Knowledge Acquisition with Implications

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Welcome students!

Professor Christian Sacarea has asked me to give an internet lecture for you.

This is an honour for me,

and I hope it will be fun for you!

In case of technical difficulties, you may send me an email under

`bernhard.ganter@tu-dresden.de`

You may also use this email address for later questions etc.

Introducing myself

I am Bernhard Ganter, a retired full professor of mathematics from TU Dresden, one of eleven „Universities of Excellence“ in Germany.

I have also been a member of the Computer Science Department, giving lectures for CS students on a regular basis.

I know Prof. Sacarea very well. We had the same Ph.D. supervisor.

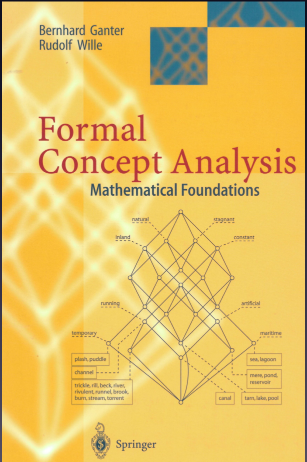
And I have visited Cluj twice, and enjoyed it very much.

Research

My main field of research is called

Formal Concept Analysis.

Actually, I have written a book on this topic (together with Rudolf Wille), which seems to be quite popular:



Prerequisites

Christian Sacarea told me that you already know most of the important things in that book,

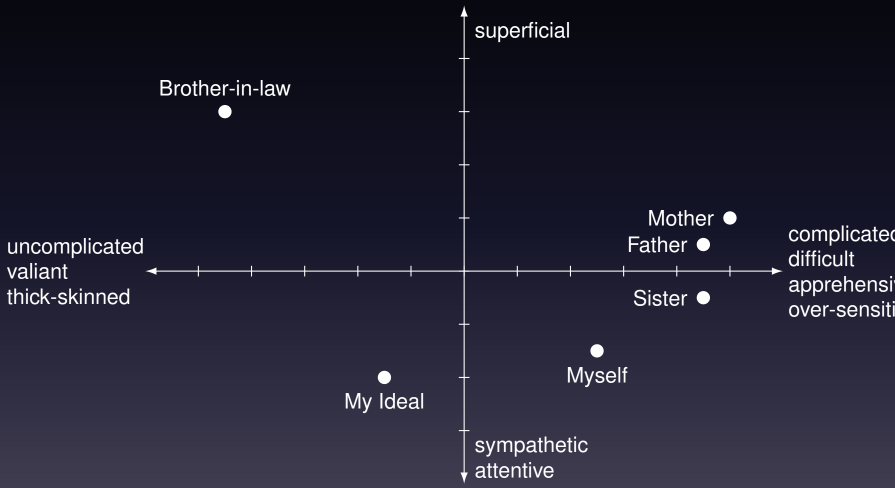
in particular, that you know what a **formal context** is, how a **formal concept** is defined, and how the formal concepts form the **concept lattice**.

This is good to know, but I shall repeat some of the important notions anyway. And I will start with some very old slides, with which I always start.

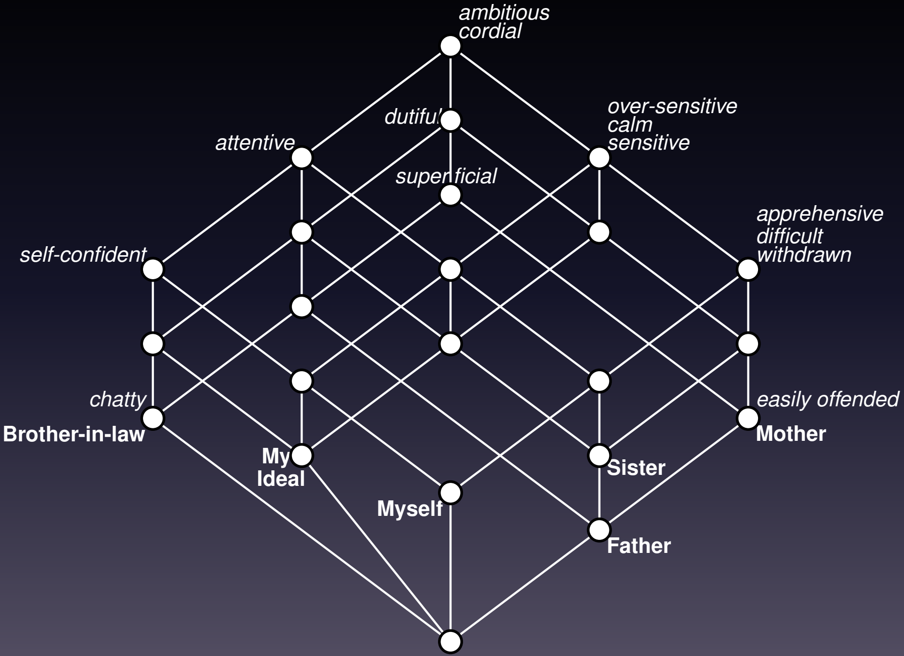
From a treatment of Anorexia nervosa

	over-sensitive	withdrawn	self-confident	dutiful	cordial	difficult	attentive	easily offended	calm	apprehensive	chatty	superficial	sensitive	ambitious
Myself	×	×	×		×	×	×		×	×			×	×
My Ideal	×		×	×	×		×		×				×	×
Father	×	×		×	×	×	×	×	×	×		×	×	×
Mother	×	×		×	×	×		×	×	×		×	×	×
Sister	×	×		×	×	×	×		×	×			×	×
Brother-in-law			×	×	×		×				×	×		×

A biplot of the interview data



A concept lattice of Anorexia data



First implications

From the concept lattice diagram we may read off some simple implications.

I have not yes defined what an „implication“ is. This will be done soon. We start with two examples. The first is

$$\textit{self-confident} \rightarrow \textit{attentive}.$$

We see that in the diagram, *self-confident* is below *attentive*. You can reach *attentive* from *self-confident* by going upwards along the lines of the diagram.

As a consequence, all persons (of that formal context) with the attribute *self-confident* also have the attribute *attentive*.

More expressive implications

Implications such as

$$\textit{self-confident} \rightarrow \textit{attentive}$$

are very simple. Let us consider a more ambitious example. In our diagram, there are only very few such.

We start with the attribute *superficial*. Which of the persons have this attribute? All, except for the *Sister* and *Myself*.

Next, we consider the attribute *withdrawn*. All persons have this attribute, except for *Brother-in-law* and *My Ideal*.

Now we ask, which persons have both attributes. Who is *superficial* and *withdrawn*?

Two-element premise

The persons being both *superficial* and *withdrawn* obviously are *Father* and *Mother*. There are several ways to see that.

We have seen that only *Sister* and *Myself* are not *superficial*. We have also seen that only *Brother-in-law* and *My Ideal* are not *withdrawn*.

So everyone else must be both *superficial* and *withdrawn*. But „everyone else“ are just *Father* and *Mother*.

In the notation of FCA this reads as follows:

$$\{\textit{superficial}, \textit{withdrawn}\}' = \{\textit{Father}, \textit{Mother}\}\ .$$

From the diagram

The same information can be read off from the concept lattice diagram.

The *superficial* persons are those which can be reached from *superficial* along a descending path in the diagram.

The *withdrawn* persons are those which can be reached from *withdrawn* along a descending path in the diagram.

There is a highest point which is under both *superficial* and *withdrawn*. It marks the formal concept

$$(\{\textit{superficial}, \textit{withdrawn}\}', \{\textit{superficial}, \textit{withdrawn}\}'').$$

Its extent is $\{\textit{Father}, \textit{Mother}\}$.

A new conclusion

But note that *Father* and *Mother* share an attribute: they both are *easily offended*!

This attribute is not implied by *withdrawn*, nor by *superficial*. We found something new:
Whoever is both *superficial* and *withdrawn* is also *easily offended*.

In short notation

$$\{\textit{superficial}, \textit{withdrawn}\} \quad \to \quad \{\textit{easily offended}\}.$$

This implication, with $\{\textit{superficial}, \textit{withdrawn}\}$ as its premise and $\{\textit{easily offended}\}$ as conclusion, holds in the formal context.

Formal definition

An **implication** over a set M is a pair of subsets $A, B \subseteq M$, usually denoted by $A \to B$.

A is called the **premise** and B is the **conclusion** of the implication $A \to B$.

The implication $A \to B$ **holds** in a formal context (G, M, I) if

- for every object $g \in G$ it is true that when g has all attributes in A, then g also has all attributes from B,
- or, equivalently: $A' \subseteq B'$,
- or, equivalently: $B \subseteq A''$.

The logic of a formal context

We have discovered a new aspect of formal contexts. Implications are the simplest instance of the *logic of a formal context*.

In logic, our implications would be seen as *Propositional Horn Formulas*. So we are in Propositional Logic, and consider the simplest fragment. Our attributes take the rôle of the propositional variables.

Can we use more expressive logics? Yes, we can, but not today! It is interesting enough to work with implications!

More expressive logics $\longrightarrow$ higher complexity.

The implications of a formal context

By the **implications of a formal context** we mean the set of all implications which hold in that formal context.

Formally:

$$\mathrm{Imp}(G, M, I) \quad := \quad \{A \to B \mid A \subseteq M,\ B \subseteq A''\}.$$

This family of implications is a well defined mathematical object and should be studied!

First ideas:

- What is interesting about implications?
- What could one do, which problems can be solved?
- Algorithms?

First ideas

Let us consider two sources of inspiring examples:

- Mathematical statements, e.g.

  Every finite, connected graph in which all vertices have even degree has a closed Eulerian circuit.

  Mathematicians are often interested in the *unsolved problems*.

- Market basket analysis, e.g.

  If a customer buys ice cream and diapers, then he/she also buys beer.

  Such *association rules* usually have additional parameters like *support* and *confidence*, since very large data sets are considered.

Exploration

Formal Concept Analysis offers a knowledge acquisition method which is, in principle, extremely simple. To explore a domain of knowledge, do this:

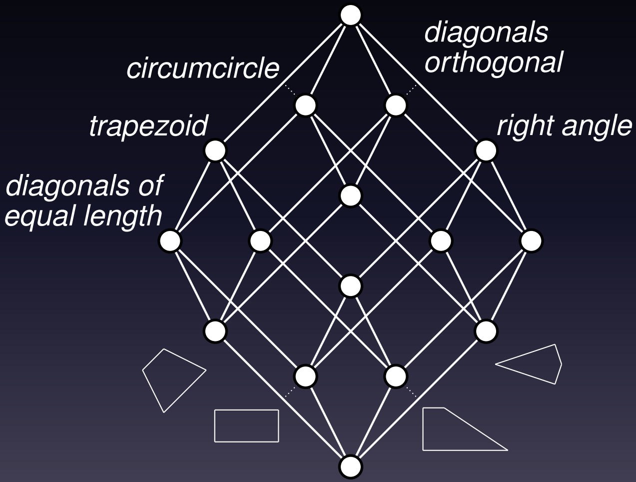
- Give as much information to the computer as you can.
- Let the computer find (and ask) the simplest question, the answer to which cannot be deduced from the given information.
- Answer that question and provide the necessary information justifying your answer.
- Repeat.

Well, some details have to be worked out. But let us first go through a very simple example.

An example from geometry: quadrilaterals

	has a circumcircle	trapezoid	has a right angle	diagonals of equal length	diagonals orthogonal
	×	×	×	×	
	×		×		×
		×	×		×
	×	×		×	×

The concept lattice



The logic of these examples

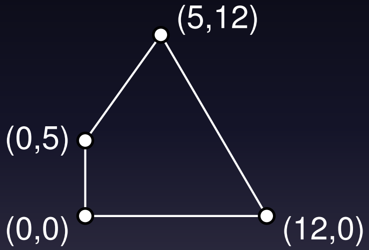
From the diagram one can read off the attribute logic for the chosen examples.

In this case, the implication logic is very simple:

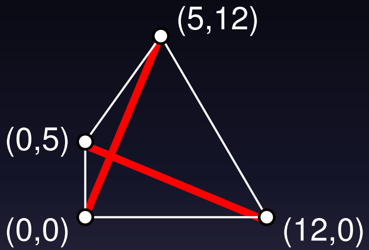
A quadrilateral has diagonals of equal length
if and only if
it is a trapezoid with a circumcircle.

It this a general rule? If not, then the chosen examples are not representative and at least one more must be added.

A counterexample

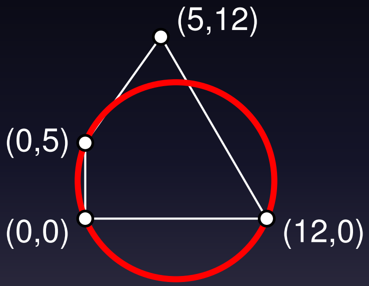


A counterexample



The diagonals are of equal length and orthogonal, and there is a right angle.

A counterexample



But the quadrilateral has no circumcircle.

The extended context of examples

	has a circumcircle	trapezoid	has a right angle	diagonals equal length	diagonals orthogonal
	×	×	×	×	
	×		×		×
		×	×		×
	×	×		×	×
			×	×	×

The new logic is valid in general!

The implications which hold for the new set of examples,

1. each trapezoid with a circumcircle
   has diagonals of equal length,
2. each trapezoid with diagonals of equal length
   has a circumcircle,
3. each quadrilateral having a circumcircle and
   diagonals of equal length
   is a trapezoid,

hold in general. Thus, in this sense the examples are representative, and the exploration is finished.

Properties of quadrilaterals (1986)

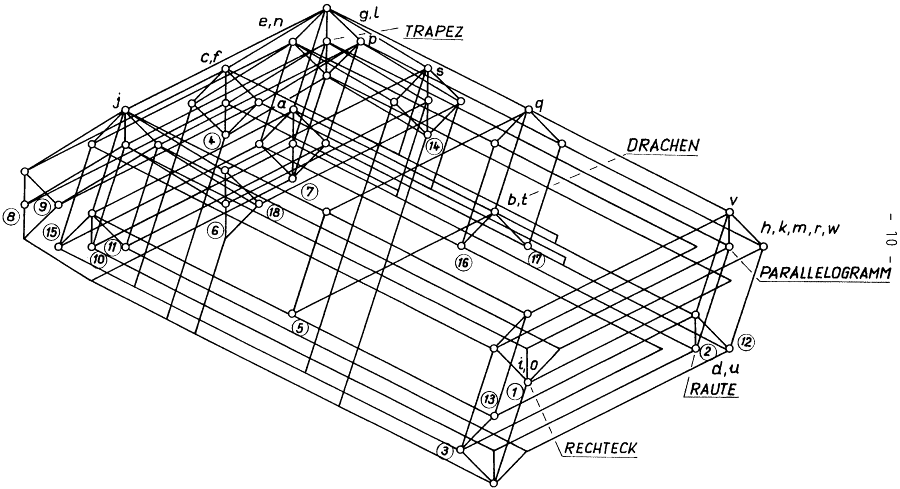


Abb. 6: Begriffsverband zum Kontext aus Abbildung 5a.

23 properties, 18 examples, 36 rules.

Attribute Exploration strategy

- Select the set M of attributes to be explored, and the domain from which the examples may be drawn.
- Make a formal context (G_0, M, I) of some examples from the domain; (G_0 may be the empty set).
- Make a list $\mathcal{L}_0$ of implications over M which are known to be true ($\mathcal{L}_0$ may be empty).
- Repeat the following:
  - Look for an „unsolved question", i.e., an implication which holds for all examples, but which does not follow from the list of verified implications.
  - If no such unsolved question exists, then the algorithm stops.
  - For the given unsolved question,
    - find a proof and then add the implication to the list of verified implications, or
    - find a counterexample and add it to the context of examples.

Questions, questions . . .

Immediately many questions come into mind about this strategy, e.g.

- What if we cannot solve a problem, can we postpone it?
- Are partial answers possible?
- Are some open problems better than others?
- What is a good algorithm for finding open problems?
- . . . .

There is a general answer to all these problems:

We need theory and algorithms!

Much of this needed theory has already been worked out, and surprisingly good algorithms have been found and implemented.

Enough for today!

Thank you for your attention!