

Lecture 5

INTRODUCTION TO FORMAL CONCEPT ANALYSIS

CONCEPTUAL SCALING

- MANY-VALUED CONTEXTS
- WHAT IS A SCALE?
- CONCEPTUAL STRUCTURE OF MANY-VALUED CONTEXTS
- BUILDING A TOSCANA SYSTEM

Multi-valued Contexts

In standard language the word “attribute” refers not only to properties which an object may have or not: attributes like “color”, “weight”, “sex”, or “grade” have *values*. We call them *many-valued attributes* in contrast to the *one-valued attributes* considered so far. (DIN 2330 calls many-valued attributes *Merkmalararten*.)

Def.: A *many-valued context* (G, M, W, I) consists of sets G , M and W and a ternary relation I between G , M and W (i.e., $I \subseteq G \times M \times W$) for which it holds that

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \text{ always implies } w = v.$$

Multi-valued Contexts

- The elements of
 - G are called *objects*, those of
 - M (*many-valued*) *attributes* and those of
 - W *attribute values*.
- $(g, m, w) \in I$ is read as “the attribute m has the value w for the object g ”.
- The many-valued attributes can be regarded as partial maps from G in W . Therefore, it seems reasonable to write $m(g) = w$ instead of $(g, m, w) \in I$.
- The *domain* of an attribute m is defined to be

$$\text{dom}(m) := \{g \in G \mid (g, m, w) \in I \text{ for some } w \in W\}$$

- An attribute m is called *complete*, if $\text{dom}(m) = G$. A many-valued context is *complete*, if all its attributes are complete.

Multi-valued Contexts

Like the one-valued contexts treated so far, many-valued contexts can be represented by tables, the rows of which are labelled by the objects and the columns labelled by the attributes:

		M	
		┌───────────┐	
		m	
		└───────────┘	
G {	g		
			$m(g)$

The entry in row g and column m then represents the attribute value $m(g)$. If the attribute m does not have a value for the object g , there will be no entry.

Multi-valued Contexts: “Drive Concepts for Motorcars”

The multi-valued context shows a comparison of the different possibilities of arranging the engine and the drive mechanism of a motorcar.¹



	De	DI	R	S	E	C	M
conventional	poor	good	good	understeering	good	medium	excellent
front-wheel	good	poor	excellent	understeering	excellent	very low	good
rear-wheel	excellent	excellent	very poor	oversteering	poor	low	very poor
mid-engine	excellent	excellent	good	neutral	very poor	low	very poor
all-wheel	excellent	excellent	good	understeering/neutral	good	high	poor

De := drive efficiency empty; DI := drive efficiency loaded; R := road holding/handling properties; S := self-steering efficiency; E := economy of space; C := cost of construction; M := maintainability;

¹Source: Schlag nach! 100 000 Tatsachen aus allen Wissensgebieten. BI Verlag Mannheim, 1982

Conceptual Scaling

How can we assign concepts to a many-valued context?

We do this in the following way:

The many-valued context is *transformed* into a one-valued one, in accordance with certain rules, which will be explained in the following. The concepts of this *derived* one-valued context are then *interpreted* as the concepts of the many-valued context. This interpretation process, however, called *conceptual scaling*, is not at all uniquely determined. The concept system of a many-valued context depends on the scaling. This may at first be confusing but has proven to be an excellent instrument for a purposeful evaluation of data.

In the process of scaling, first of all each attribute of a many-valued context is interpreted by means of a context. This context is called *conceptual scale*.

Conceptual Scaling

Def.: A *scale* for the attribute m of a many-valued context is a (one-valued) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called *scale values*, the attributes are called *scale attributes*.

$$\mathbb{S}_R :=$$

	++	+	--
excellent	×	×	
good		×	
very poor			×

Every context can be used as a scale. Formally there is no difference between a scale and a context. However, we will use the term “scale” only for contexts which have a clear conceptual structure and which bear meaning. Some particularly simple contexts are used as scales over and over and again.

Conceptual Scaling: “Drive Concepts for Motorcars”

	De	DI	R	S	E	C	M
conventional front-wheel	poor	good	good	understeering	good	medium	excellent
rear-wheel	good	poor	excellent	understeering	excellent	very low	good
mid-engine	excellent	excellent	very poor	oversteering	poor	low	very poor
all-wheel	excellent	excellent	good	neutral	very poor	low	very poor
	excellent	excellent	good	understeering/neutral	good	high	poor

The following one-valued context is obtained as the derived context of the multi-valued context above, if we use the following scales:

$$S_{De} := S_{DI} :=$$

	++	+	-
excellent	×	×	
good		×	
poor			×

$$S_R :=$$

	++	+	--
excellent	×	×	
good		×	
very poor			×

$$S_S :=$$

	u	o	n	u/n
understeering	×			
oversteering		×		
neutral			×	
understeering/neutral				×

$$S_C :=$$

	vl	l	m	h
very low	×	×		
low		×		
medium			×	
high				×

$$S_E := S_M :=$$

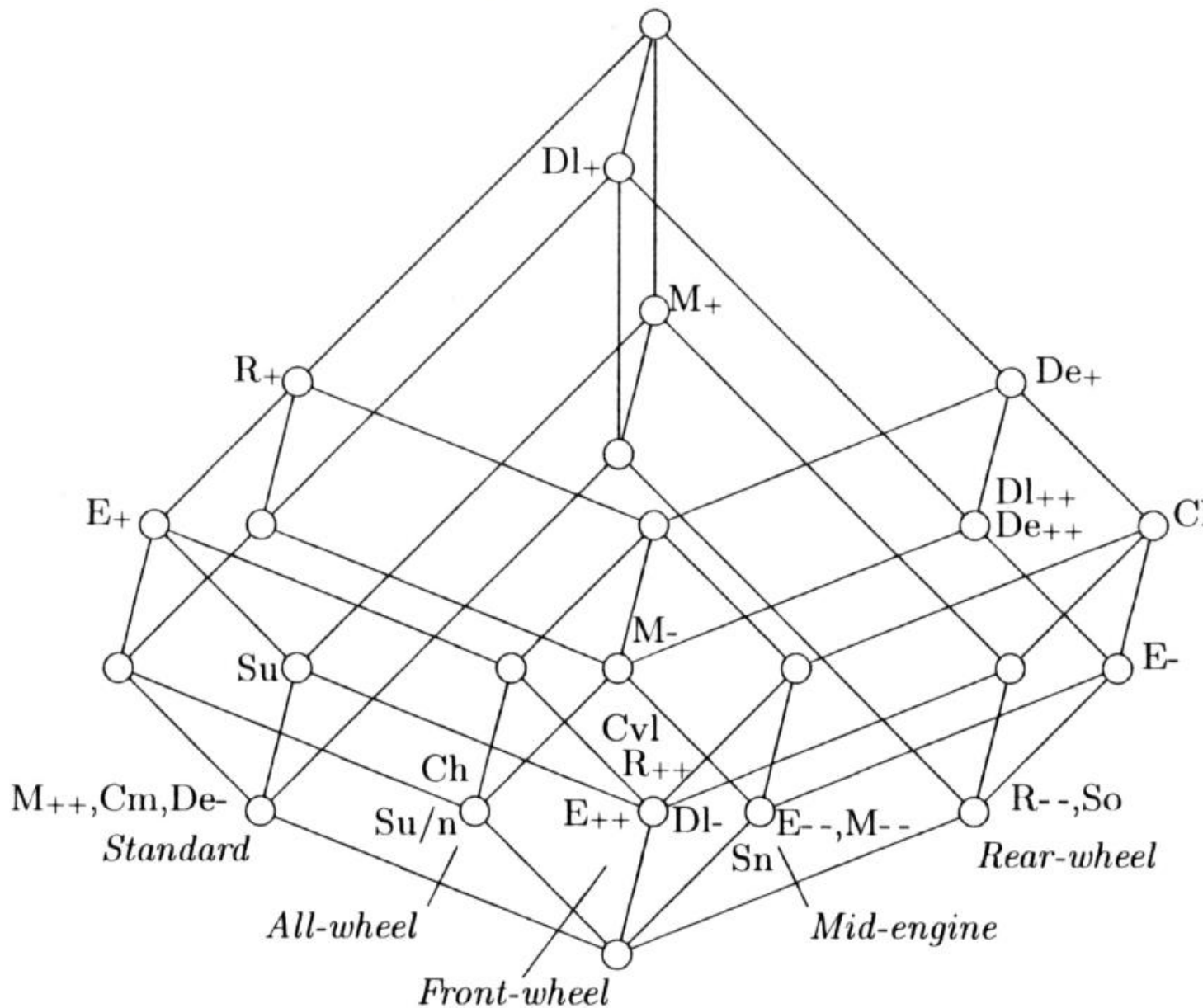
	++	+	-	--
excellent	×	×		
good		×		
poor			×	
very poor			×	×

Conceptual Scaling: “Drive Concepts for Motorcars”

	De			DI			R			S				E				C				M			
	++	+	-	++	+	-	++	+	--	u	o	n	u/n	++	+	-	--	vl	l	m	h	++	+	-	--
conventional			x		x			x		x					x					x		x	x		
front-wheel		x				x	x	x		x				x	x			x	x				x		
rear-wheel	x	x		x	x				x		x					x			x				x		
mid-engine	x	x		x	x			x				x			x	x			x				x	x	
all-wheel	x	x		x	x			x				x		x						x			x		

If we had used the scale \mathbb{S}_E for the attributes De , DI , and R as well, the derived context would have only turned out slightly different.

Conceptual Scaling: "Drive Concepts for Motorcars"



De := drive efficiency empty
 Dl := drive efficiency loaded
 R := road holding/handling properties
 S := self-steering efficiency
 E := economy of space
 C := cost of construction
 M := maintainability

Conceptual Scaling

In the case of *plain scaling* the derived one-valued context is obtained from the many-valued context (G, M, W, I) and the scale contexts $\mathcal{S}_m, m \in M$ as follows:

The object set G remains unchanged, every many-valued attribute m is replaced by the scale attributes of the scale \mathcal{S}_m . If we imagine a many-valued context as represented by a table, we can visualize plain scaling as follows: Every attribute value $m(g)$ is replaced by the row of the scale context \mathcal{S}_m which belongs to $m(g)$.

	De			DI			R			S			E			C			M		
conventional	poor			good			good			understeering			good			medium			excellent		
front-wheel	good			poor			excellent			understeering			excellent			very low			good		
rear-wheel	excellent			excellent			very poor			oversteering			poor			low			very poor		
mid-engine	excellent			excellent			good												very poor		
all-wheel	excellent			excellent			good			under									poor		

	De			DI			R			S			E			C			M						
	++	+	-	++	+	-	++	+	--		++	+	-		++	+	-	l	m	h	++	+	-	--	
conventional			x	x			x			x			x			x			x			x			
front-wheel		x				x	x			x			x			x			x			x			
rear-wheel	x	x		x	x			x		x				x			x		x				x		
mid-engine	x	x		x	x				x		x				x	x		x				x	x		
all-wheel	x	x		x	x				x			x		x							x				

Conceptual Scaling

A detailed description is given in the following definition, for which we first introduce an abbreviation: The attribute set of the derived context is the disjoint union of the attribute sets of the scales involved. In order to make sure that the sets are disjoint, we replace the attribute set of the scale \mathbb{S}_m by

$$\dot{M}_m := \{m\} \times M_m.$$

Def.: If (G, M, W, I) is a many-valued context and $\mathbb{S}_m, m \in M$ are scale contexts, then the *derived context with respect to plain scaling* is the context (G, N, J) with

$$N := \bigcup_{m \in M} \dot{M}_m,$$

and

$$gJ(m, n) :\iff m(g) = w \text{ and } wI_m n.$$

Conceptual Scaling

The formal definition of a context permits turning relations originating from any domain into contexts and examining their concept lattices, i.e., even contexts where an *interpretation* of the sets G and M as “objects” or “attributes” appears artificial.

This is the case with many contexts from mathematics, and in this way we obtain concept lattices which often have structural properties occurring very rarely with empirical data sets.

Nevertheless, these contexts are also of great importance for data analysis. They can be used for example, as “ideal structures” or as scales for the scaling introduced before.

The scales which are used by far most frequently, the *elementary scales* will be introduced now.

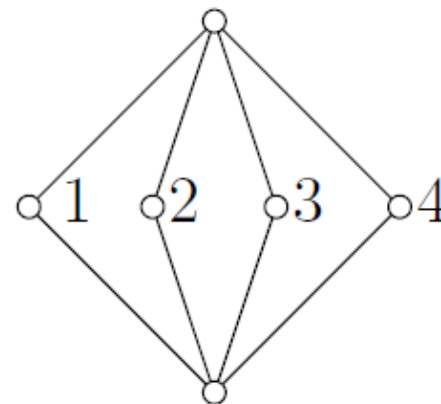
We will use the abbreviation $\mathbf{n} := \{1, \dots, n\}$.

Elementary Scales

Nominal Scales: $\mathbb{N}_n := (\mathbf{n}, \mathbf{n}, =)$

Nominal scales are used to scale attributes, the values of which *mutually exclude* each other. If an attribute, for example, has the values $\{\textit{masculine}, \textit{feminine}, \textit{neuter}\}$, the use of a nominal scale suggests itself. We thereby obtain a *partition* of the objects into extents. In this case, the classes correspond to the values of the attribute.

	1	2	3	4
1	×			
2		×		
3			×	
4				×



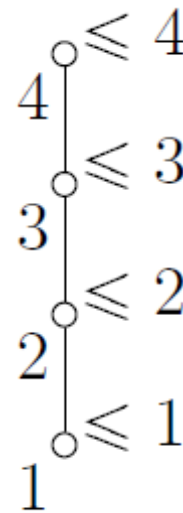
The Nominal Scale \mathbb{N}_4 .

Elementary Scales

Ordinal Scales: $\mathbb{O}_n := (\mathbf{n}, \mathbf{n}, \leq)$

Ordinal scales scale many-valued attributes, the values of which are *ordered* and where each value implies the weaker ones. If an attribute has for instance the values $\{\text{loud}, \text{very loud}, \text{extremely loud}\}$ ordinal scaling suggests itself. The attribute values then result in a chain of extents, interpreted as a *hierarchy*.

	1	2	3	4
1	×	×	×	×
2		×	×	×
3			×	×
4				×



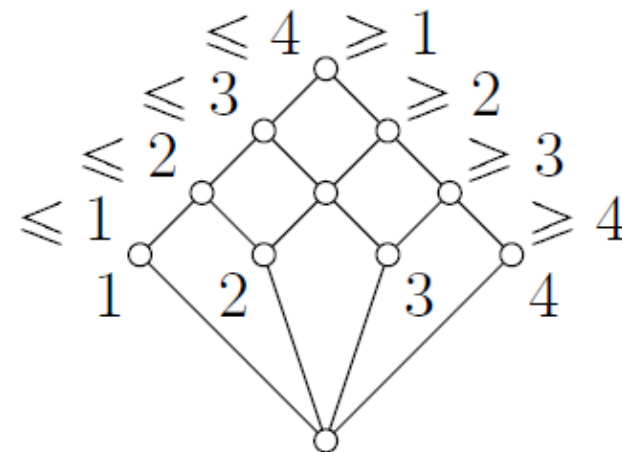
The Ordinal Scale \mathbb{O}_4 .

Elementary Scales

Interordinal Scales: $\mathbb{I}_n := (\mathbf{n}, \mathbf{n}, \leq) \mid (\mathbf{n}, \mathbf{n}, \geq)$

Questionnaires often offer opposite pairs as possible answers, as for example *active–passive*, *talkative–taciturn*, etc., allowing a choice of intermediate values. In this case, we have a *bipolar* ordering of the values. This kind of attributes lend themselves to scaling by means of an interordinal scale. The extents of the interordinal scale are precisely the intervals of values, in this way, the *betweenness relation* is reflected conceptually.

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 1	≥ 2	≥ 3	≥ 4
1	x	x	x	x	x			
2		x	x	x	x	x		
3			x	x	x	x	x	
4				x	x	x	x	x



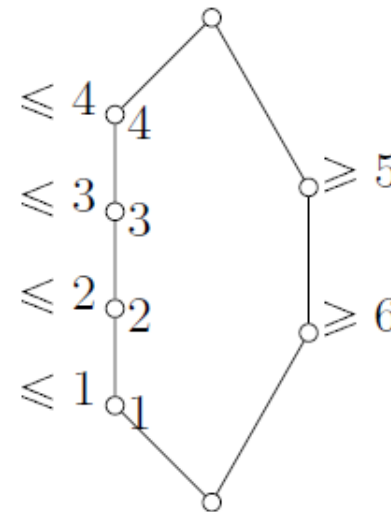
The Interordinal Scale \mathbb{I}_4 .

Elementary Scales

Biordinal Scales: $\mathbb{M}_{n,m} := (\mathbf{n}, \mathbf{n}, \leq) \cup (\mathbf{m}, \mathbf{m}, \geq)$

Colloquially we often use opposite pairs not in the sense of an interordinal scale, but simpler: each object is assigned one of the two poles, allowing graduations. The values $\{\text{very low}, \text{low}, \text{loud}, \text{very loud}\}$ for example suggest this way of scaling: *loud* and *low* mutually exclude each other, *very loud* implies *loud*, *very low* implies *low*. We also find this kind of *partition with a hierarchy* in the names of the school marks: an *excellent* performance obviously is also *very good*, *good*, and *satisfactory*, but not *unsatisfactory* or a *fail*.

	≤ 1	≤ 2	≤ 3	≤ 4	≥ 5	≥ 6
1	x	x	x	x		
2		x	x	x		
3			x	x		
4				x		
5					x	
6					x	x



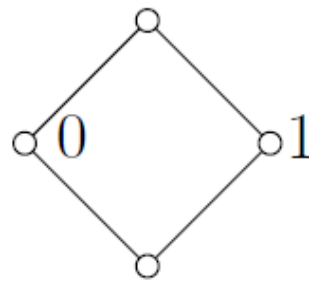
The Biordinal Scale $\mathbb{M}_{4,2}$.

Elementary Scales

The **Dichotomic Scale**: $\mathbb{D} := (\{0, 1\}, \{0, 1\}, =)$

The dichotomic scale constitutes a special case, since it is isomorphic to the scales \mathbb{N}_2 and $\mathbb{M}_{1,1}$ and closely related to \mathbb{I}_2 . It is frequently used to scale attributes with the values of the kind $\{\text{yes}, \text{no}\}$.

	0	1
0	×	
1		×



The Dichotomic Scale \mathbb{D} .

Elementary Scales

A special case of plain scaling which frequently occurs is the case that all many-valued attributes can be interpreted with respect to the same scale or family of scales. Thus we speak of a *nominally scaled context*, if all scales \mathcal{S}_m are nominal scales, etc.

We call a many-valued context *nominal*, if the nature of the data suggests nominal scaling; a many-valued context is called an *ordinal context* if for each attribute the set of values is ordered in a natural way.

Elementary Scales: Example “Forum Romanum”

Forum Romanum		B	GB	M	P
1	Arch of Septimus Severus	*	*	**	*
2	Arch of Titus	*	**	**	
3	Basilica Julia			*	
4	Basilica of Maxentius	*			
5	Phocas column		*	**	
6	Curia				*
7	House of the Vestals			*	
8	Portico of Twelve Gods		*	*	*
9	Temple of Antonius and Fausta	*	*	***	*
10	Temple of Castor and Pollux	*	**	***	*
11	Temple of Romulus		*		
12	Temple of Saturn			**	*
13	Temple of Vespasian			**	
14	Temple of Vesta		**	**	*

Example of an ordinal context: Ratings of monuments on the Forum Romanum in different travel guides (B = Baedeker, GB = Les Guides Bleus, M = Michelin, P = Polyglott). The context becomes ordinal through the number of stars awarded. If no star has been awarded, this is rated zero.

Elementary Scales: Example “Forum Romanum”

Forum Romanum		B *	GB * **		*	M ** ***		GB *
1	Arch of Septimus Severus	×	×		×	×		×
2	Arch of Titus	×	×	×	×	×		
3	Basilica Julia				×			
4	Basilica of Maxentius	×						
5	Phocas column		×		×	×		
6	Curia							×
7	House of the Vestals				×			
8	Portico of Twelve Gods		×		×			×
9	Temple of Antonius and Fausta	×	×		×	×	×	×
10	Temple of Castor and Pollux	×	×	×	×	×	×	×
11	Temple of Romulus		×					
12	Temple of Saturn				×	×		×
13	Temple of Vespasian				×	×		
14	Temple of Vesta		×	×	×	×		×

Elementary Scales: Example "Forum Romanum"

