# Knowledge Discovery with FCA

## Lecture 1: Introduction to Formal Concept Analysis

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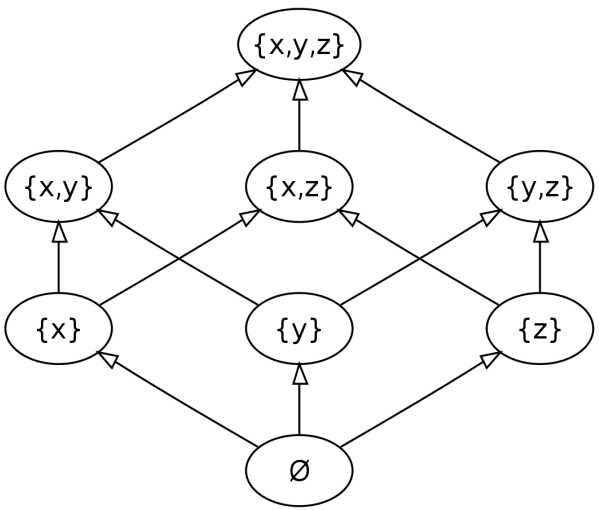
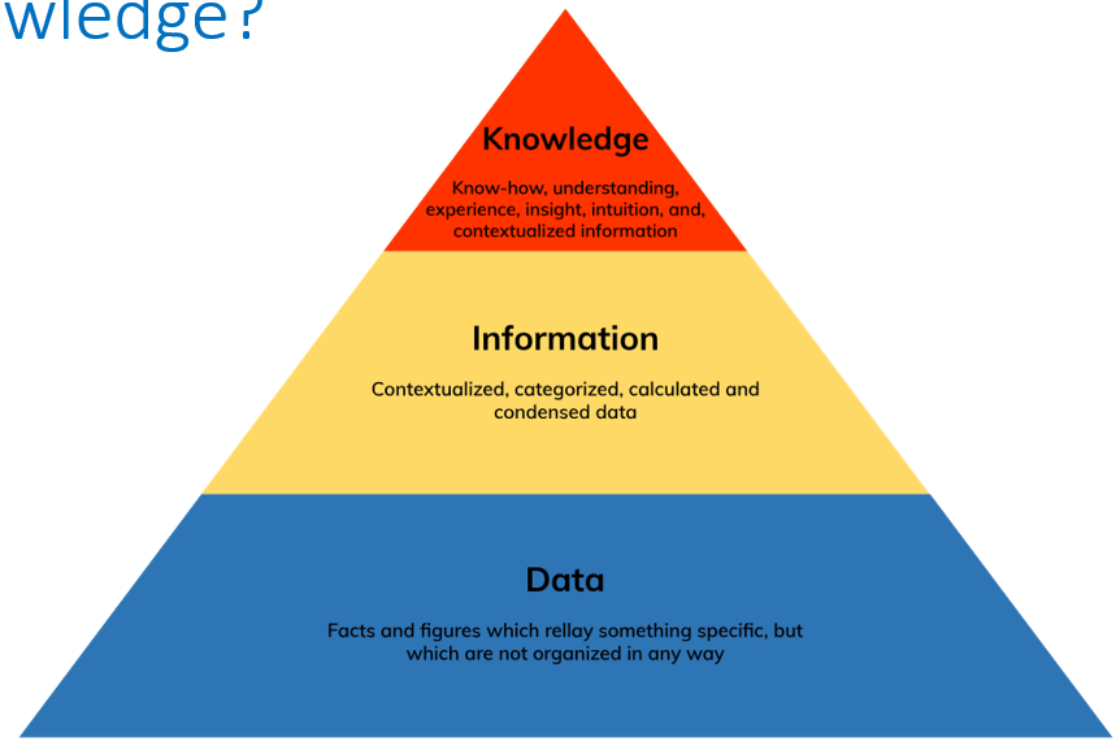


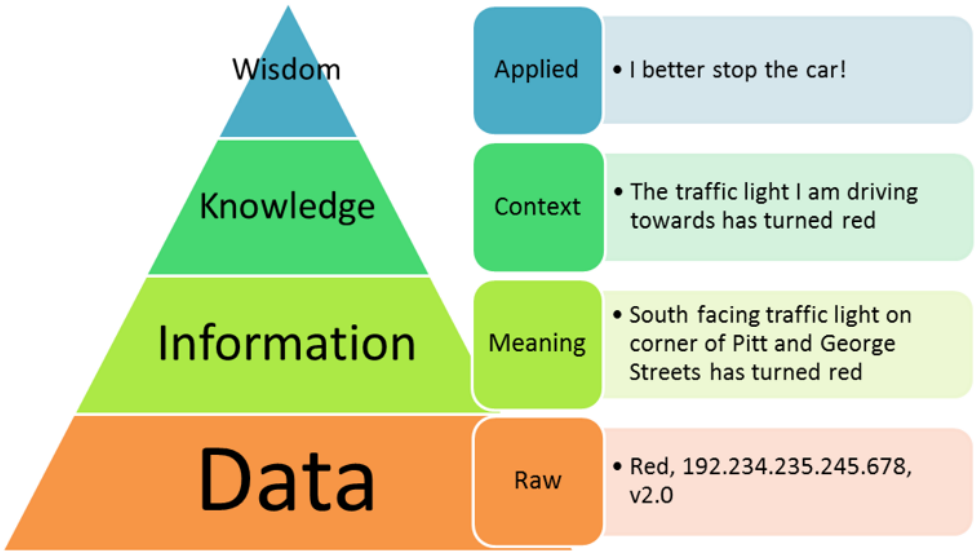
# What is Knowledge?

- Data
- Information
- Knowledge



# What is Knowledge

- Data Mining
- Information Retrieval
- Knowledge Discovery
  - ❑ Knowledge Mining
  - ❑ Knowledge Processing
  - ❑ Knowledge Representation

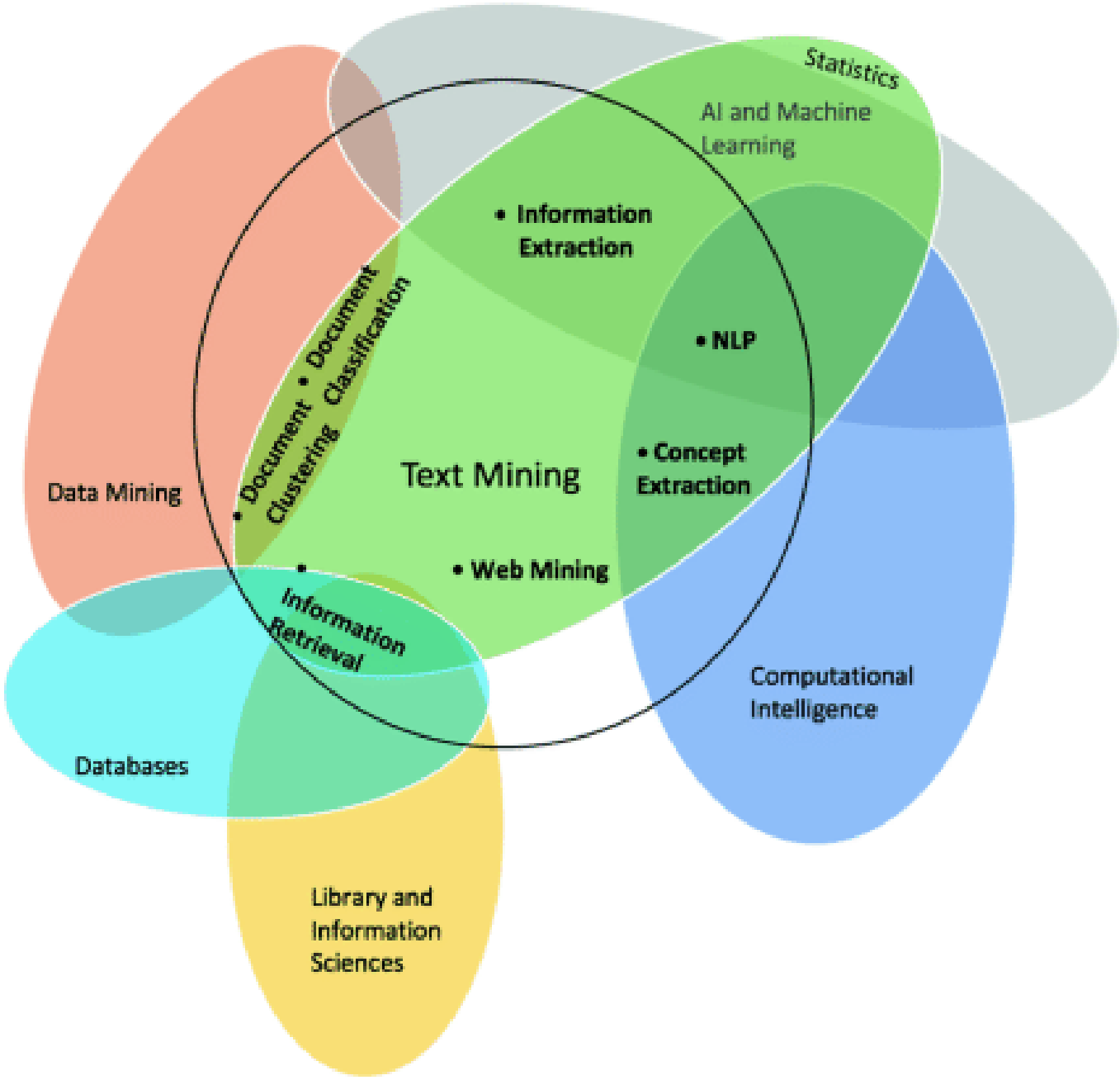


# Data Mining

- Discover patterns in data

# Data Mining so far

1. Define what you are looking for: identify the problem you need to solve
2. Select your approach: clustering, rough sets, rule based mining, SVM, etc.
3. Extract patterns
4. Analyze the results
5. Get Knowledge!, i.e., answer your problem


Statistics
AI and Machine Learning
• Information Extraction
• Document Classification
• Document Clustering
• NLP
• Concept Extraction
Text Mining
Data Mining
• Web Mining
Information Retrieval
Computational Intelligence
Databases
Library and Information Sciences

# Top 10 Algorithms in Data Mining

- The aim of data mining is to make sense of large amounts of mostly unsupervised data, in some domain.
- C4.5, k-Means, SVM, Apriori, EM, PageRank, AdaBoost, kNN, Naive Bayes, and CART

# But what is Knowledge?

# Declarative Knowledge

- Declarative knowledge is to know about something.
- It includes concepts, facts, and objects.
- It is also called descriptive knowledge and expressed in declarative sentences.
- It is simpler than procedural language.

# Procedural Knowledge

- It is also known as imperative knowledge.
- Procedural knowledge is a type of knowledge which is responsible for knowing how to do something.
- It can be directly applied to any task.
- It includes rules, strategies, procedures, agendas, etc.
- Procedural knowledge depends on the task on which it can be applied.

# META-KNOWLEDGE

Knowledge about the other types of knowledge is called Meta-Knowledge.

# Heuristic Knowledge

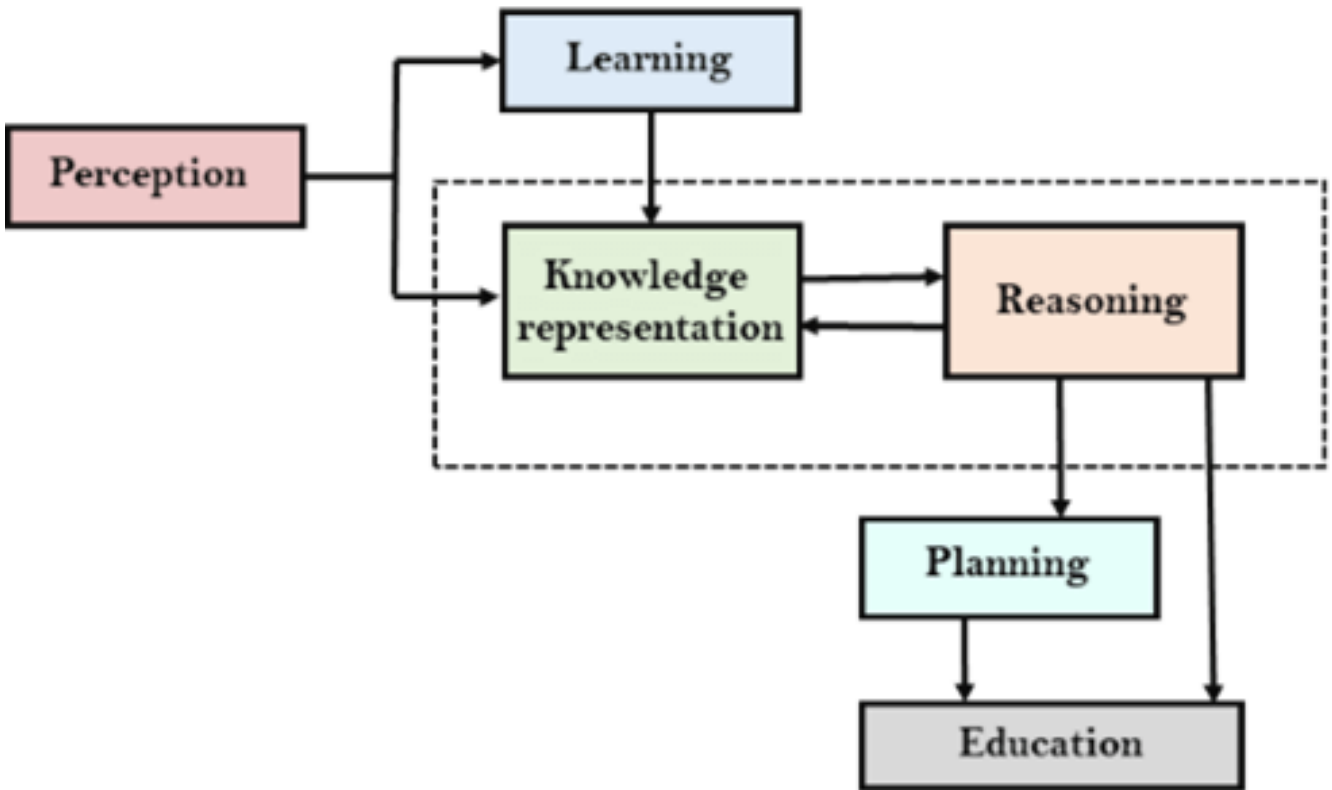
- Heuristic knowledge is representing knowledge of some experts in a filed or subject.
- Heuristic knowledge is rules of thumb based on previous experiences, awareness of approaches, and which are good to work but not guaranteed.

# Structural Knowledge

- Structural knowledge is basic knowledge to problem-solving.
- It describes relationships between various concepts such as kind of, part of, and grouping of something.
- It describes the relationship that exists between concepts or objects.

# AI SYSTEM COMPONENTS

- Perception
- Learning
- Knowledge Representation and Reasoning
- Planning
- Execution



# Approaches to Knowledge Representation

## Simple relational knowledge

- It is the simplest way of storing facts which uses the relational method, and each fact about a set of the object is set out systematically in columns.
- This approach of knowledge representation is famous in database systems where the relationship between different entities is represented.
- In this format, this approach has little opportunity for inference.
- Is there any other possibility?

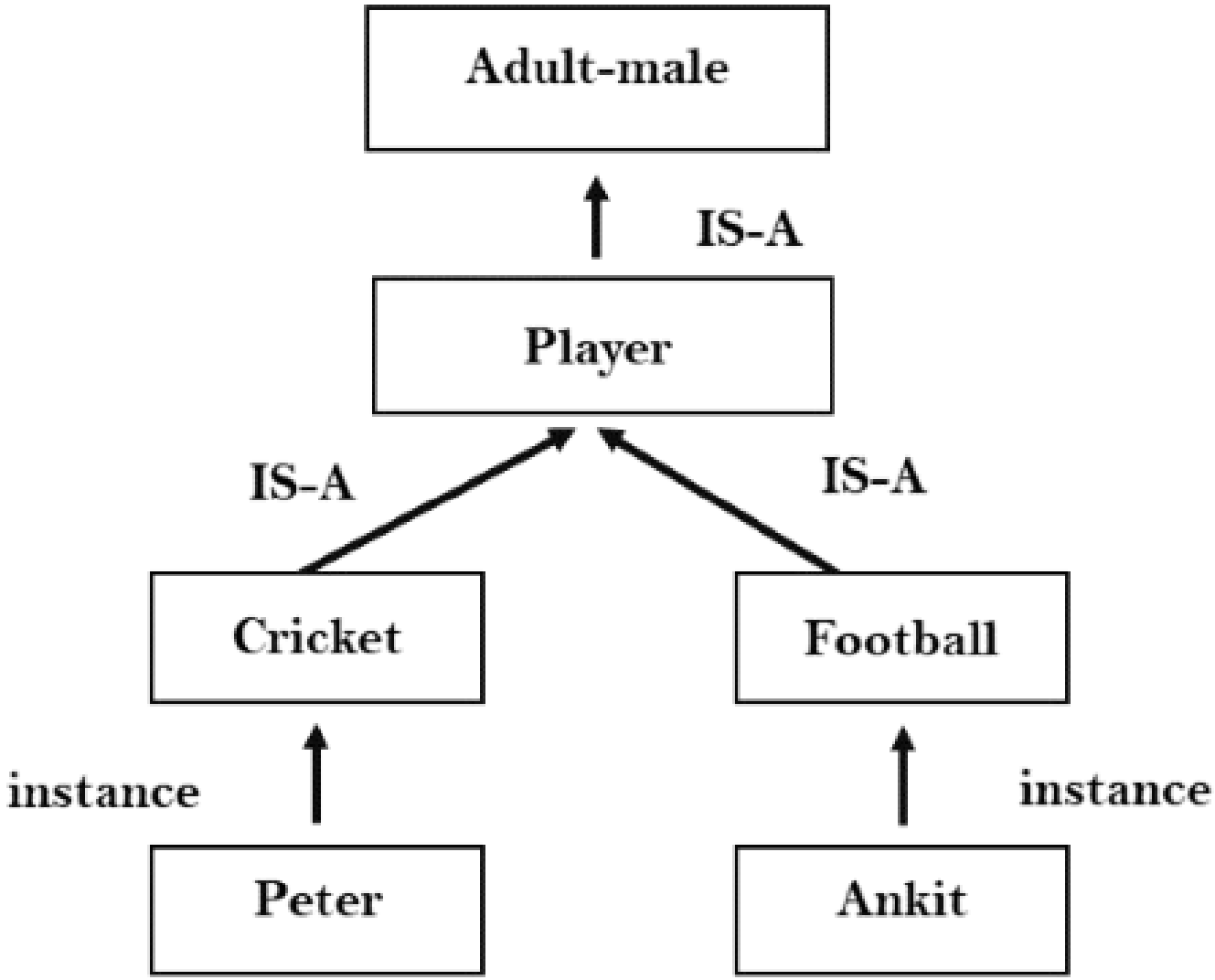
# Simple relational knowledge - Example

| Player | Weight | Age |
|---|---|---|
| Player1 | 65 | 23 |
| Player2 | 58 | 18 |
| Player3 | 75 | 24 |

# Inheritable Knowledge

- In the inheritable knowledge approach, all data must be stored into a hierarchy of classes.
- All classes should be arranged in a generalized form or a hierarchal manner.
- In this approach, we apply inheritance property.
- Elements inherit values from other members of a class.
- This approach contains inheritable knowledge which shows a relation between instance and class, and it is called instance relation.
- Every individual frame can represent the collection of attributes and its value.
- In this approach, objects and values are represented in Boxed nodes.
- We use Arrows which point from objects to their values.

# Inheritable knowledge - Example



# Inheritable knowledge

- Hierarchy!
- Graph based knowledge discovery tools, e.g. Conceptual Graphs

# Inferential Knowlegde

- Inferential knowledge approach represents knowledge in the form of formal logics.
- This approach can be used to derive more facts.
- It guarantees correctness.
- Based on FO Logic

# Procedural Knowledge

- Procedural knowledge approach uses small programs and codes which describes how to do specific things, and how to proceed.
- In this approach, one important rule is used which is If-Then rule.
- In this knowledge, we can use various coding languages such as LISP language and Prolog language.
- We can easily represent heuristic or domain-specific knowledge using this approach.

# What is Formal Concept Analysis?

## Branch of Applied Mathematics and Artificial Intelligence

- Based on Lattice Theory developed by Garrett Birkhoff and others in the 1930s
- Employs algebra in order to formalize notions of concept and conceptual hierarchy
- Term Formal Concept Analysis (short: FCA) introduced by Rudolf Wille in the 1980s.

# Why Formal Concept Analysis

**Because...**

The methods of Formal Concept Analysis offers an algebraic approach to data analysis and knowledge processing.

- Strengths of FCA are
- ... a solid mathematical and philosophical foundation,
- ... more than 2000 research publications,
- ... experience of several hundred application projects,
- ... an expressive and intuitive graphical representation,
- Due to its elementary yet powerful formal theory, FCA can express other methods, and therefore has the potential to unify the methodology of data analysis.


# Applications

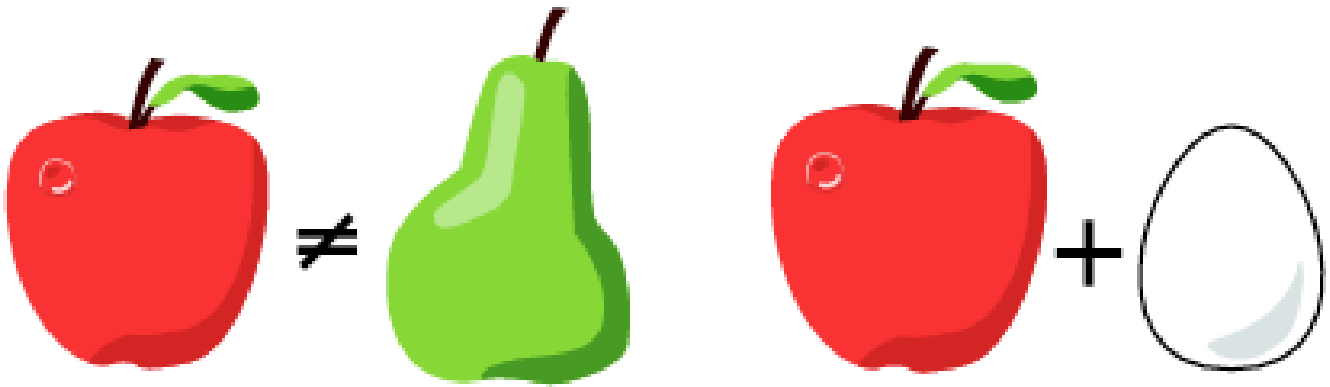
Formal Concept Analysis has recently been applied in

- Description Logics, for checking completeness of knowledge bases,
- Linguistics, for the investigation of thesauri and ontologies,
- Software Engineering, for modelling type hierarchies with role types,
- Biomathematics, for analysing gene expression data, item Machine Learning, for discovering website duplicates,
- Data Mining, for pattern matching problems,
- Rough Set Theory, for studying granular data,
- Web Usage Mining, for discovering usage patterns
- Medicine, etc...

# What is Formal Concept Analysis?

## Formal Concept Analysis (FCA) is a...

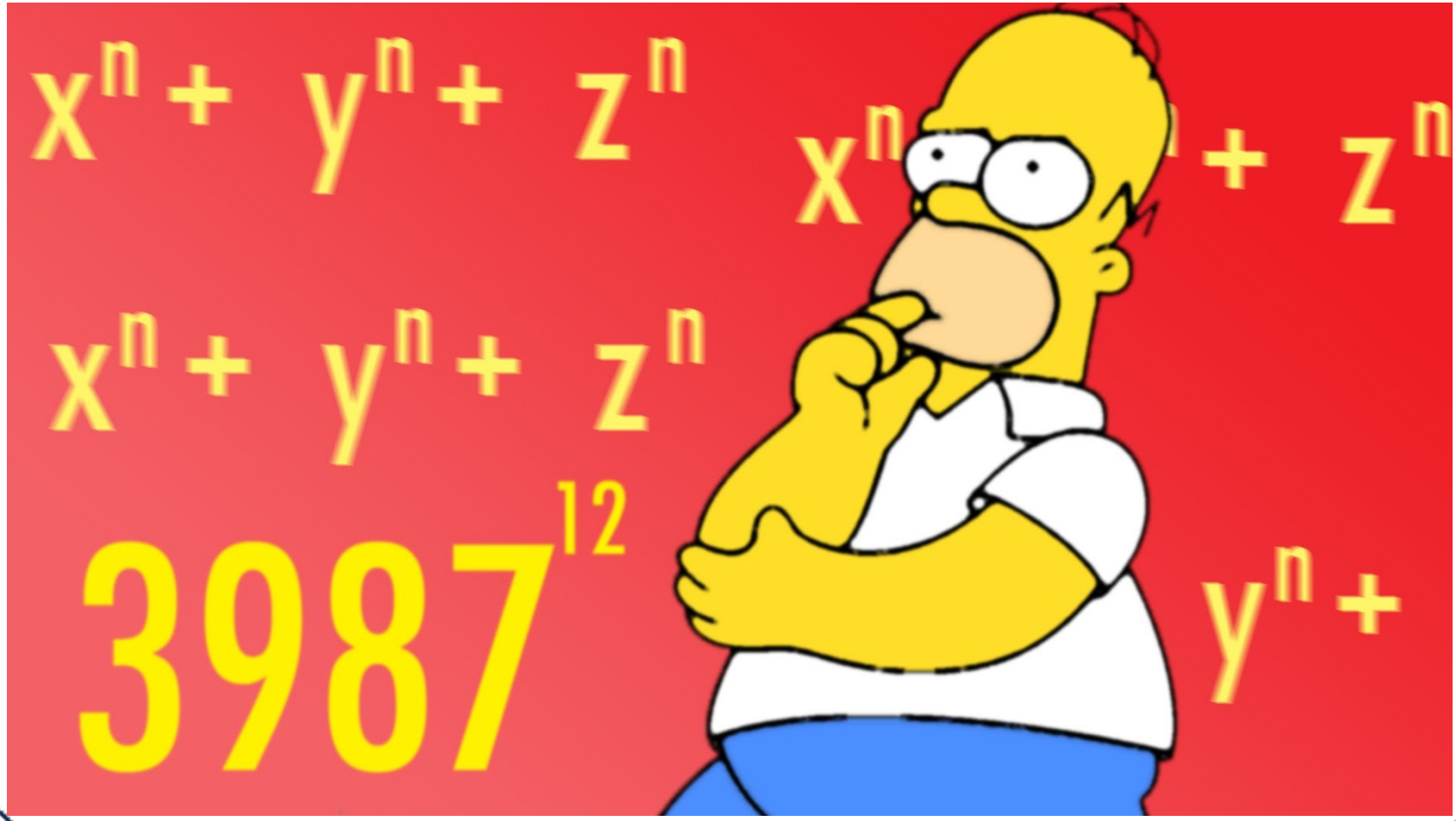
- mathematization/formalization of the philosophical understanding of concepts
- human-centered method to structure and analyze data
- method to visualize data and its inherent structures, implications and dependencies



# Agenda

1. Concept Lattices
   - What is a concept?
   - Formal Context
   - Derivation Operators
   - Formal Concept
   - Concept Lattice
   - Computing All Concepts
   - Drawing Concept Lattices
   - Clarifying and Reducing a Formal Context
   - Interlude: ConExp, FCA Tools Bundle

# What is a concept?



# What is a concept?

Consider the concept bird. What drives us to call something a bird?

- Every object with certain attributes is called bird:
  - A bird has feathers.
  - A bird has two legs.
  - A bird has a bill. ...
- All objects having these attributes are called birds:
  - Duck, goose, owl and parrot are birds.
  - Penguins are birds, too.
  - ...

# What is a concept?

# What is a concept?

We need

- objects
- attributes
- ...
- what else?
- What makes a concept to be a concept?

# What is a concept in FCA?

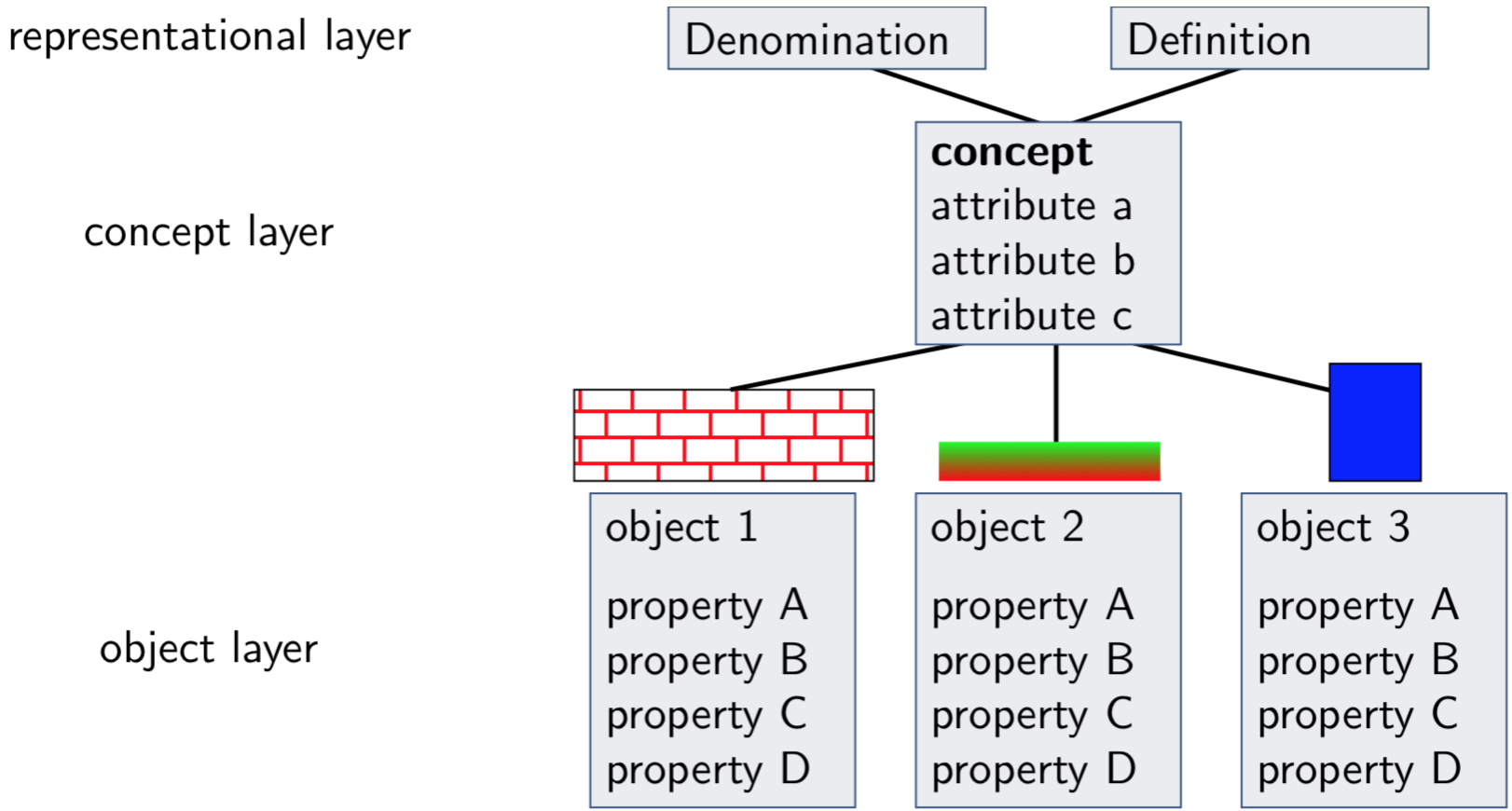
Formal Concept Analysis models concepts as units of thought that consist of two parts:

- The concept extent comprises all objects that belong to the concept.
- The concept inten contains all attributes that all of the objects have in common.

FCA is used, amongst others, in data analysis, information retrieval, data mining and software engineering.

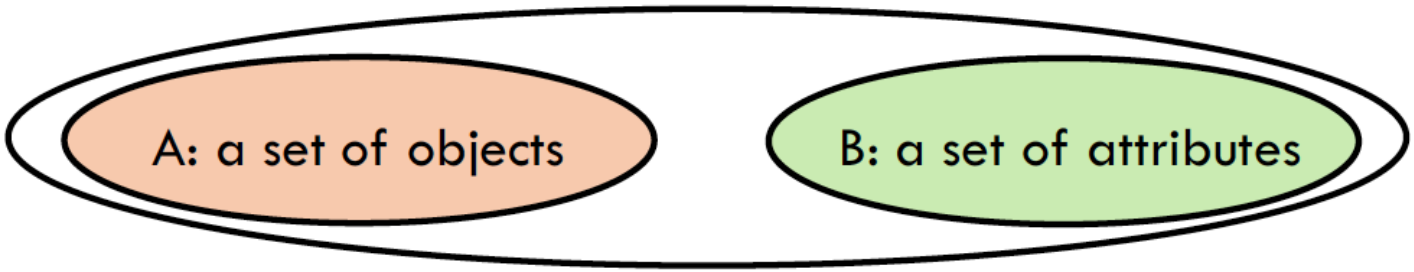
# What is a concept?

FCA is working on the conceptual layer. The representational layer plays only a minor role.



# What is a (formal) concept?

- So, a **formal concept** is constituted by two parts




- ... having a certain relation:
  - every **object** belonging to this concept has <u>all</u> the attributes in B
  - every **attribute** belonging to this concept is shared by <u>all</u> objects in A
- **A** is called the **concept**'s **extent**, **B** is called the concept's **intent**

# THE UNIVERSE OF DISCOURSE

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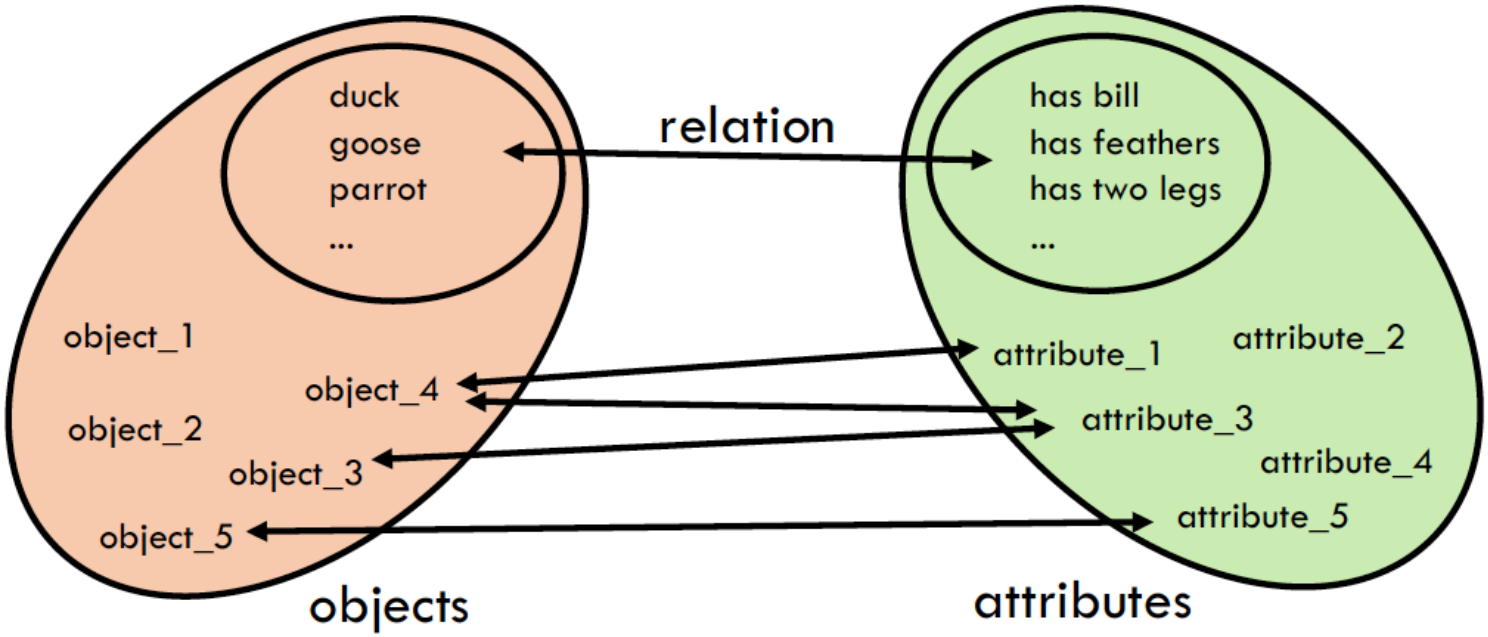
- A repertoire of **objects** and **attributes** (which might or might not be related) constitutes the „context“ of our considerations.



# The (formal) context

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| 𝕂 | small | medium | big | 2legs | 4legs | feathers | hair | fly | hunt | run | swim | mane | hooves |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| dove | x | | | x | | x | | x | | | | | |
| hen | x | | | x | | x | | | | | | | |
| duck | x | | | x | | x | | x | | | x | | |
| goose | x | | | x | | | | | | | | | |
| owl | x | | | x | | | | | | | | | |
| hawk | x | | | x | | | | | | | | | |
| eagle | | x | | x | | x | | x | x | | | | |
| fox | | x | | | x | | x | | x | x | | | |
| dog | | x | | | x | | | | | | | | |
| wolf | | x | | | x | | | | | | | x | |
| cat | x | | | | x | | | | | | | | |
| tiger | | | x | | x | | x | | x | | | | |
| lion | | | x | | x | | x | | x | x | | | |
| horse | | | x | | x | | x | | | x | | | x |
| zebra | | | | | x | | x | | | x | | | |
| cow | | | | | x | | x | | | | | | |

set of **attributes** (M)

crosses indicate **incidence relation** (I) between G and M

(G,M,I) is called **formal context**

set of **objects** (G)

# The (formal) context

**Definition**

*A formal context is a triple $(G, M, I)$, where $G$ is a set of objects, $M$ is a set of attributes, and $I$ is a relation between $G$ and $M$.*

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- What is a relation?
- What types of relations do we know?

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- What is a relation?
- What types of relations do we know?

We read $(g, m) \in I$ as object $g$ has attribute $m$.

# Example 1

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| Sequoia Natl. Park | × | × | × | | | × | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

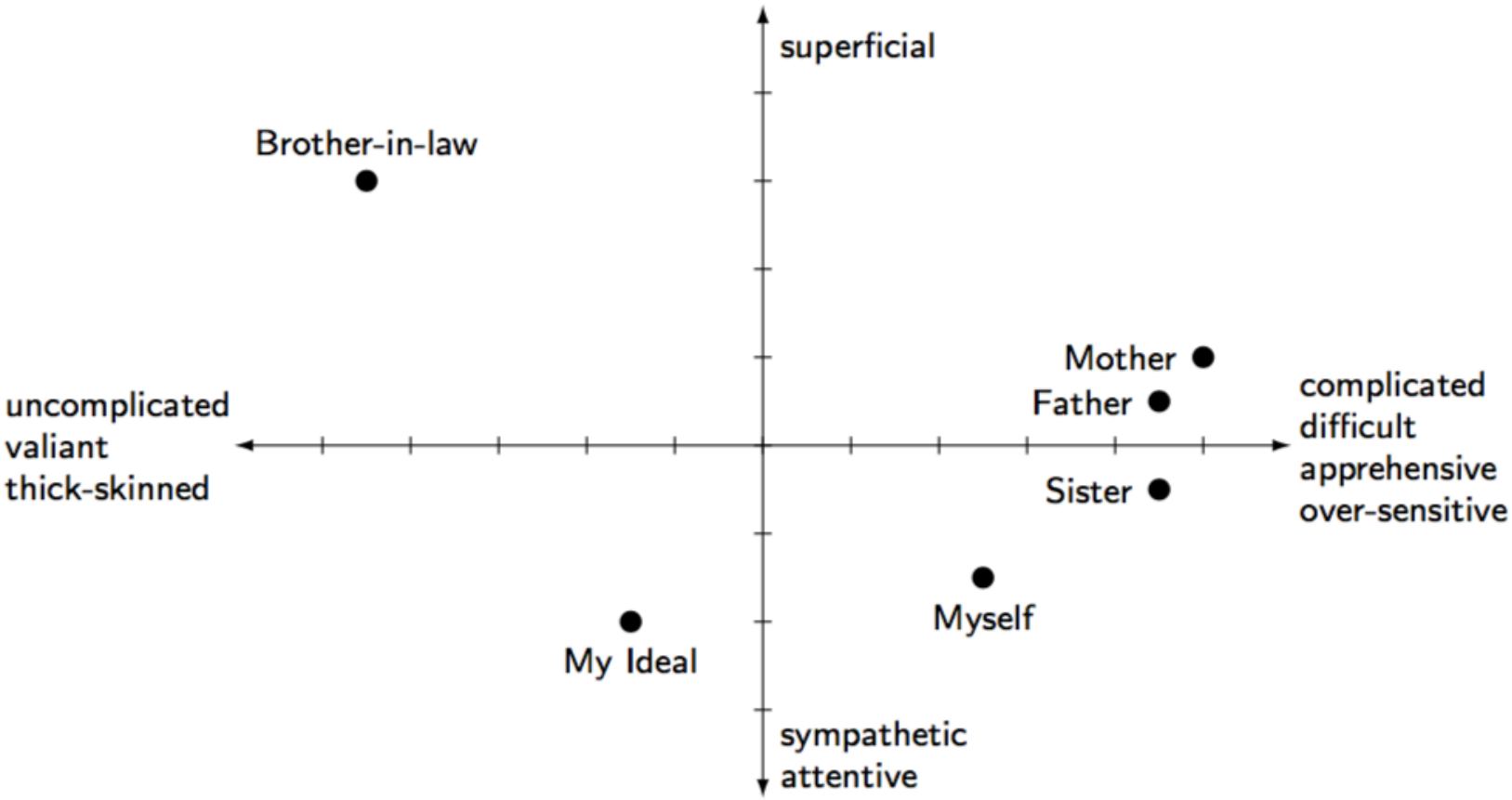

# EXAMPLE 2

| | over-sensitive | withdrawn | self-confident | dutiful | cordial | difficult | attentive | easily offended | calm | apprehensive | chatty | superficial | sensitive | ambitious |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Myself | × | × | × | | × | × | × | | × | × | | | × | × |
| My Ideal | × | | × | × | × | | × | | × | | | | × | × |
| Father | × | × | | × | × | × | × | × | × | × | | × | × | × |
| Mother | × | × | | × | × | × | | × | × | × | | × | × | × |
| Sister | × | × | | × | × | × | × | | × | × | | | × | × |
| Brother-in-law | | | × | × | × | | × | | | | × | × | | × |

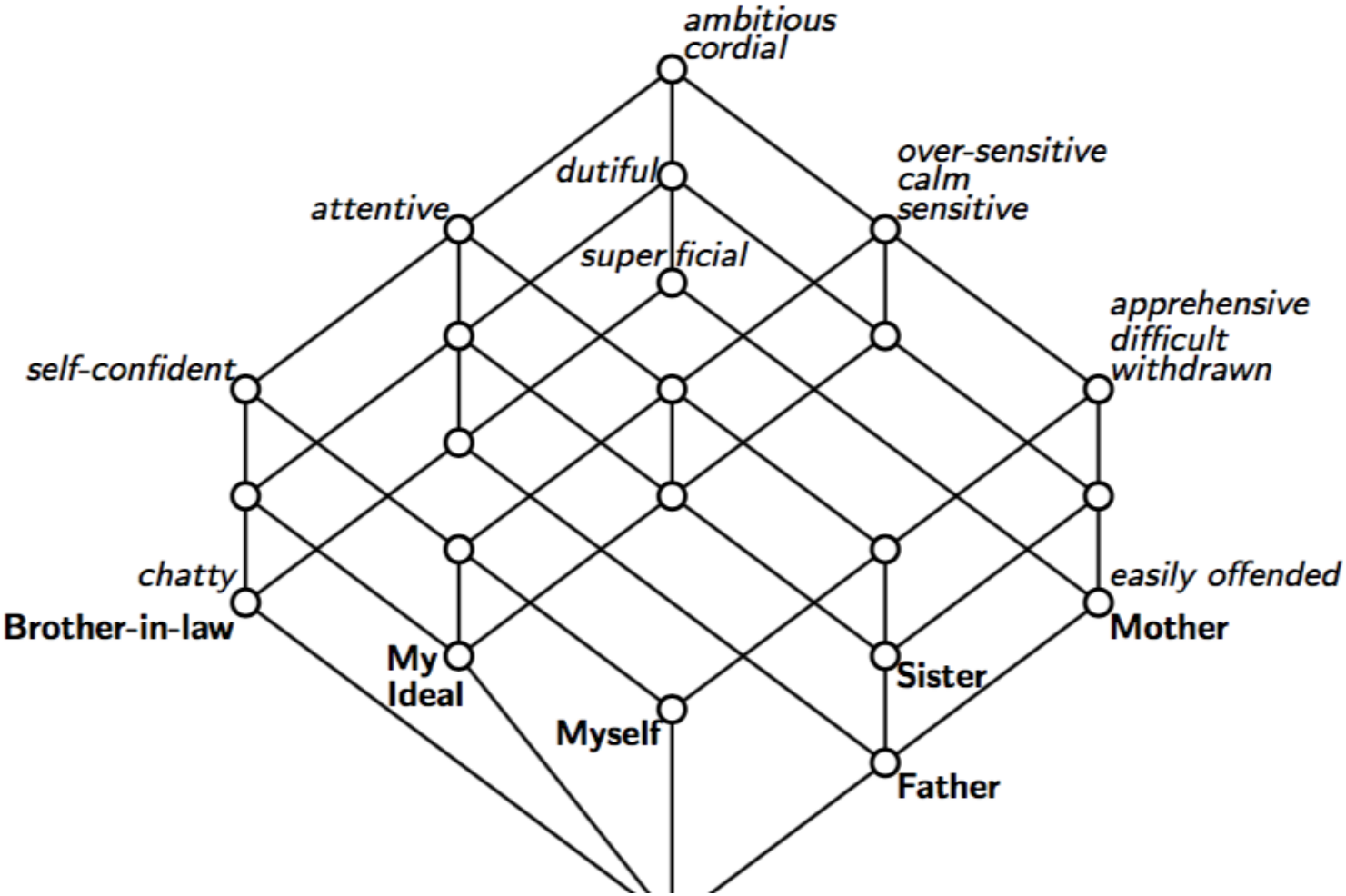
# Data representation

- By a formal context we represent data...
- Nice... but...
- However, why FCA and not SQL, Data Mining, etc?

# Data biplot of interview data



# CONCEPT LATTICE OF INTERVIEW DATA



# UNFOLDING DATA IN A CONCEPT LATTICE

## THE BASIC PROCEDURE OF FORMAL CONCEPT ANALYSIS:

- Data is represented in a very basic data type, called formal context.
- Each formal context is transformed into a mathematical structure called concept lattice. The information contained in the formal context is preserved.
- The concept lattice is the basis for further data analysis. It may be represented graphically to support communication, or it may be investigated with with algebraic methods to unravel its structure.

# What is a concept lattice?

- Graphical diagram
- Is it a graph? Yes/No?

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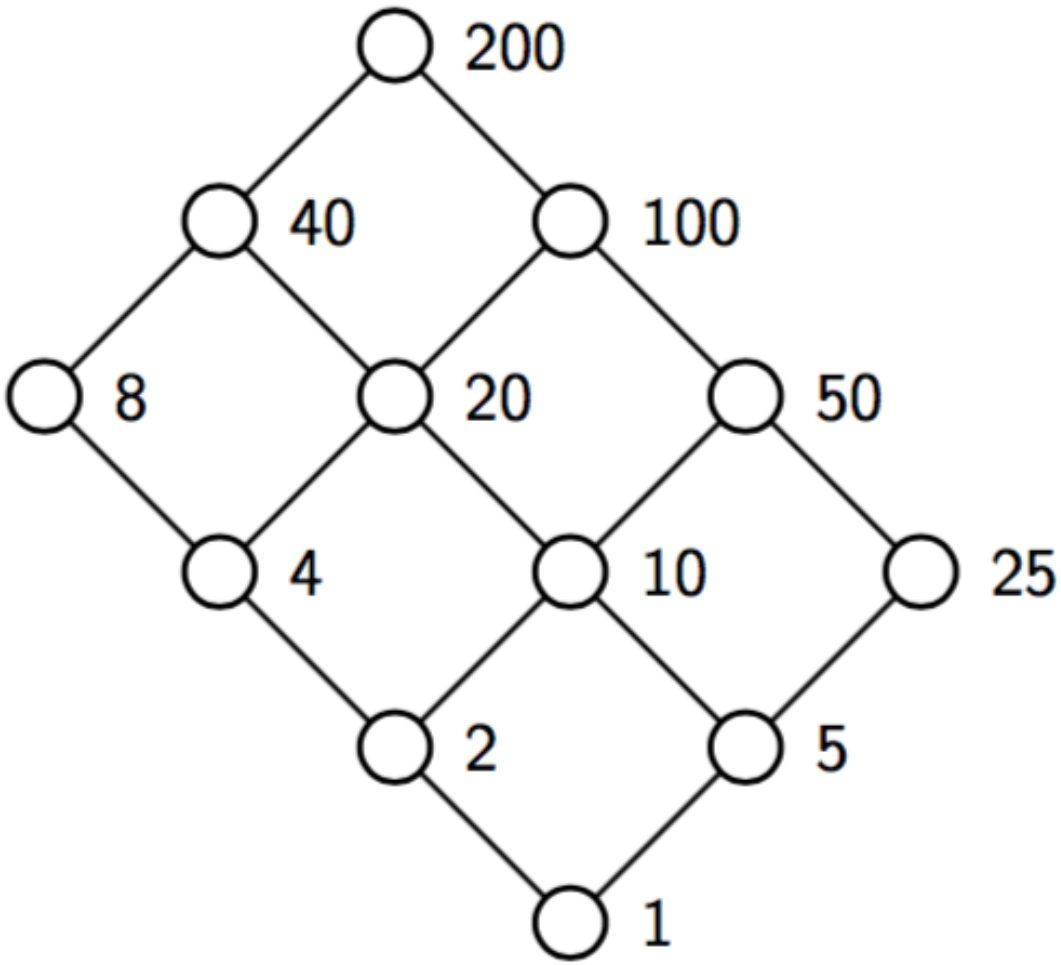
- Is it a graph? Yes/No?
- Order diagram?
- What is an order?

# What is a concept lattice?

- Graphical diagram
- Is it a graph? Yes/No?
- Order diagram?
- What is an order?
- What is a lattice?

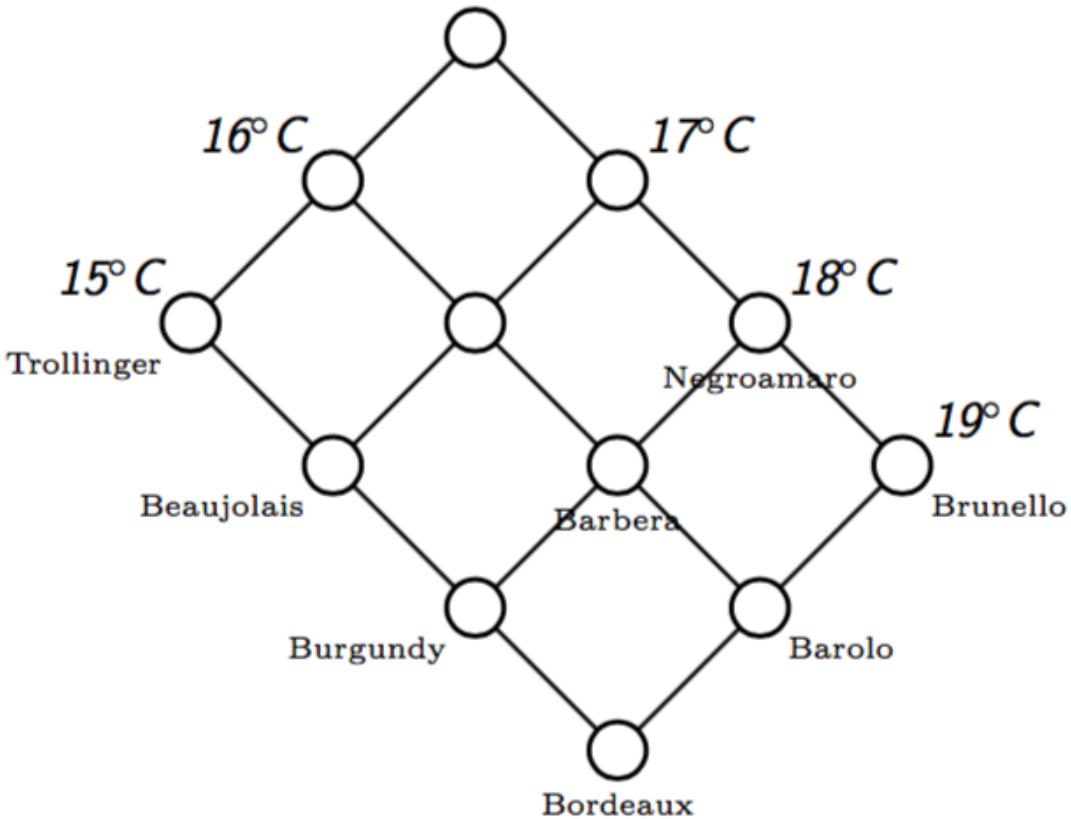
# More examples

## Divisor lattice of 200



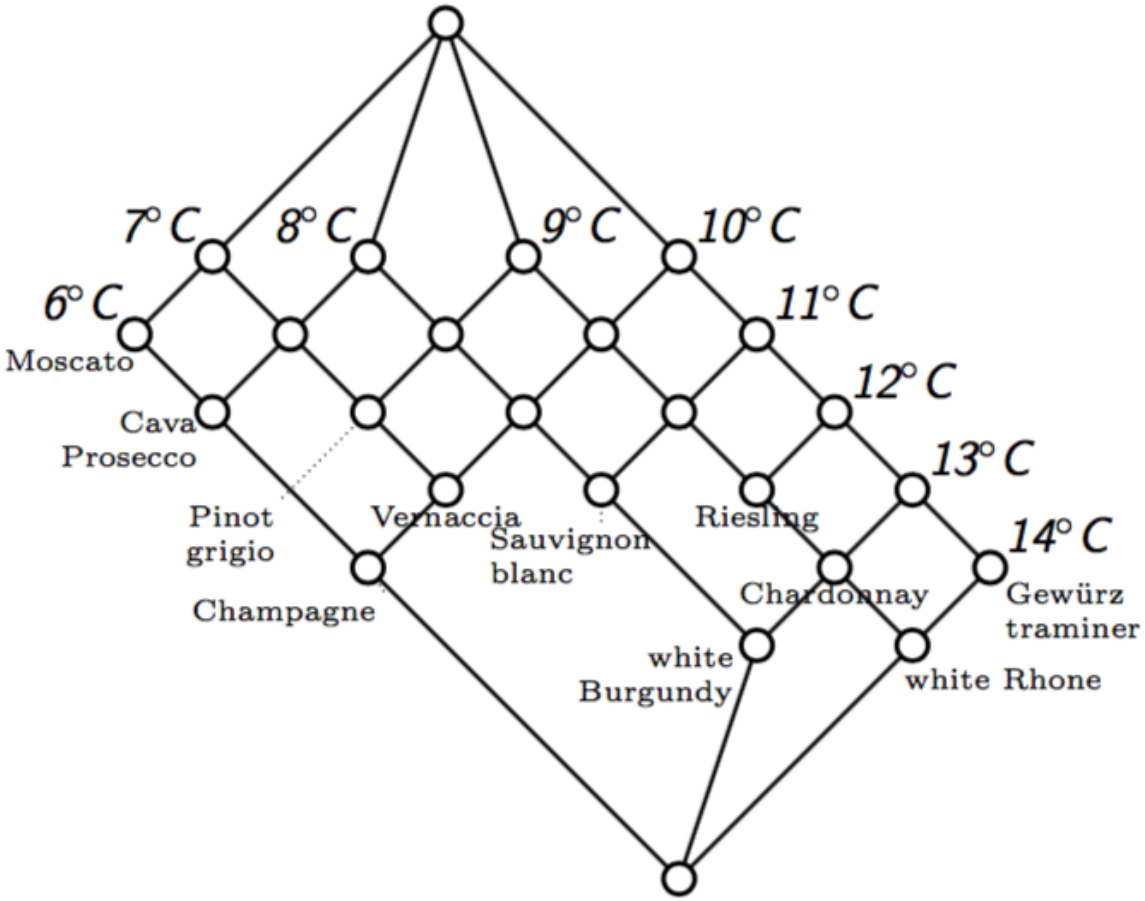
# More examples

## Recommended serving temperature for red wines



# More examples

## Recommended serving temperature for white wines



# How do we compute concepts?

For the mathematical definition of formal concepts we introduce the derivation operator $'$.

For a set of objects $A \subseteq G$, $A'$ is defined as:

$$A' = \{\text{all attributes in M common to the objects of A}\}.$$

For a set of attributes $B \subseteq M$, $B'$ is defined as:

$$B' = \{\text{objects in G having all attributes of B}\}.$$

- We are looking for pairs $(A, B)$ of objects $A$ and attributes $B$ that satisfy the conditions

$$A' = B \text{ and } B' = A$$

and we call these pairs formal concepts.

# Derivation operators

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| Sequoia Natl. Park | × | × | × | | | × | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

$A$

# Derivation Operators

For $A \subseteq G$ we define $A' := \{m \in M \mid \forall g \in A : (g, m) \in I\}$.

For $B \subseteq M$ we define $B' := \{g \in G \mid \forall m \in B : (g, m) \in I\}$.

($X'$ is spoken "X prime")

$A'$

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| $A$ { Sequoia Natl. Park | × | × | × | | | × | | × |
| $A$ { Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

# Derivation Operators: Properties

For $A, A_1, A_2 \subseteq G$

- $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$
- $A \subseteq A''$
- $A' = A'''$

holds.

For $B, B_1, B_2 \subseteq M$

- $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$
- $B \subseteq B''$
- $B' = B'''$

holds.

$A'$

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
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| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| $A$ { Sequoia Natl. Park | × | × | × | | | × | | × |
| $A$ { Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

# Formal Concept

**Def.:** A *formal concept* is a pair $(A, B)$ with

- $A \subseteq G$ and $B \subseteq M$
- $A' = B$
- $B' = A$

$A$ is the *extent* and $B$ the *intent* of the concept.

intent

extent

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| Sequoia Natl. Park | × | × | × | | | × | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

# Formal Concept

**Lemma:** $(A, B)$ is a formal concept iff $A \subseteq G$, $B \subseteq M$ and $A$ and $B$ are both maximal with respect to $A \times B \subseteq I$.
I.e., every concept corresponds to a maximal rectangle in the relation $I$.

**Def.:** The set of all concepts of $(G, M, I)$ is depicted as $\mathfrak{B}(G, M, I)$.

intent

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| Sequoia Natl. Park | × | × | × | | | × | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

extent

# Formal Concept: Subconcept and Superconcept

The blue concept is a *subconcept* of the yellow concept because

- the blue extent is contained in the yellow extent
- ($\Leftrightarrow$ the yellow intent is contained in the blue intent)

**Def.:**

$(A_1, B_1) \leqslant (A_2, B_2)$
$:\Leftrightarrow A_1 \subseteq A_2$
$(\Leftrightarrow B_1 \supseteq B_2)$

| National Parks in California | NPS Guided Tours | Hiking | Horseback Riding | Swimming | Boating | Fishing | Bicycle Trail | Cross Country Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | × | × | |
| Channel Islands Natl. Park | | × | | × | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | × | |
| Devils Postpile Natl. Mon. | × | × | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | × | × | × | | × | × | |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | × | × | × | | | × | | × |
| Lassen Volcanic Natl. Park | × | × | × | × | × | × | | × |
| Lava Beds Natl. Mon. | × | × | | | | | | |
| Muir Woods Natl. Mon. | | × | | | | | | |
| Pinnacles Natl. Mon. | | × | | | | | | |
| Point Reyes Natl. Seashore | × | × | × | × | | × | × | |
| Redwood Natl. Park | × | × | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | × | × | × | × | × | × | | |
| Sequoia Natl. Park | × | × | × | | | × | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | × | × | × | × | × | × | | |
| Yosemite Natl. Park | × | × | × | × | × | × | × | × |

# Concept Lattice

## (Recapitulation: Partial Order)

**Def. (recap.):** $(A_1, B_1) \leqslant (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$

**Def.:** The set of all concepts $\mathfrak{B}(G, M, I)$ together with the partial order $\leqslant$ is the *concept lattice* of the context $(G, M, I)$ and is depicted with $\underline{\mathfrak{B}}(G, M, I)$.

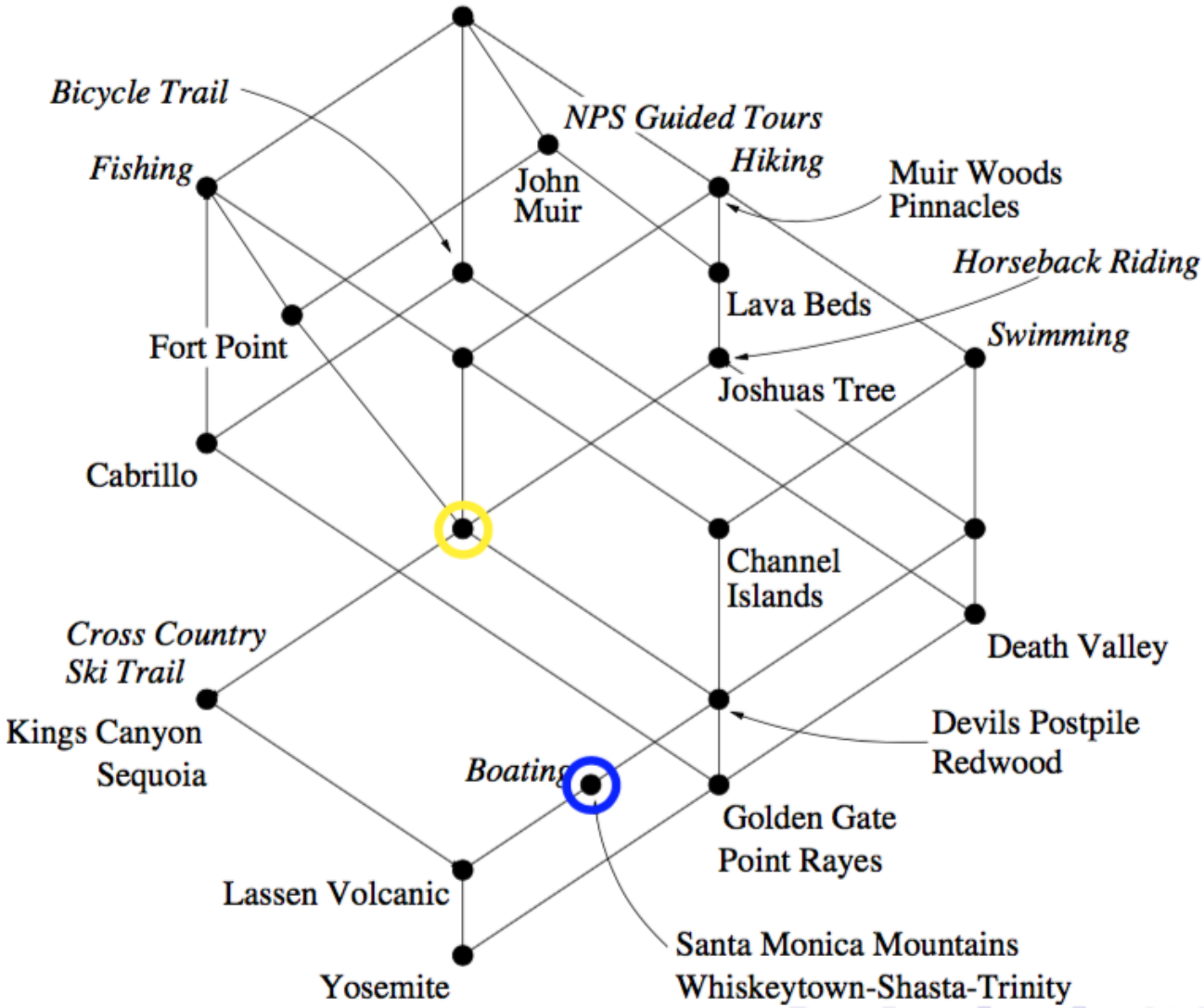
**Def.:** A binary relation $R \subseteq P \times P$ on a set $P$ is a *partial order* if it is

- reflexive (i.e., $xRx$ for all $x \in P$),
- antisymmetric (i.e., $xRy$ and $yRx$ implies $x = y$ for all $x, y \in P$), and
- transitive (i.e., $xRy$ and $yRz$ implies $xRz$ for all $x, y, z \in P$).

# Concept Lattice: as Line Diagram

The *concept lattice* for the national park context.

| National Parks in California | Bicycle Trail | Cross Country Ski Trail | Fishing | Hiking | Horseback Riding | Swimming | Boating | Bicycle Trail |
|---|---|---|---|---|---|---|---|---|
| Cabrillo Natl. Mon. | | | | | | | | × |
| Channel Islands Natl. Park | | | × | | | × | | |
| Death Valley Natl. Mon. | × | × | × | × | | | | × |
| Devils Postpile Natl. Mon. | | | × | × | | × | | |
| Fort Point Natl. Historic Site | × | | | | | × | | |
| Golden Gate Natl. Recreation Area | × | | × | × | | × | × | × |
| John Muir Natl. Historic Site | × | | | | | | | |
| Joshua Tree Natl. Mon. | × | × | × | | | | | |
| Kings Canyon Natl. Park | | × | × | | | | | × |
| Lassen Volcanic Natl. Park | | | | × | × | × | × | × |
| Lava Beds Natl. Mon. | × | | | | | | | |
| Muir Woods Natl. Mon. | | | × | | | | | |
| Pinnacles Natl. Mon. | | | | | | | | |
| Point Reyes Natl. Seashore | × | | × | × | | × | | × |
| Redwood Natl. Park | × | | × | × | | × | | |
| Santa Monica Mts. Natl. Recr. Area | | | | × | | × | × | |
| Sequoia Natl. Park | × | × | × | | | | | × |
| Whiskeytown-Shasta-Trinity Natl. Recr. Area | | | | × | | × | × | |
| Yosemite Natl. Park | | | | × | | × | × | × |



# Concept Lattice: Dual Context

**Def.:** Let $(G, M, I)$ be a context. Then $(M, G, I^{-1})$ with
$(m, g) \in I^{-1} \iff (g, m) \in I$
is the *dual context* of $(G, M, I)$.

**Theorem:** Its concept lattice is isomorphic to $(\mathfrak{B}(G, M, I), \geqslant)$.

*Remark:* In general, $G$ and $M$ need not be disjunct, they can even be identical.