EXACT SOLUTIONS FOR DIFFERENTIAL EQUATIONS IN CELESTIAL MECHANICS

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Abstract. There are many problems in Celestial Mechanics which can be reduced to differential equations, either linear or nonlinear. We expose some of them for which it is possible to obtain the exact solutions. The corresponding differential equations are derived from various models, such as the three-body circular restricted problem or the inverse problem of Dynamics.

These equations can be exposed to the students from high school or from the faculties of sciences in order to understand the importance of the study of differential equations. They will also learn to apply their knowledge to solving problems related to phenomena of real world.

1. INTRODUCTION

The students meet differential equations in high school, at Mathematics or Physics classes. Later on, those who follow the Faculties of Sciences study in detail the theory of existence and uniqueness of the solutions, and various classes of such equations. They learn methods of finding the solutions and do many exercises.

It is very important for them to be motivated by understanding that many real-world phenomena can be modeled through differential equations, and that the methods of integration give a deeper insight into these processes. Examples of second-order linear differential equations, which model electrical circuits, can be found in [1] and [2]. In this paper we present differential equations from the field of Celestial Mechanics. They are solved using only elementary methods, so they are appropriate for undergraduate students, as well as for good high school pupils.

2. GRAVITATIONAL ORBITS ON THE OX AXIS

A classic problem in Celestial Mechanics is the free fall of a particle toward a gravitational source. Following [7], we consider the simple case when the particle moves on a straight line, taken here as Ox axis, to the source situated at $x = 0$. The equation of motion in the presence of the Newtonian attraction law is for $x \geq 0$

$$\ddot{x} + \frac{k}{x^2} = 0, \tag{1}$$

where $k > 0$ corresponds to an attractive force. Multiplying by $\dot{x}$ and integrating we get

$$\frac{1}{2}\dot{x}^2 - \frac{k}{x} = E, \tag{2}$$

where E is a constant, representing the energy. We consider negative energy and we denote $h = -E > 0$.

Since $\dot{x}^2 \geq 0$ and $k > 0$, it follows that $E + k/x \geq 0$, hence $x \leq k/h$. From (2) we get

$$\dot{x} = \pm\sqrt{2}\sqrt{\frac{k}{x} - h}, \quad x > 0. \tag{3}$$

The equation (3) can be written as

$$dt = \pm\frac{dx}{\sqrt{2}\sqrt{\frac{k}{x} - h}},$$

or, substituting $u = hx/k$ $(0 \leq u \leq 1)$

$$\frac{h\sqrt{2h}}{k}dt = \pm\frac{du}{\sqrt{\frac{1}{u} - 1}}. \tag{4}$$

We integrate equation (4), using the substitution $v = \sqrt{\frac{1}{u} - 1}$ to obtain a rational function, and we get for $x_0 = x(0)$

$$F(hx(t)/k) - F(hx_0/k) = \pm\left(\frac{h\sqrt{2h}}{k}\right)t, \tag{5}$$

where

$$F(u) = -\sqrt{u(1-u)} + 1/2 \ \arcsin\,(2u - 1). \tag{6}$$

The $+$ sign corresponds to $\dot{x} > 0$, and the $-$ sign to $\dot{x} < 0$.

In [7], a problem is solved using the result (5)-(6) and real data:

Nuclear waste is left to fall into the Sun, starting with zero velocity. Find the time needed for the waste to reach the Sun's surface.

3. A SIMPLE DIFFERENTIAL EQUATION IN THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

This problem is exposed in [8]. Two bodies, named primaries, move in circular orbits about their center of mass. A third body, with infinitesimal mass, moves in space in the field generated by the two bodies of finite mass, without influencing them. The model roughly applies to a satellite in the gravitational field of the Earth and Moon (whose orbits are almost circular), or to an asteroid in the field of Jupiter and the Sun. It can be also seen as describing the motion of a spacecraft in the field of Jupiter and one of its moons, e. g. Europa.

The two main bodies have a total mass that is normalized to one. Their masses are denoted by $m_1 = 1 - \mu$ and $m_2 = \mu$, respectively. These bodies rotate in the plane Oxy counterclockwise about their common centre of mass and with the angular velocity normalized to one. The third body (satellite, asteroid or spacecraft) moves in three dimensional space and its motion is assumed not to affect the primaries.

A rotating coordinate system is chosen so that the origin is at the centre of mass and the primaries are fixed on Ox at $(-\mu, 0, 0)$ and $(1 - \mu, 0, 0)$, respectively. Let (x, y, z) be the position of the infinitesimal body in the rotating frame.

The system of equations which describes the motion of the infinitesimal body is

$$\begin{aligned} &\ddot{x} - 2\dot{y} = \Omega_x \\ &\ddot{y} + 2\dot{x} = \Omega_y \\ &\ddot{z} = \Omega_z, \end{aligned} \tag{7}$$

with

$$\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu\,(1-\mu)}{2}, \tag{8}$$

where

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x-1+\mu)^2 + y^2 + z^2. \tag{9}$$

The dots denote differentiation with respect to the time.

We simplify the problem by considering the primaries of equal masses $m_1 = m_2 = 1/2$ and by imposing to the third body to move only on Oz. We are left with the equation

$$\ddot{z} + \frac{8z}{\left(4z^2+1\right)^{3/2}} = 0. \tag{10}$$

It admits obviously the trivial solution $z(t) = 0$. We multiply by $\dot{z}$ and integrate to get

$$\dot{z} = \pm\sqrt{c_1 + \frac{4}{\sqrt{4z^2+1}}},$$

hence

$$t = \pm\int_{z_0}^{z(t)} \frac{\sqrt[4]{4u^2+1}}{\sqrt{4 + c_1\sqrt{4u^2+1}}}du,$$

where $z_0 = z(0)$.

4. HOMOGENEOUS POTENTIALS IN THE INVERSE PROBLEM OF DYNAMICS

We consider the following version of the inverse problem for one material point of unit mass, moving in the Oxy inertial Cartesian plane. Given a family of curves

$$f(x,y)=c, \tag{11}$$

find the potentials $V(x,y)$ under whose action, for appropriate initial conditions, the particle will describe the curves of that family. The equations of the motion are

$$\begin{aligned}\ddot{x}&=-V_x\\ \ddot{y}&=-V_y,\end{aligned} \tag{12}$$

where the dots denote derivatives with respect to the time t, and the subscripts partial derivatives.

We emphasize that in this version of the inverse problem a *family* of curves (11) is given, which is in fact determined by the ratio f_y/f_x. Using the functions

$$\begin{aligned}&\gamma=\tfrac{f_y}{f_x},\quad \Gamma=\gamma\gamma_x-\gamma_y,\quad \kappa=\tfrac{1}{\gamma}-\gamma,\\ &\lambda=\tfrac{\Gamma_y-\gamma\Gamma_x}{\gamma\Gamma},\quad \mu=\lambda\gamma+\tfrac{3\Gamma}{\gamma}.\end{aligned} \tag{13}$$

For families of straight lines one has $\Gamma=0$, and details of this special case can be found in [4]. Bozis [5] obtained, for families not consisting in straight lines, a partial differential equation of second order satisfied by the potentials V giving rise to the family (11), namely

$$-V_{xx}+\kappa V_{xy}+V_{yy}=\lambda V_x+\mu V_y. \tag{14}$$

The case of planar orbits in one-variable conservative fields is exposed in [3].

The equation (14) becomes an ordinary one if the potential is homogeneous of degree m,

$$V(x,y)=x^m v(z),$$

where $z=y/x$, and the function f which defines the family is also homogeneous, hence

$$\gamma(x,y)=g(z).$$

It is shown in [6] that the ordinary differential equation satisfied by v is

$$Q_2v''+Q_1v'+Q_0v=0, \tag{15}$$

where

$$\begin{aligned}&Q_2=1-z^2-kz,\\ &Q_1=2mz+k(m-1)+\tfrac{g''}{gg'}(1+zg)(z-g)+\tfrac{g'}{g}(z^2+2zg+3)-2g,\\ &Q_0=-m\left(m+1+\tfrac{g''}{gg'}(1+zg)+\tfrac{zg'}{g}\right),\quad k=\tfrac{1}{g}-g.\end{aligned}$$

The coefficient of v'', Q_2, does not vanish except for two cases: $g=z$ leading to the family of concentric circles $x^2+y^2=c$, and $g=-1/z$, leading to the excluded case of straight lines $y/x = c$. Hence, for all the families different from these two, the differential equation in v is of second order.

Let us consider at first the family of concentric circles $x^2+y^2 = c$, for which $g = z$. Equation (15) becomes of first order

$$(m+2)(z^2+1)v' - m(m+2)zv = 0. \tag{16}$$

For $m = -2$, the equation is identically satisfied, which means that $V(x,y) = v(y/x)/x^2$ is compatible with the family of circles for arbitrary v. For $m \neq -2$, the equation (16) becomes

$$\frac{v'}{v} = m\frac{z}{z^2+1}$$

with the solution $v = c_1(z^2+1)^{m/2}$, hence $V(x,y) = c_1(x^2+y^2)^{m/2}$.

We shall look for potentials homogeneous of degree 4 which satisfy equation (15) and give rise to families of hyperbolae or ellipses.

For the family of hyperbolae $f = 3x^2 - 2y^2$ with $g(z) = -2/3z$, equation (15) reads

$$\left(10z^3 - 15z\right)v'' - \left(66z^2 - 9\right)v' + 144zv = 0, \tag{17}$$

and has the solution $v = c_1(4z^2+9)z^{8/5} + c_2(8z^4+36z^2+3)$. The potential will be $V(x,y) = x^4 v(y/x)$.

For the family of ellipses $f = x^2 + 2y^2$, with $g(z) = 2z$, equation (15) reads

$$\left(2z^3 + z\right)v'' + \left(6z^2 + 9\right)v' - 48zv = 0. \tag{18}$$

Its solution is $v = c_1(4z^2+1)/z^8 + c_2(8z^4+12z^2+5)$ and the potential will be $V(x,y) = x^4 v(y/x)$.

Equations (17) and (18) were solved using the *Maple* symbolic algebra system. It is a good opportunity to remind the students the general form of solutions of homogeneous second-order linear differential equation and to let them check the linear independence of the fundamental solutions by calculating the corresponding Wronskian.

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