Septimiu Crivei

INJECTIVE MODULES RELATIVE TO TORSION THEORIES

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Relative to Torsion Theories

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To my family

Contents

Preface

Torsion theories for abelian categories have been introduced in the 1960's with the motivation and purpose of unifying a common behavior observed for abelian groups, modules over certain domains or even modules in general. Thus the work of P. Gabriel [45], J.-M. Maranda [73] or S.E. Dickson [39] set the base for an extensive study of torsion theories for the years to come, with significant contributions by many authors. Of special interest are the hereditary torsion theories on the category R-Mod of left R-modules, in the view of the bijective correspondence between them and the localized subcategories of R-Mod, fact that shows the importance of the former in the study of the category R-Mod. The concept of non-commutative localization and Gabriel-Popescu Theorem have become important tools and a lot of results have gained a more natural interpretation in this torsion-theoretic language.

Injectivity and its various generalizations have been intensively studied throughout the years, especially in the attempt to characterize rings by their modules. Many types of injectivity may be characterized by a Baer-type criterion by restricting to a subset of left ideals of the ring, but valuable information may be obtained if that subset of left ideals is a Gabriel filter. The bijection between the Gabriel filters on R-Mod and the hereditary torsion theories on R-Mod suggests that the latter ones are a good framework for studying injectivity, not only as an instrument for localization, but also for its intrinsic properties. Thus results on torsion-theoretic injectivity have been present in the general study of torsion theories right from the beginning.

The purpose of the present work is to offer a presentation of injectivity relative to a hereditary torsion theory τ, with emphasis on the concepts of minimal τ-injective module and τ-completely decomposable module. The core of the book is the author's Ph.D. thesis on this topic.

Since the intention was to make it self-contained from the torsion-theoretic point of view, the book begins with a general chapter on torsion theories. This gathers in the first part some of the most important properties of arbitrary torsion theories and afterwards continues with results in the setting of hereditary torsion theories, insisting on the topics needed in the next stages. Chapter 2 introduces injectivity relative to a hereditary torsion theory τ and the notion of relative injective hull of a module and studies the class of τ-injective modules as well as connections between this relative injectivity and the usual injectivity. The next two chapters contain the main results of the book. Thus Chapter 3 deals with minimal τ-injective modules, that are the torsion-theoretic analogues of indecomposable injective modules. They are used to get information on (the structure of) τ-injective hulls of modules. Throughout Chapter 4 we study some direct sum decompositions and especially τ-completely decomposable modules, that is, direct sums of minimal τ-injective modules. Thus we obtain some (τ-complete) decompositions for τ-injective hulls of modules and we deal with a few important problems on direct summands or essential extensions of τ-completely decomposable modules. The final chapter presents results on τ-quasi-injective modules and generalizes connections between conditions involving τ(-quasi)-injectivity to the setting of a τ-natural class.

The prerequisites of the reader should be some general Ring and Module Theory and basic Homological Algebra, completed with just few notions from Theory of Categories, since I have tried to avoid a categorical language. The book have been thought as an introduction to the concept of torsion-theoretic injectivity, but hopefully some parts may also be useful for the interested researcher.

I would like to thank Professors Ioan Purdea and Andrei Mărcuş for their valuable comments and suggestions.

I am grateful to my father, Professor Iuliu Crivei, for numerous illuminating discussions on the topic and constant encouragement and support.

Septimiu Crivei

Cluj–Napoca, April 2004

Notations

Throughout the text we denote by R an associative ring with non-zero identity and by τ a hereditary torsion theory on the category R-Mod of left R-modules, except for the first chapter, where τ may be an arbitrary torsion theory on R-Mod. All modules are left unital R-modules, unless stated otherwise. By a homomorphism we understand an R-homomorphism. By a class of modules we mean a non-empty class of modules.

ACC	the ascending chain condition
$\mathrm{Ann}_R B$	the left annihilator in R of a subset $B \neq \emptyset$ of a module A, that is, $\mathrm{Ann}_R B = \{r \in R \mid rb = 0\,, \forall b \in B\}$
$\mathrm{Ann}_A I$	the left annihilator in a module A of a subset $I \neq \emptyset$ of R, that is, $\mathrm{Ann}_A I = \{a \in A \mid ra = 0\,, \forall r \in I\}$
$(A : a)$	the set $\{r \in R \mid ra \in A\}$ for some element a of a module A
$\mathrm{Spec}(R)$	the set of all prime ideals of a commutative ring R
$\dim R$	the (Krull) dimension of a commutative ring R
$\dim p$	the dimension of $p \in \mathrm{Spec}(R)$, that is, $\dim R/p$
$B \leq A$	B is a submodule of a module A
$B \trianglelefteq A$	B is an essential submodule of a module A
$\mathrm{Soc}(A)$	the socle of a module A
$\mathrm{Rad}(A)$	the (Jacobson) radical of a module A
$\mathrm{End}_R(A)$	the set of endomorphisms of a module A
Hom_R	the Hom functor

Ext^i_R	the i-th Ext functor
Tor^R_i	the i-th Tor functor
$E(A)$	the injective hull of a module A
$t(A)$	the (unique) maximal τ-torsion submodule of a module A
$E_\tau(A)$	the τ-injective hull of a module A
$Q_\tau(A)$	the τ-quasi-injective hull of a module A
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$	the sets of natural numbers, integers, rational numbers
$\mathbb{N}^*$, $\mathbb{Z}^*$, $\mathbb{Q}^*$	the sets $\mathbb{N} \setminus \{0\}$, $\mathbb{Z} \setminus \{0\}$, $\mathbb{Q} \setminus \{0\}$

Chapter 1

Introduction to torsion theories

This introductory chapter briefly presents the context of torsion theories, with emphasis on the definitions and properties that will be used later on.

For additional information on torsion theories the reader is referred to [49] and [107].

1.1 Radicals and Gabriel filters

In this first section we introduce some important notions that are closely connected to torsion theories.

Definition 1.1.1 A functor $r : R\text{-Mod} \to R\text{-Mod}$ is called a *preradical* on R-Mod if:

(i) For every module A, $r(A) \leq A$.

(ii) For every homomorphism $f : A \to B$, $r(f) : r(A) \to r(B)$ is the restriction of f to $r(A)$.

Throughout the text all the preradicals will be on R-Mod.

Definition 1.1.2 Let r_1 and r_2 be preradicals. Define two functors $r_1 \circ r_2$, $(r_1 : r_2) : R\text{-Mod} \to R\text{-Mod}$, on objects by

$$(r_1 \circ r_2)(A) = r_1(r_2(A)),$$

$$(r_1 : r_2)(A)/r_1(A) = r_2(A/r_1(A))\,,$$

for every module A, and on morphisms by taking the corresponding restrictions.

Definition 1.1.3 A preradical r is called:
(i) *idempotent* if $r \circ r = r$.
(ii) *radical* if $(r : r) = r$.

The following two results are immediate.

Lemma 1.1.4 *Let r_1 and r_2 be preradicals. Then $r_1 \circ r_2$ and $(r_1 : r_2)$ are preradicals.*

Lemma 1.1.5 *Let r be a preradical. Then:*
(i) r is idempotent if and only if $r(r(A)) = r(A)$ for every module A.
(ii) r is a radical if and only if $r(A/r(A)) = 0$ for every module A.
(iii) r is left exact if and only if $r(A) = A \cap r(B)$ for every module A and every submodule B of A.
(iv) If r is a left exact functor, then r is idempotent.

Proposition 1.1.6 *Let r be a radical, A be a module and $B \leq r(A)$. Then $r(A/B) = r(A)/B$.*

Proof. Denote by $p : A \to A/B$ and $q : A/B \to A/r(A)$ the natural homomorphisms. Then we have $p(r(A)) \subseteq r(A/B)$ and since $B \subseteq r(A)$, we deduce that $r(A)/B \subseteq r(A/B)$. We also have $q(r(A/B)) \subseteq r(A/r(A)) = 0$, whence $r(A/B) \subseteq r(A)/B$. Therefore $r(A/B) = r(A)/B$. □

Corollary 1.1.7 *Let r_1 and r_2 be radicals. Then $r_1 \circ r_2$ is a radical.*

Proof. Let A be a module. We have $(r_1 \circ r_2)(A) \subseteq r_2(A)$, whence we get $r_2(A/(r_1 \circ r_2)(A)) = r_2(A)/(r_1 \circ r_2)(A)$ by Proposition 1.1.6. Then

$$(r_1 \circ r_2)(A/(r_1 \circ r_2)(A)) = r_1(r_2(A)/(r_1(r_2(A))) = 0\,.$$

Thus $r_1 \circ r_2$ is a radical. □

Example 1.1.8 (1) For every abelian group G, let $t(G)$ be the set of elements of G having finite order (torsion elements). Then t is a left exact radical on $\mathbb{Z}$-Mod.

(2) For every abelian group G, let $d(G)$ be the sum of its divisible subgroups. Then d is an idempotent radical on $\mathbb{Z}$-Mod, that in general is not left exact.

(3) For every module A, let $\mathrm{Soc}(A)$ be its socle, that is, the sum of its simple submodules. Then Soc is a left exact preradical, that in general is not a radical.

(4) For every module A, let $\mathrm{Rad}(A)$ be its Jacobson radical, that is, the intersection of its maximal submodules. Then Rad is a preradical, that in general is not idempotent or a radical.

(5) For every module A, let $Z(A)$ be the *singular submodule* of A, that is,

$$Z(A) = \{x \in A \mid \mathrm{Ann}_R x \trianglelefteq R\},$$

and let $Z_2(A)$ be such that

$$Z_2(A)/Z(A) = Z(A/Z(A)),$$

that is,

$$Z_2(A) = \{x \in A \mid x + Z(A) \in Z(A/Z(A))\}.$$

Then Z is a left exact preradical, that in general is not a radical. But Z_2 is a left exact radical, called the *singular radical* of A.

(6) A left ideal I of R is called *dense* if the right annihilator of $(I : r)$ in R is zero for every $r \in R$. For every module A, let

$$D(A) = \{x \in A \mid \mathrm{Ann}_R x \text{ dense in } R\}.$$

Then D is a left exact radical.

Definition 1.1.9 A non-empty set $F(R)$ of left ideals of R is called a *Gabriel filter* if:

(i) For every $I \in F(R)$ and every $a \in R$, we have $(I : a) \in F(R)$.

(ii) For every $J \in F(R)$ and every left ideal I of R with $(I : a) \in F(R)$ for each $a \in J$, we have $I \in F(R)$.

Remark. If $F(R)$ is a Gabriel filter, then $R \in F(R)$. Indeed, if $I \in F(R)$ and $a \in I$, then $R = (I : a) \in F(R)$.

Proposition 1.1.10 *Let $F(R)$ be a Gabriel filter. Then:*

(i) For every $J \in F(R)$ and every left ideal I of R with $J \subseteq I$, we have $I \in F(R)$.

(ii) For every $I, J \in F(R)$, we have $I \cap J \in F(R)$.

(iii) For every $I, J \in F(R)$, we have $IJ \in F(R)$.

Proof. (i) Let $J \in F(R)$ and let I be a left ideal of R such that $J \subseteq I$. Then for every $a \in J$, we have $(I : a) = R \in F(R)$. Thus $I \in F(R)$ by Definition 1.1.9 (ii).

(ii) Let $I, J \in F(R)$. Then for every $a \in J$, $((I \cap J) : a) = (J : a) \in F(R)$. Thus $I \cap J \in F(R)$ by Definition 1.1.9 (ii).

(iii) Let $I, J \in F(R)$. Then for every $a \in J$, we have $J \subseteq (IJ : a)$. Hence by (i) we have $(IJ : a) \in F(R)$ for every $a \in J$. Then $IJ \in F(R)$ by Definition 1.1.9 (ii). $\square$

Remark. A non-empty set of left ideals of R satisfying the first two conditions of Proposition 1.1.10 is called a *filter*.

1.2 Torsion theories. Basic facts

Definition 1.2.1 A pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of modules is called a *torsion theory* if the following conditions hold:

(i) $\mathrm{Hom}_R(A, B) = 0$ for every $A \in \mathcal{T}$ and every $B \in \mathcal{F}$.

(ii) If $\mathrm{Hom}_R(A, B) = 0$ for every $B \in \mathcal{F}$, then $A \in \mathcal{T}$.

(iii) If $\mathrm{Hom}_R(A, B) = 0$ for every $A \in \mathcal{T}$, then $B \in \mathcal{F}$.

The class $\mathcal{T}$ is called the *torsion class* of τ and its members are called *τ-torsion modules*, whereas the class $\mathcal{F}$ is called the *torsionfree class* of τ and its members are called *τ-torsionfree modules.*

Every class of modules $\mathcal{A}$ generates and cogenerates a torsion theory in the following sense.

Definition 1.2.2 Let $\mathcal{A}$ be a class of modules.

Consider the classes of modules

$$\mathcal{F}_1 = \{Y \mid \mathrm{Hom}_R(A, Y) = 0\,, \forall\, A \in \mathcal{A}\}\,,$$

$$\mathcal{T}_1 = \{X \mid \mathrm{Hom}_R(X, F) = 0\,, \forall\, F \in \mathcal{F}_1\}\,.$$

Then $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion theory called the *torsion theory generated by $\mathcal{A}$.*

Consider the classes of modules

$$\mathcal{T}_2 = \{X \mid \mathrm{Hom}_R(X, A) = 0\,, \forall\, A \in \mathcal{A}\}\,,$$

$$\mathcal{F}_2 = \{Y \mid \mathrm{Hom}_R(T, Y) = 0\,, \forall\, T \in \mathcal{T}_2\}\,.$$

Then $(\mathcal{T}_2, \mathcal{F}_2)$ is a torsion theory called the *torsion theory cogenerated by $\mathcal{A}$.*

Remark. $\mathcal{T}_1$ is the least torsion class containing $\mathcal{A}$, whereas $\mathcal{F}_2$ is the least torsionfree class containing $\mathcal{A}$.

Let us set now some terminology on a class of modules.

Definition 1.2.3 A class $\mathcal{A}$ of modules is called:

(i) *closed under submodules* if for every $A \in \mathcal{A}$ and every submodule B of A, we have $B \in \mathcal{A}$.

(ii) *closed under homomorphic images* (respectively *isomorphic copies*) if for every $A \in \mathcal{A}$ and every epimorphism (respectively isomorphism) $f : A \to B$, we have $B \in \mathcal{A}$.

(iii) *closed under direct sums* (respectively *direct products*) if $\bigoplus_{i\in I} A_i \in \mathcal{A}$ (respectively $\prod_{i\in I} A_i \in \mathcal{A}$) for every $A_i \in \mathcal{A}$ $(i \in I)$.

(iv) *closed under extensions* if for every exact sequence $0 \to A' \to A \to A'' \to 0$ with $A', A'' \in \mathcal{A}$, we have $A \in \mathcal{A}$.

(v) *closed under injective hulls* if for every $A \in \mathcal{A}$, we have $E(A) \in \mathcal{A}$.

Remark. We have mentioned only the most used closedness conditions, the others being defined in a similar way.

Lemma 1.2.4 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then $\mathcal{T}$ is closed under direct sums and $\mathcal{F}$ is closed under direct products.*

Proof. Clear by the properties $\mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$ and $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$. □

Theorem 1.2.5 *Let $\mathcal{T}$ and $\mathcal{F}$ be classes of modules. Then $\tau = (\mathcal{T}, \mathcal{F})$ is a torsion theory if and only if*

(i) $\mathcal{T} \cap \mathcal{F} = \{0\}$.

(ii) $\mathcal{T}$ is closed under homomorphic images.

(iii) $\mathcal{F}$ is closed under submodules.

(iv) Every module A has a submodule $t(A)$ such that $t(A) \in \mathcal{T}$ and $A/t(A) \in \mathcal{F}$.

Proof. Assume first that $\tau = (\mathcal{T}, \mathcal{F})$ is a torsion theory. Then clearly (i) holds. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence. Then it induces for every $F \in \mathcal{F}$ the exact sequence $0 \to \mathrm{Hom}_R(A'', F) \to \mathrm{Hom}_R(A, F)$. Hence if $A \in \mathcal{T}$, then $A'' \in \mathcal{T}$. The initial exact sequence also induces for every $T \in \mathcal{T}$ the exact sequence $0 \to \mathrm{Hom}_R(T, A') \to \mathrm{Hom}_R(T, A)$. Hence if $A \in \mathcal{F}$, then $A' \in \mathcal{F}$.

For the fourth condition, take a module A and let

$$t(A) = \sum_{i\in I} \{B_i \leq A \mid B_i \in \mathcal{T}\}\,.$$

Since there exists a natural epimorphism $f : \bigoplus_{i\in I} B_i \to t(A)$, it follows by (ii) and Lemma 1.2.4 that $t(A) \in \mathcal{T}$.

Suppose that $A/t(A) \notin \mathcal{F}$. Then there exists $T \in \mathcal{T}$ and a non-zero homomorphism $g : T \to A/t(A)$. It follows that $B/t(A) = \mathrm{Im} g \in \mathcal{T}$. On the other hand, the exact sequence

$$0 \to t(A) \to B \to B/t(A) \to 0$$

induces for every $F \in \mathcal{F}$ the exact sequence

$$\mathrm{Hom}_R(B/t(A), F) \to \mathrm{Hom}_R(B, F) \to \mathrm{Hom}_R(t(A), F)$$

Since the first and the last term are zero, we have $\mathrm{Hom}_R(B, F) = 0$ for every $F \in \mathcal{F}$, whence $B \in \mathcal{T}$. But then $B = t(A)$ and $g = 0$, a contradiction.

Conversely, suppose now that the classes $\mathcal{T}$ and $\mathcal{F}$ satisfy the conditions (i)-(iv).

First, let $T \in \mathcal{T}$ and suppose that there exists $F \in \mathcal{F}$ such that $\mathrm{Hom}_R(T, F) \neq 0$, say $0 \neq f : T \to F$. Then $\mathrm{Im} f \in \mathcal{T}$ by (ii) and $\mathrm{Im} f \in \mathcal{F}$ by (iii), whence $\mathrm{Im} f = 0$ by (i), a contradiction.

Secondly, let A be a module such that $\mathrm{Hom}_R(A, F) = 0$ for every $F \in \mathcal{F}$. By (iv) there exists a submodule $t(A)$ of A such that $t(A) \in \mathcal{T}$ and $A/t(A) \in \mathcal{F}$. Then $\mathrm{Hom}_R(A, A/t(A)) = 0$, whence $A = t(A) \in \mathcal{T}$.

Thus the pair $(\mathcal{T}, \mathcal{F})$ satisfies the second condition from the definition of a torsion theory. Similarly, one shows that it fulfils the third one. □

Theorem 1.2.6 *Let $\mathcal{T}$ and $\mathcal{F}$ be classes of modules. Then:*

(i) $\mathcal{T}$ is a torsion class for some torsion theory if and only if it is closed under homomorphic images, direct sums and extensions.

(ii) $\mathcal{F}$ is a torsionfree class for some torsion theory if and only if it is closed under submodules, direct products and extensions.

Proof. (i) First suppose that $\mathcal{T}$ is a torsion class for some torsion theory τ. Then it is closed under homomorphic images and direct sums by Theorem 1.2.5 and Lemma 1.2.4. Now let $0 \to A' \to A \to A'' \to 0$ be an exact sequence with $A', A'' \in \mathcal{T}$. Then it induces for every τ-torsionfree module F the exact sequence

$$\mathrm{Hom}_R(A'', F) \to \mathrm{Hom}_R(A, F) \to \mathrm{Hom}_R(A', F)$$

where the first and the last term are zero because $A', A'' \in \mathcal{T}$. Hence $\mathrm{Hom}_R(A, F) = 0$ and thus $A \in \mathcal{T}$.

Conversely, suppose that $\mathcal{T}$ is closed under homomorphic images, direct sums and extensions. Consider the torsion theory $(\mathcal{T}_1, \mathcal{F}_1)$ generated by $\mathcal{T}$. We will show that $\mathcal{T} = \mathcal{T}_1$. Let $A \in \mathcal{T}_1$. Then we have $\mathrm{Hom}_R(A, F) = 0$ for every $F \in \mathcal{F}_1$. Since $\mathcal{T}$ is closed under homomorphic images and direct sums, there exists a largest submodule T of A that belongs to $\mathcal{T}$, namely the sum of all submodules of A belonging to $\mathcal{T}$. We will prove that $T = A$ and to this end it is enough to show that $A/T \in \mathcal{F}$. Let $f \in \mathrm{Hom}_R(U, A/T) \neq 0$ for some $U \in \mathcal{T}$. Suppose that $f \neq 0$. Then $\mathrm{Im} f = B/T \in \mathcal{T}$ for some submodule B such that $T \subset B \subseteq A$. Since $\mathcal{T}$ is closed under extensions, we have $B \in \mathcal{T}$. But this contradicts the maximality of T. Hence $\mathrm{Hom}_R(U, A/T) = 0$ for every $U \in \mathcal{T}$, so that $A/T \in \mathcal{F}$, whence $T = A$, that finishes the proof.

(ii) First suppose that $\mathcal{F}$ is a torsionfree class for some torsion theory τ. Then it is closed under submodules and direct products by Theorem 1.2.5 and Lemma 1.2.4. Now let $0 \to A' \to A \to A'' \to 0$ be an exact sequence with $A', A'' \in \mathcal{F}$. Then it induces for every τ-torsion module T the exact sequence

$$\mathrm{Hom}_R(T, A') \to \mathrm{Hom}_R(T, A) \to \mathrm{Hom}_R(T, A'')$$

where the first and the last term are zero because $A', A'' \in \mathcal{F}$. Hence $\mathrm{Hom}_R(T, A) = 0$ and thus $A \in \mathcal{F}$.

Conversely, suppose that $\mathcal{F}$ is closed under submodules, direct products and extensions. Consider the torsion theory $(\mathcal{T}_2, \mathcal{F}_2)$ cogenerated by $\mathcal{F}$ and show, dually to (i), that $\mathcal{F} = \mathcal{F}_2$. □

Remark. A torsion class (or a torsionfree class) uniquely determines a torsion theory.

We define some types of torsion theories that will be of special interest.

Definition 1.2.7 A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called:
(i) *stable* if $\mathcal{T}$ is closed under injective hulls.
(ii) *hereditary* if $\mathcal{T}$ is closed under submodules.

We will see in Proposition 1.4.8 that every torsion theory on R-Mod is stable provided R is commutative noetherian. But for the moment, we can give the following properties on stable torsion theories.

Proposition 1.2.8 *A torsion theory τ is stable if and only if $t(A)$ is a direct summand of every injective module A.*

Proof. First, let A be an injective module. By the stability of τ, $E(t(A))$ is τ-torsion, hence $E(t(A)) = t(A)$. Now A is a direct summand of $t(A)$.

Conversely, let A be a τ-torsion module. Then $t(E(A))$ is a direct summand of $E(A)$. Since $A \subseteq t(E(A))$ and $A \trianglelefteq E(A)$, we must have $t(E(A)) = E(A)$. Hence $E(A)$ is τ-torsion and, consequently, τ is stable. $\square$

Proposition 1.2.9 *Let τ be a stable torsion theory and let A be a module. Then $E(A/t(A)) = E(A)/E(t(A))$ and $E(A) \cong E(t(A)) \oplus E(A/t(A))$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & t(A) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/t(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E(t(A)) \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d, "p"] & E(A)/E(t(A)) \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & E(t(A))/t(A) \arrow[r] \arrow[d] & E(A)/A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since the middle row clearly splits, we have $E(A) \cong E(t(A)) \oplus E(A)/E(t(A))$, hence $E(A)/E(t(A))$ is injective. Let us show that $A/t(A) \trianglelefteq E(A)/E(t(A))$.

In general, for a module M and $N \leq M$, we have

$$N \trianglelefteq M \Longleftrightarrow \forall 0 \neq x \in M\,, \exists r \in R : 0 \neq rx \in N$$

$$\Longleftrightarrow \forall\, 0 \neq x \in M\,, \mathrm{Ann}_R x \neq \mathrm{Ann}_R(x+N)\,.$$

Now suppose that $A/t(A)$ is not essential in $E(A)/E(t(A))$. Since τ is stable, we have $E(t(A)) = t(E(A))$, hence $E(A)/E(t(A))$ is τ-torsionfree. Then there exists $0 \neq x \in E(A)/E(t(A))$ such that $\mathrm{Ann}_R x = \mathrm{Ann}_R q(x)$. Also there exists $y \in E(A)$ such that

$$\mathrm{Ann}_R y = \mathrm{Ann}_R x = \mathrm{Ann}_R q(x) = \mathrm{Ann}_R p(y)\,,$$

contradiction with the fact that $A \trianglelefteq E(A)$. Therefore $A/t(A) \trianglelefteq E(A)/E(t(A))$ and consequently $E(A/t(A)) = E(A)/E(t(A))$. Now we also have $E(A) \cong E(t(A)) \oplus E(A/t(A))$. □

Proposition 1.2.10 *Let τ be a torsion theory.*

(i) If τ is stable, then every indecomposable injective module is either τ-torsion or τ-torsionfree.

(ii) If R is left noetherian and every indecomposable injective module is either τ-torsion or τ-torsionfree, then τ is stable.

Proof. (i) Let A be an indecomposable injective module. If A is not τ-torsionfree, we have $t(A) \trianglelefteq A$, whence $t(A) = A$, because τ is stable. Thus A is τ-torsion.

(ii) Let A be a τ-torsion module. Since R is left noetherian, we have $E(A) = \bigoplus_{i \in I} E_i$ for some indecomposable injective modules E_i. For each i we clearly have $A \cap E_i \neq 0$. Then each E_i cannot be τ-torsionfree, hence each E_i is τ-torsion. Thus $E(A)$ is τ-torsion and consequently τ is stable. □

Hereditary torsion theories can be also characterized in terms of some closedness property of the torsionfree class.

Proposition 1.2.11 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then τ is hereditary if and only if $\mathcal{F}$ is closed under injective hulls.*

Proof. Assume that τ is hereditary. Let $A \in \mathcal{F}$. Then $t(E(A)) \cap A \in \mathcal{T}$, whence $t(E(A)) \cap A \subseteq t(A) = 0$. It follows that $t(E(A)) = 0$, that is, $E(A) \in \mathcal{F}$.

Conversely, assume that $\mathcal{F}$ is closed under injective hulls. Let $A \in \mathcal{T}$ and let $B \leq A$. The exact sequence $0 \to B \to A \to A/B \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(A, E(B/t(B))) \to \mathrm{Hom}_R(B, E(B/t(B))) \to \mathrm{Ext}^1_R(A/B, E(B/t(B)))$$

The first term is zero because $A \in \mathcal{T}$ and $E(B/t(B)) \in \mathcal{F}$ by hypothesis, whereas the last term is zero by the injectivity of $E(B/t(B))$. Hence $\mathrm{Hom}_R(B, E(B/t(B))) = 0$. But then we must have $t(B) = B$. Consequently, τ is hereditary. $\square$

Corollary 1.2.12 *Let $\mathcal{A}$ be a class of modules closed under submodules and homomorphic images. Then the torsion theory τ generated by $\mathcal{A}$ is hereditary.*

Proof. Let F be a τ-torsionfree module. Suppose that $t(E(F)) \neq 0$. Since $\mathcal{A}$ is closed under homomorphic images, it follows that $t(E(F))$ contains a non-zero submodule $A \in \mathcal{A}$. Then $F \cap A \neq 0$. Since $\mathcal{A}$ is closed under submodules, we have $F \cap A \in \mathcal{A}$, whence $t(F) \neq 0$, a contradiction. Therefore $t(E(F)) = 0$, that is, $E(F)$ is τ-torsionfree. Now by Proposition 1.2.11, τ is hereditary. $\square$

For a hereditary torsion theory τ, τ-torsion and τ-torsionfree modules can be characterized as follows.

Proposition 1.2.13 *Let τ be a hereditary torsion theory.*

(i) A module A is τ-torsion if and only if $\mathrm{Hom}_R(A, E(B)) = 0$ *for every τ-torsionfree module B.*

(ii) A module B is τ-torsionfree if and only if $\mathrm{Hom}_R(A, E(B)) = 0$ *for every τ-torsion module A.*

Proof. (i) If A is τ-torsion, then $\mathrm{Hom}_R(A, E(B)) = 0$ for every τ-torsionfree module B by Proposition 1.2.11.

Conversely, the exact sequence $0 \to t(A) \to A \to A/t(A) \to 0$ induces for every τ-torsionfree module B the exact sequence

$$0 \to \mathrm{Hom}_R(A/t(A), E(B)) \to \mathrm{Hom}_R(A, E(B)) \to \mathrm{Hom}_R(t(A), E(B)) \to 0$$

with the middle term zero by hypothesis. Hence $\mathrm{Hom}_R(A/t(A), E(B)) = 0$ for every τ-torsionfree B. In particular, $\mathrm{Hom}_R(A/t(A), E(A/t(A))) = 0$, whence we get $t(A) = A$.

(ii) If B is τ-torsionfree, then $\mathrm{Hom}_R(A, E(B)) = 0$ for every τ-torsion module A by Proposition 1.2.11.

Conversely, the exact sequence $0 \to B \to E(B) \to E(B)/B \to 0$ induces for every τ-torsion module A the exact sequence

$$0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, E(B)) \to \mathrm{Hom}_R(A, E(B)/B)$$

with the third term zero by hypothesis. Hence $\mathrm{Hom}_R(A, B) = 0$ for every τ-torsion module A, that is, B is τ-torsionfree. $\square$

In the following two results, we will see how a hereditary torsion theory can be generated or cogenerated.

Theorem 1.2.14 *Every hereditary torsion theory τ is generated by the class $\mathcal{A}$ of cyclic modules R/I which are torsion modules.*

Proof. Note that a module F is τ-torsionfree if and only if $\mathrm{Hom}_R(R/I, F) = 0$ for every $R/I \in \mathcal{A}$. $\square$

Theorem 1.2.15 *A torsion theory τ is hereditary if and only if it is cogenerated by a (τ-torsionfree) injective module.*

Proof. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory. Denote

$$E = \prod \{E(R/I) \mid I \leq_R R \text{ such that } R/I \in \mathcal{F}\}\,.$$

Clearly, E is injective and $E \in \mathcal{F}$, hence we have $\mathrm{Hom}_R(M, E) = 0$ for every $M \in \mathcal{T}$. Moreover, if $M \notin \mathcal{T}$, then there exists a non-zero homomorphism $f : C \to F$ for some cyclic submodule C of M and some $F \in \mathcal{F}$. Since $\mathrm{Im}f \in \mathcal{F}$ is cyclic, f induces a homomorphism $C \to E$ that can be extended to a non-zero homomorphism $M \to E$. Hence $M \in \mathcal{T}$ if and only if $\mathrm{Hom}_R(M, E) = 0$, that is, τ is cogenerated by E. $\square$

Example 1.2.16 (1) Denote by 0 the class consisting only of the zero module. The pairs $\chi = (R-\mathrm{Mod}, 0)$ and $\xi = (0, R-\mathrm{Mod})$ are hereditary torsion theories, called the *improper* and the *trivial torsion theory* on R-Mod respectively.

(2) Let $(G, +)$ be an abelian group. Denote by $t(G)$ the set of elements of G having finite order (torsion elements). Let $\mathcal{T}$ and $\mathcal{F}$ be the classes of all abelian groups G such that $t(G) = G$ (torsion groups) and $t(G) = 0$ (torsionfree groups) respectively. Then the pair $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on $\mathbb{Z}$-Mod.

(3) Let $\mathcal{D}$ be the class of all divisible (injective) abelian groups and let $\mathcal{R}$ be the class of all reduced abelian groups, that is, abelian groups without a non-trivial divisible direct summand. Then the pair $(\mathcal{D}, \mathcal{R})$ is a torsion theory on $\mathbb{Z}$-Mod, which is not hereditary, because the class $\mathcal{D}$ is not closed under subgroups (for instance, $\mathbb{Q}$ is divisible, whereas $\mathbb{Z}$ is not).

(4) A module A is called *semiartinian* if every non-zero homomorphic image of A contains a simple submodule. Let τ_D be the torsion theory generated by the class of semisimple (or even simple) modules. Then τ_D is a hereditary torsion theory, called the *Dickson torsion theory*. Its torsion and torsionfree classes are respectively

$$\mathcal{T}_D = \{A \mid A \text{ is semiartinian}\},$$

$$\mathcal{F}_D = \{A \mid \mathrm{Soc}(A) = 0\}.$$

(5) A module A is called *singular* if $Z(A) = A$ and *nonsingular* if $Z(A) = 0$. Let τ_G be the torsion theory generated by all modules of the form A/B,

where $B \trianglelefteq A$. Then τ_G is a stable hereditary torsion theory, called the *Goldie torsion theory*. Its torsion and torsionfree classes are respectively

$$\mathcal{T}_G = \{A \mid Z_2(A) = A\}\,,$$

$$\mathcal{F}_G = \{A \mid A \text{ is nonsingular}\}\,.$$

If R is nonsingular, then $Z(A) = Z_2(A)$ for every module A, hence $\mathcal{T}_G$ consists of all singular modules.

(6) Let τ_L be the torsion theory cogenerated by $E(R)$. Then τ_L is a hereditary torsion theory, called the *Lambek torsion theory*. Its torsion class is

$$\mathcal{T}_L = \{A \mid \mathrm{Ann}_R x \text{ dense in } R\,, \forall x \in A\}\,.$$

(7) A finite strictly increasing sequence $p_0 \subset p_1 \subset \cdots \subset p_n$ of prime ideals of a commutative ring R is said to be a *chain of length* n. The supremum of the lengths of all chains of prime ideals of R is called the *(Krull) dimension of* R and it is denoted by $\dim R$ [8, p.89]. If that supremum does not exist, then the dimension of R is considered to be infinite. For a commutative ring R and $p \in \mathrm{Spec}(R)$, $\dim R/p$ is called the *dimension of* p and it is denoted by $\dim p$ [41, p.227].

Let n be a positive integer and let R be a commutative ring with $\dim R \geq n$. Let τ_n be the torsion theory generated by the class of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing an ideal $p \in \mathrm{Spec}(R)$ with $\dim p \leq n$ (or equivalently, the torsion theory generated by all modules of Krull dimension at most n). Then τ_n is a hereditary torsion theory. Note that τ_0 is the hereditary torsion theory generated by the class of all simple modules, i.e. the Dickson torsion theory τ_D.

1.3 Some bijective correspondences

In this section we will show how torsion theories are connected to radicals and to Gabriel filters.

Theorem 1.3.1 *There is a bijective correspondence between:*
(i) torsion theories in R-Mod.
(ii) idempotent radicals on R-Mod.

Proof. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory in R-Mod. For every module A, let $t_\tau(A)$ be the sum of all τ-torsion submodules of A. Clearly, $t = t_\tau$ is a preradical on R-Mod. Moreover, $t(t(A)) = t(A)$ and, since $A/t(A) \in \mathcal{F}$, we have $t(A/t(A)) = 0$. Thus t is an idempotent radical by Lemma 1.1.5.

Conversely, let r be an idempotent radical on R-Mod. Denote

$$\mathcal{T}_r = \{A \mid r(A) = A\}\,,$$

$$\mathcal{F}_r = \{A \mid r(A) = 0\}\,.$$

We claim that $\tau_r = (\mathcal{T}_r, \mathcal{F}_r)$ is a torsion theory in R-Mod. To this end, apply Theorem 1.2.5.

Now denote by F the correspondence $\tau \mapsto t_\tau$ and by G the correspondence $r \mapsto \tau_r$. We will show that F and G are inverses one to the other.

For every torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, we have

$$G(F(\tau)) = (\mathcal{T}_{t_\tau}, \mathcal{F}_{t_\tau}) = (\mathcal{T}, \mathcal{F}) = \tau\,,$$

because

$$A \in \mathcal{T} \Longleftrightarrow t_\tau(A) = A \Longleftrightarrow A \in \mathcal{T}_{t_\tau}$$

and a torsion theory is determined by its torsion class. Hence $G \circ F = 1$.

For every idempotent radical r, we show that

$$F(G(r)) = t_{\tau_r} = r\,.$$

Denote $t = t_{\tau_r}$ and let A be a module. Since $r(r(A)) = r(A)$, we have $r(A) \in \mathcal{T}_r$, hence $r(A) \leq t(A)$. Moreover, $r(A/r(A)) = 0$, hence $A/r(A) \in \mathcal{F}_r$. But $r(A) \leq t(A)$, so that it follows by Proposition 1.1.6 that $0 = t(A/r(A)) = t(A)/r(A)$, whence $r(A) = t(A)$. Thus $r = t$ and consequently $F \circ G = 1$. $\square$

Hence for any torsion theory τ, there is an associated idempotent radical t_τ (usually denoted simply by t), called the *torsion radical* associated to τ. For every module A, $t(A)$ will be the unique maximal τ-torsion submodule of A.

Throughout the text, t will denote the idempotent radical corresponding to a torsion theory τ.

Proposition 1.3.2 *A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is hereditary if and only if the corresponding torsion radical t is left exact.*

Proof. Assume that τ is hereditary. Let A be a module and $B \leq A$. Since $t(A) \in \mathcal{T}$, we have $t(A) \cap B \in \mathcal{T}$, whence $t(A) \cap B \in t(B)$. But $t(B) \subseteq t(A) \cap B$, so that we have $t(A) \cap B = t(B)$. Now by Lemma 1.1.5, t is left exact.

Conversely, assume that t is left exact. Let $A \in \mathcal{F}$. Using Lemma 1.1.5, we have $0 = t(A) = t(E(A)) \cap A$, whence $t(E(A)) = 0$, that is, $E(A) \in \mathcal{F}$. Now by Proposition 1.2.11, τ is hereditary. $\square$

Theorem 1.3.3 *There is a bijective correspondence between:*

*(i) hereditary torsion theories in R-*Mod*.*

*(ii) left exact radicals on R-*Mod*.*

(iii) Gabriel filters of left ideals of R.

Proof. *(i)* $\Longleftrightarrow$ *(ii)* By Proposition 1.3.2 and Theorem 1.3.1.

(i) $\Longleftrightarrow$ *(iii)* Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in R-Mod. Define

$$F(R) = \{I \leq_R R \mid R/I \in \mathcal{T}\}.$$

We will prove that $F(R)$ is a Gabriel filter.

First, let $I \in F(R)$ and $a \in R$. Define a monomorphism

$$f : R/(I : a) \to R/I\,, \quad f(r + (I : a)) = ra + I\,.$$

Since $R/I \in \mathcal{T}$ and τ is hereditary, it follows that $R/(I : a) \in \mathcal{T}$, whence $(I : a) \in F(R)$.

Secondly, let $J \in F(R)$ and let I be a left ideal of R such that $(I : a) \in F(R)$ for every $a \in J$. Consider the exact sequence

$$0 \to (I + J)/I \to R/I \to R/(I + J) \to 0\,.$$

Since $R/J \in \mathcal{T}$ and there is a natural epimorphism $R/J \to R/(I + J)$, it follows that $R/(I + J) \in \mathcal{T}$. On the other hand, $(I + J)/I \cong J/(I \cap J)$. But $J/(I \cap J) \in \mathcal{T}$, because for every $a \in J$ we have $((I \cap J) : a) = (I : a) \in F(R)$. Since $\mathcal{T}$ is closed under extensions, it follows that $R/I \in \mathcal{T}$.

Conversely, let $F(R)$ be a Gabriel filter of left ideals of R. Define

$$\mathcal{T} = \{A \mid (0 : x) \in F(R) \text{ for every } x \in A\}\,.$$

We will prove that $\mathcal{T}$ is a hereditary torsion class, that is, it is closed under submodules, homomorphic images, direct sums and extensions.

Clearly, $\mathcal{T}$ is closed under submodules.

Let $A \in \mathcal{T}$ and let $f : A \to B$ be an epimorphism. For every $y \in B$, there exists $x \in A$ such that $f(x) = y$. Then clearly, $(0 : x) \subseteq (0 : y)$ and since $(0 : x) \in F(R)$, it follows that $(0 : y) \in F(R)$. Hence $B \in \mathcal{T}$.

Let $(A_i)_{i \in I}$ be a family of modules, where each $A_i \in \mathcal{T}$. Let $(a_i)_{i \in I} \in \bigoplus_{i \in I} A_i$. Then

$$(0 : (a_i)_{i \in I}) = \bigcap_{i \in I} (0 : a_i) = \bigcap_{i \in K} (0 : a_i)$$

for some finite $K \subseteq I$, whence $(0 : (a_i)_{i \in I}) \in F(R)$. Hence $\bigoplus_{i \in I} A_i \in \mathcal{T}$.

Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence with $A', A'' \in \mathcal{T}$. Assume for simplicity that $A'' = A/A'$. Let $x \in A$. Then $(A' : x) = (0 : x + A') \in F(R)$. For every $r \in (A' : x)$, we have $rx \in A'$, whence $((0 : x) : r) = (0 : rx) \in F(R)$. But then $(0 : x) \in F(R)$, hence $A \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a hereditary torsion class.

Finally, let us show that we have a bijective correspondence. If we start with a hereditary torsion class $\mathcal{T}$, then we have

$$F(R) = \{I \leq_R R \mid R/I \in \mathcal{T}\}\,.$$

Since $(0:x) \in F(R)$ for some $x \in A$ implies $Rx \cong R/(0:x) \in \mathcal{T}$, we use the properties of $\mathcal{T}$ to get

$$\{A \mid (0:x) \in F(R)\,, \forall x \in A\} =$$

$$= \{A \mid \text{every cyclic submodule of } A \text{ is in } \mathcal{T}\} = \mathcal{T}\,.$$

Now if we start with a Gabriel filter $F(R)$, then we have

$$\mathcal{T} = \{A \mid (0:x) \in F(R) \text{ for every } x \in A\}\,.$$

By the properties of $F(R)$, we get

$$\{I \leq_R R \mid R/I \in \mathcal{T}\} = \{I \leq_R R \mid (I:r) \in F(R)\,, \forall r \in R\} = F(R)\,.$$

Hence the correspondence is bijective. $\square$

We have seen that a torsion theory is completely determined by its torsion class or by its torsionfree class. Now by Theorem 1.3.3, a hereditary torsion theory τ is also completely determined by the associated left exact radical or by the associated Gabriel filter, that consists of the τ-dense left ideals of R. Even if some of the results to be given hold for an arbitrary torsion theory, we are going to use the better framework of hereditary torsion theories, hence of left exact radicals and of Gabriel filters. So, *from now on, we will consider only hereditary torsion theories and τ will always denote such a torsion theory.*

Example 1.3.4 (1) The Gabriel filter associated to the improper torsion theory $\chi = (R - \mathrm{Mod}, 0)$ consists of all left ideals of R, whereas the Gabriel filter associated to the trivial torsion theory $\xi = (0, R - \mathrm{Mod})$ consists only of R.

(2) If $(\mathcal{T}, \mathcal{F})$ is the usual torsion theory for abelian groups (see Example 1.2.16 (2)), then its associated Gabriel filter consists of all non-zero ideals of $\mathbb{Z}$, that is, $n\mathbb{Z}$ for each $n \in \mathbb{N}^*$.

Let us now see how to compare torsion theories.

Proposition 1.3.5 *The following statements are equivalent for two torsion theories τ and σ:*

(i) Every τ-torsion module is σ-torsion.

(ii) Every σ-torsionfree module is τ-torsionfree.

Proof. Denote by t_τ and t_σ the idempotent radicals associated to the torsion theories τ and σ respectively.

$(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (i)$ Let A be a τ-torsion module. Since $A/t_\sigma(A)$ is σ-torsionfree, we have $t_\tau(A/t_\sigma(A)) = 0$. By Proposition 1.1.6 it follows that $t_\tau(A)/t_\sigma(A) = 0$, whence $t_\sigma(A) = t_\tau(A) = A$. □

Definition 1.3.6 If τ and σ are two torsion theories for which the equivalent conditions of Proposition 1.3.5 hold, it is said that σ is a *generalization* of τ and it is denoted by $\tau \leq \sigma$.

Example 1.3.7 (1) Clearly, we have $\xi \leq \tau \leq \chi$ for every torsion theory τ.

(2) Consider the Lambek torsion theory τ_L and the Goldie torsion theory τ_G. Since every dense left ideal of R is essential in R [82, p.246], we have $\tau_L \leq \tau_G$.

(3) Let R be commutative. Consider the Dickson torsion theory τ_D and the torsion theories τ_n $(n \in \mathbb{N})$. Then

$$\tau_D = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \dots$$

1.4 τ-dense and τ-closed submodules

Recall that τ is a hereditary torsion theory on R-Mod and t is its associated left exact radical.

Two special types of submodules play a key part in the context of torsion theories.

Definition 1.4.1 A submodule B of a module A is called *τ-dense* (respectively *τ-closed*) in A if A/B is τ-torsion (respectively τ-torsionfree).

Proposition 1.4.2 *Let A be a module and $B, B' \leq A$.*

(i) If $B \subseteq B'$, then B is τ-dense in A if and only if B is τ-dense in B' and B' is τ-dense in A.

(ii) If B, B' are τ-dense in A, then $B \cap B'$ is τ-dense in A.

(iii) If A is τ-torsionfree and B is τ-dense in A, then $B \trianglelefteq A$.

Proof. (i) and (ii) Straightforward.

(iii) Let $a \in A \backslash B$. Then $Ra/(Ra \cap B) \cong (Ra + B)/B$ is τ-torsion, because B is τ-dense in A and, consequently, τ-dense in $Ra + B$. Then $Ra \cap B \neq 0$, because $Ra \subseteq A$ is τ-torsionfree. Hence B is essential in A. $\square$

Proposition 1.4.3 *Let A be a module and $B, B' \leq A$.*

(i) $t(A)$ is a τ-closed submodule of A.

(ii) If B is τ-closed in A, then $t(A) \subseteq B$ and $t(B) = t(A)$.

(iii) If $B \subseteq B'$ and B is τ-closed in A, then B is τ-closed in B'.

(iv) If $B \subseteq B'$, B is τ-closed in B' and B' is τ-closed in A, then B is τ-closed in A.

(v) The class of τ-closed submodules of A is closed under intersections.

(vi) $t(A)$ coincides with the intersection of all τ-closed submodules of A.

Proof. (i), (iii) and (iv) Clear.

(ii) If $t(A) \nsubseteq B$, then $(t(A) + B)/B \cong t(A)/(t(A) \cap B) \subseteq A/B$ would be τ-torsion. Hence $t(A) \subseteq B$ and then $t(B) = t(A)$.

(v) Let $(B_i)_{i\in I}$ be a family of τ-closed submodules of A. Since there is a canonical monomorphism $A/\bigcap_{i\in I} B_i \cong \prod_{i\in I} A/B_i$, it follows by Theorem 1.2.6 that $\bigcap_{i\in I} B_i$ is τ-closed in A.

(vi) Since $t(A)$ is τ-closed in A, it belongs to the above intersection. Conversely, if B is τ-closed in A and $x \in t(A)$, then $(0 : x) \subseteq (0 : x + B)$ implies that $x + B$ is a τ-torsion element of A/B. Then $x \in B$, so that $t(A) \subseteq B$. □

Definition 1.4.4 Let A be a module and $B \leq A$. The unique minimal τ-closed submodule of A containing B is called the τ-*closure* of B in A.

Remark. Note that the intersection of all τ-closed submodules of A containing B is τ-closed in A by Proposition 1.4.3.

Proposition 1.4.5 *Let A be a module, $B \leq A$ and let B' be the τ-closure of B in A. Then $B'/B = t(A/B)$.*

Proof. If $t(A/B) = C/B$ for some $C \leq A$, then C/B is τ-closed in A/B. Thus C is τ-closed in A and contains B, hence $B \subseteq C$. If $B' \neq C$, then $C/B' \subseteq A/B'$ is τ-torsionfree. But it is also τ-torsion as a homomorphic image of C/B, a contradiction. Hence $B' = C$ and consequently, $B'/B = t(A/B)$. □

Let us denote by $\mathcal{C}_\tau(A)$ the set of all τ-closed submodules of a module A.

Proposition 1.4.6 *Let A be a module and $B \leq A$. Then:*

(i) $\mathcal{C}_\tau(A)$ is a complete lattice.

(ii) There exists a canonical embedding of $\mathcal{C}_\tau(B)$ into $\mathcal{C}_\tau(A)$.

(iii) There exists a canonical embedding of $\mathcal{C}_\tau(A/B)$ into $\mathcal{C}_\tau(A)$.

(iv) The lattices $\mathcal{C}_\tau(A)$ and $\mathcal{C}_\tau(A/t(A))$ are isomorphic.

Proof. (i) For a family of τ-closed submodules of a module A, the infimum is their intersection and the supremum is the τ-closure of their sum. Note that $\mathcal{C}_\tau(A)$ has a least element, namely $t(A)$, and a greatest element, namely A.

(ii) We claim that the canonical embedding is given by taking the τ-closure in A. Indeed, let X, Y be τ-closed submodules of B having the same τ-closure in A, say X'. Since B/X and B/Y are τ-torsionfree and X'/X and X'/Y are τ-torsion, it follows that $X = X' \cap B = Y$.

(iii) Note that if C/B is a τ-closed submodule of A/B, then C is τ-closed in A.

(iv) Define

$$F : \mathcal{C}_\tau(A/t(A)) \to \mathcal{C}_\tau(A)\,, \quad F(B/t(A)) = B\,.$$

Then F is injective by (iii) and it is easy to check that F is a lattice homomorphism. Now let B be a τ-closed submodule of A. Since $t(A)/(B \cap t(A)) \cong (B + t(A))/B \subseteq A/B$ is both τ-torsion and τ-torsionfree, we must have $t(A) \subseteq B$. Then $B/t(A)$ is τ-closed in $A/t(A)$ and $F(B/t(A)) = B$. Thus F is surjective. Therefore F is a lattice isomorphism. $\square$

The following easy lemma will be frequently used.

Lemma 1.4.7 *Let R be commutative and $p \in \operatorname{Spec}(R)$. Then p is is either τ-dense or τ-closed in R.*

Proof. Assume that p is not τ-dense in R. Suppose that $t(R/p) \neq 0$. Let $0 \neq a \in t(R/p)$. Then $Ra \subseteq t(R/p)$ is τ-torsion. But $Ra \cong R/\operatorname{Ann}_R a = R/p$, a contradiction. $\square$

Proposition 1.4.8 *If R is commutative noetherian, then every hereditary torsion theory on R-Mod is stable.*

Proof. Let A be a τ-torsion module. By hypothesis, every injective module is a direct sum of indecomposable injective modules, that is, a direct sum of modules of the form $E(R/p)$ for $p \in \operatorname{Spec}(R)$. So we may assume without loss of generality that $E(A) \cong E(R/p)$ for some $p \in \operatorname{Spec}(R)$. It follows that R/p cannot be τ-torsionfree, hence it is τ-torsion by Lemma 1.4.7. Now let

$0 \neq x \in E(A) \cong E(R/p)$. Then there exists a natural power n such that $\mathrm{Ann}_R x = p^n$ [101, Proposition 4.23]. We have

$$Rx \cong R/\mathrm{Ann}_R x = R/p^n\,.$$

Since p is τ-dense in R, it follows by Proposition 1.1.10 that p^n is τ-dense in R. Hence Rx is τ-torsion for every $x \in E(A)$ and consequently, $E(A)$ is τ-torsion. $\square$

The following noetherian-type notions related to τ-dense and τ-closed submodules will be needed.

Definition 1.4.9 A torsion theory τ is called *noetherian* if for every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R the union of which is τ-dense in R, there exists a positive integer k such that I_k is τ-dense in R.

Definition 1.4.10 A module is called *τ-noetherian* if it has ACC on τ-closed submodules.

Example 1.4.11 Every τ-torsion and every τ-cocritical module is τ-noetherian.

Proposition 1.4.12 *Let A be a module and $B \leq A$. Then A is τ-noetherian if and only if B and A/B are τ-noetherian.*

Proof. If A is τ-noetherian, then B and A/B are both τ-noetherian by Proposition 1.4.6 (ii) and (iii).

Conversely, suppose that both B and A/B are τ-noetherian. Let $A_1 \subseteq A_2 \subseteq \dots$ be an ascending chain of τ-closed submodules of A. We have $(B + A_i)/A_i \cong B/(B \cap A_i)$, hence $B \cap A_i$ is τ-closed in B for each i. Thus we have the ascending chain $B \cap A_1 \subseteq B \cap A_2 \subseteq \dots$ of τ-closed submodules of B. Then there exists $j \in \mathbb{N}^*$ such that $B \cap A_i = B \cap A_j$ for each $i \geq j$. For each i, denote by C_i the τ-closure of $B + A_i$ in A_i.

We claim that the correspondence $A_i \mapsto C_i$ is injective. Supposing the contrary, we have $C_i = C_k$ for some $i > k$. Consider the natural epimorphism $g : A_i/A_k \to (B + A_i)/(B + A_k)$. Then

$$\mathrm{Ker} g \cong (A_i \cap (B + A_k))/A_k \cong (A_k + (B \cap A_i))/A_k = A_k/A_k = 0\,,$$

hence g is an isomorphism. It follows that

$$A_i/A_k \cong (B + A_i)/(B + A_k) \subseteq C_i/(B + A_k)$$

is τ-torsion, that contradicts the fact that A_k is τ-closed in A.

Now C_i/B is τ-closed in A/B for each i, hence there exists $l \in \mathbb{N}^*$ such that $C_i = C_l$ for each $i \geq l$. Therefore $A_i = A_l$ for each $i \geq l$. Thus A is τ-noetherian. $\square$

Recall that a module A is said to *have finite uniform (Goldie) dimension* if A does not contain an infinite direct sum of non-zero submodules or equivalently if there exists a natural number n such that A contains an essential submodule $U_1 \oplus \cdots \oplus U_n$ for some uniform submodules U_i of A [40, p.40].

Proposition 1.4.13 *Every τ-torsionfree τ-noetherian module has finite uniform dimension.*

Proof. Let A be a τ-torsionfree τ-noetherian module. Suppose that there is an infinite set $(A_j)_{j\in\mathbb{N}^*}$ of non-zero submodules of A whose sum is direct. For each j, let B_j be the τ-closure of $A_1 \oplus \cdots \oplus A_j$ in A. Clearly, each B_j is a τ-closed submodule of A and $B_j \subseteq B_{j+1}$ for each j. Since A is τ-noetherian, there exists $k \in \mathbb{N}^*$ such that $B_k = B_{k+1}$. Now if $a \in A_{k+1}$, then $a \in B_{k+1} = B_k$, hence we have $Ia \subseteq A_1 \oplus \cdots \oplus A_k$ for some τ-dense left ideal I of R. Since $Ia \subseteq A_{k+1}$ and the sum $A_1 + \cdots + A_{k+1}$ is direct, we must have $Ia = 0$. But A is τ-torsionfree, hence $a = 0$, so that $A_{k+1} = 0$, a contradiction. $\square$

Theorem 1.4.14 *The following statements are equivalent:*

(i) R is τ-noetherian.

(ii) If $I_1 \subset I_2 \subset \dots$ is a strictly ascending chain of left ideals of R having union I, then there exists $k \in \mathbb{N}^*$ such that I_k is τ-dense in I.

(iii) If $t(R) \subset I_1 \subset I_2 \subset \dots$ is a strictly ascending chain of left ideals of R having union I, then there exists $k \in \mathbb{N}^*$ such that I_k is τ-dense in I.

(iv) Every direct sum of τ-torsionfree injective modules is injective.

Proof. $(i) \Longrightarrow (ii)$ Let $I_1 \subset I_2 \subset \dots$ be a strictly ascending chain of left ideals of R having union I. For each I_j denote by I'_j the τ-closure of I_j in R. Then $I'_1 \subset I'_2 \subset \dots$ is an ascending chain of τ-closed left ideals of R. Since R is τ-noetherian, there exists $k \in \mathbb{N}^*$ such that $I'_j = I'_k$ for every $j \geq k$. Then

$$I = \bigcup_{j \in \mathbb{N}^*} I_j \subseteq \bigcup_{j \in \mathbb{N}^*} I'_j = I'_k \,,$$

whence it follows that

$$I/I_k \subseteq I'_k/I_k = t(R/I_k)\,.$$

Thus I_k is τ-dense in I.

$(ii) \Longrightarrow (iii)$ Clear.

$(iii) \Longrightarrow (iv)$ Let $(A_l)_{l \in L}$ be a family of τ-torsionfree injective module and denote $A = \bigoplus_{l \in L} A_l$. Let I be a left ideal of R and let $f : I \to A$ be a homomorphism. Since A is τ-torsionfree, we have $t(I) \subseteq \mathrm{Ker} f$, hence f can be extended to a homomorphism $g : I + t(R) \to A$ by taking $g(r) = 0$ for every $r \in t(R)$. Therefore without loss of generality we may assume that $t(R) \subseteq I$.

Let us define a transfinite sequence $(J_\alpha)_\alpha$ of left ideals of R that contain I in the following way:

- $J_0 = t(R)$;
- if α is not a limit ordinal and $I/J_{\alpha-1}$ is finitely generated, then take $J_\alpha = I$;

• if α is not a limit ordinal and $I/J_{\alpha-1}$ is not finitely generated, then take some elements $r_{\alpha s} \in I$ such that

$$J_{\alpha-1} \subset J_{\alpha-1} + Rr_{\alpha 1} \subset J_{\alpha-1} + Rr_{\alpha 1} + Rr_{\alpha 2} \subset \dots$$

and take

$$J_\alpha = J_{\alpha-1} + \sum_{s \in \mathbb{N}^*} Rr_{\alpha s}\,;$$

• if α is a limit ordinal, then take

$$J_\alpha = \bigcup_{\beta<\alpha} J_\beta\,.$$

We use transfinite induction on the least ordinal γ such that $J_\gamma = I$. For $\gamma = 0$, we have $I = J_0 = t(R)$, hence $f = 0$, that trivially extends to a homomorphism $g : R \to A$.

First, suppose that γ is not a limit ordinal. By the induction hypothesis, there exist a finite subset $F_\alpha \subseteq L$ and a homomorphism $f_\alpha : R \to \bigoplus_{l \in F_{\alpha-1}} A_l$ that extends $f|_{J_{\alpha-1}}$.

If $I/J_{\alpha-1}$ is finitely generated, $\mathrm{Im} f \subseteq \bigoplus_{l \in F_\alpha} A_l$ for some finite subset F_α such that $F_{\alpha-1} \subseteq F_\alpha \subseteq L$. Thus f extends to a homomorphism $g : R \to A$.

If I/J_α is not finitely generated, then by the definition of J_α, there exists a strictly ascending chain

$$t(R) \subset J_{\alpha-1} = I_1 \subset I_2 \subset \dots \subseteq I$$

of left ideals of R such that I is their union and each I_l/I_1 is finitely generated. By hypothesis, there exists $k \in \mathbb{N}^*$ such that I_k is τ-dense in I. Since $I_k/J_{\alpha-1}$ is finitely generated, we use the above case to get a homomorphism $g : R \to A$ that extends $f|_{I_k}$. Since $I_k \subseteq \mathrm{Ker}(f - g|_I)$, $f - g|_I$ induces a homomorphism $I/I_k \to A$, which must be zero because I_k is τ-dense in I. Thus g extends f.

Secondly, suppose that α is a limit ordinal. It is enough to show that $\mathrm{Im} f$ is contained in a finite direct sum of A_l's. Suppose the contrary. Then we construct a sequence $(\alpha_n)_{n \in \mathbb{N}^*}$ such that for every $n \in \mathbb{N}^*$ we have: $\alpha_n < \alpha$,

$n < m$ implies $\alpha_n < \alpha_m$ and $f(J_{\alpha_n})$ is not contained in any direct sum consisting of n elements of the family $(A_l)_{l\in L}$. Then $\bigcup_{n\in\mathbb{N}^*} J_{\alpha_n} = I$, because otherwise there exists an ordinal $\beta < \alpha$ such that $\bigcup_{n\in\mathbb{N}^*} J_{\alpha_n} \subseteq J_\beta$, that contradicts the induction hypothesis. Now proceed as above to deduce that f extends to a homomorphism $g : R \to A$. But this is a contradiction.

$(iv) \Longrightarrow (i)$ Suppose that we have a strictly ascending chain $I_1 \subset I_2 \subset \cdots \subset$ of τ-closed left ideals of R and denote by I their union. Note that $I(1 + I_j) \neq 0$ for each $j \in \mathbb{N}^*$. Now consider the homomorphism

$$f : I \to \bigoplus_{j\in\mathbb{N}^*} E(R/I_j)\,, \quad f(r) = (r + I_j)_{j\in\mathbb{N}^*}\,.$$

By hypothesis, $\bigoplus_{j\in\mathbb{N}^*} E(R/I_j)$ is injective, hence there exists $x \in \bigoplus_{j\in\mathbb{N}^*} E(R/I_j)$ such that $f(r) = rx$ for every $r \in I$. Then $f(I)$ is contained in a finite direct sum of $E(R/I_j)$. But this contradicts the fact that $I(1 + I_j) \neq 0$ for each $j \in \mathbb{N}^*$. Thus R is τ-noetherian. □

We can show now that the condition on R to be τ-noetherian assures a direct sum decomposition for any τ-torsionfree injective module.

Theorem 1.4.15 *Let R be τ-noetherian. Then every τ-torsionfree injective module is a direct sum of indecomposable injective modules.*

Proof. Let A be a τ-torsionfree injective module. If $0 \neq x \in A$, then Rx is clearly τ-torsionfree and τ-noetherian by Proposition 1.4.12. Hence by Proposition 1.4.13, Rx has finite uniform dimension, so that Rx has a uniform submodule, hence A has a uniform submodule. Now by Zorn's Lemma, there exists a maximal independent family $(A_i)_{i\in I}$ of uniform submodules of A. By the injectivity of A, we have $E(A_i) \subseteq A$ for each i and, by the fact that each A_i is indecomposable, it follows that the sum $\sum_{i\in I} E(A_i)$ is direct. By Theorem 1.4.14, $\bigoplus_{i\in I} E(A_i)$ is injective, hence it is a direct summand of A. On the other hand, we have $\bigoplus_{i\in I} E(A_i) \trianglelefteq A$. Hence we must have $A = \bigoplus_{i\in I} E(A_i)$. □

1.5 τ-cocritical modules and a generalization

A certain subclass of the class of τ-torsionfree modules is of particular importance. That is the class of τ-cocritical modules.

Definition 1.5.1 A non-zero module A is called *τ-cocritical* if A is τ-torsionfree and every non-zero submodule of A is τ-dense in A.

Proposition 1.5.2 *Let A be a non-zero module. Then the following statements are equivalent:*

(i) A is either τ-torsion or τ-cocritical.

(ii) Every non-zero proper submodule B of A is not τ-closed in A.

Proof. $(i) \Longrightarrow (ii)$ This is clear.

$(ii) \Longrightarrow (i)$ Suppose that A is not τ-torsion, that is, $t(A) \neq A$. If $t(A) \neq 0$, then by hypothesis $A/t(A)$ is not τ-torsionfree, a contradiction. Hence $t(A) = 0$, i.e. A is τ-torsionfree. Now let B be a non-zero proper submodule of A. Then A/B is not τ-torsionfree, hence $t(A/B) \neq 0$. Let $t(A/B) = C/B$. Then $A/C \cong (A/B)/t(A/B)$ is τ-torsionfree, whence $A = C$ by hypothesis. Hence A/B is τ-torsion and thus A is τ-cocritical. $\square$

Recall that a non-zero module A is said to be *uniform* in case each of its non-zero submodules is essential in A.

Proposition 1.5.3 *The following statements are equivalent for a module A:*

(i) A is τ-cocritical.

(ii) A is uniform and A contains a τ-dense τ-cocritical submodule.

(iii) A is τ-torsionfree and A contains a τ-dense τ-cocritical submodule.

(iv) Every non-zero cyclic submodule of A is τ-cocritical.

Proof. $(i) \Longrightarrow (ii)$ Every proper submodule of A is τ-dense and, consequently, essential by Proposition 1.4.2. Hence A is uniform. Trivially, A is a τ-dense τ-cocritical submodule of itself.

(ii) $\Longrightarrow$ *(iii)* Denote $B = t(A)$ and let C be a τ-dense τ-cocritical submodule of A. Then we have $C \trianglelefteq A$. It follows that if $B \neq 0$, then $B \cap C \neq 0$. So, if $B \neq 0$, then the τ-cocritical module C would have a τ-torsion submodule, a contradiction. Hence $t(A) = B = 0$.

(iii) $\Longrightarrow$ *(i)* Let B be a non-zero submodule of A and let C be a τ-dense τ-cocritical submodule of A. Then $C \trianglelefteq A$ and $B \cap C$ is a non-zero τ-dense submodule of A. Now consider the exact sequence

$$0 \to (B + C)/B \to A/B \to A/(B + C) \to 0$$

Here $(B + C)/B \cong C/(B \cap C)$ is τ-torsion and $A/(B + C)$ is τ-torsion as a homomorphic image of A/C. Then B is τ-dense in A. Thus A is τ-cocritical.

(i) $\Longrightarrow$ *(iv)* Let B be a non-zero submodule of A. Then B is τ-torsionfree. Also, every non-zero submodule of B is τ-dense in A and, consequently, τ-dense in B. Hence B is τ-cocritical.

(iv) $\Longrightarrow$ *(i)* Since every non-zero cyclic submodule of A is τ-torsionfree, A is also τ-torsionfree. Now assume that A is not τ-cocritical. Then there exists a proper τ-closed submodule B of A. Let $a \in A \setminus B$. Then $Ra/(Ra \cap B) \cong (Ra + B)/B \subseteq A/B$ is τ-torsionfree. But $Ra \cap B \neq 0$, because A is uniform, contradicting the fact that Ra is τ-cocritical. Thus A is τ-cocritical. □

Proposition 1.5.4 *Let A be a uniform module having a τ-cocritical submodule. Then A has a unique maximal τ-cocritical submodule.*

Proof. Since A is uniform and has a τ-cocritical submodule, A must be τ-torsionfree. Denote by A_i the τ-cocritical submodules of A, where $1 \leq i \leq \omega$ and ω is some ordinal. Then $M = \sum_{i \leq \omega} A_i$ is clearly τ-torsionfree. We claim that M is the unique maximal τ-cocritical submodule of A and we prove it by transfinite induction on ω. If $\omega = 1$, the assertion holds. Now take $\omega > 1$ and suppose that $C = \sum_{i < \omega} A_i$ is τ-cocritical. If $A_\omega \subseteq C$, then $M = C$ and we are done. So, assume further that $A_\omega \nsubseteq C$. Let B be a non-zero submodule of M and consider the exact sequence

$$0 \to (B + C)/B \to M/B \to M/(B + C) \to 0$$

Since A is uniform, we have $B \cap C \neq 0$ and $(B + C) \cap A_\omega \neq 0$. Since C and A_ω are τ-cocritical, it follows that $(B + C)/B \cong C/(B \cap C)$ and

$$M/(B + C) = (B + C + A_\omega)/(B + C) \cong A_\omega/((B + C) \cap A_\omega)$$

are τ-torsion. Then B is τ-dense in M. Thus M is τ-cocritical. $\square$

Proposition 1.5.5 *Let $(A_i)_{i\in I}$ be a set of τ-cocritical submodules of a module A such that $\sum_{i\in I} A_i$ is τ-dense in A and let B be a τ-closed submodule of A. Then there exists $J \subseteq I$ such that the sum $\sum_{j\in J} A_j$ is direct and $B \oplus (\bigoplus_{j\in J} A_j)$ is τ-dense in A.*

Proof. We may suppose that $B \neq A$. Let $\mathcal{M}$ be the family of all subsets $K \subseteq I$ such that the sum $\sum_{k\in K} A_k$ is direct and $B \cap (\sum_{k\in K} A_k) = 0$. Then $\mathcal{M} \neq \emptyset$, because $\emptyset \in \mathcal{M}$. Every chain of elements in $(\mathcal{M}, \subseteq)$ has the union in $\mathcal{M}$. Then by Zorn's Lemma, $\mathcal{M}$ has a maximal element, say J, that clearly has the requested properties. $\square$

We should note that there are torsion theories τ and rings R such that there is no τ-cocritical module and rings R such that for every proper torsion theory τ there is a τ-cocritical module.

Definition 1.5.6 A ring R is said to be *left seminoetherian* if for every proper torsion theory τ on R-Mod there exists a τ-cocritical module.

Example 1.5.7 (1) There is no χ-cocritical module.

Alternatively, if R is an infinite direct product of copies of a field, then there is no τ_D-cocritical module [3].

(2) [94] Every left noetherian ring is left seminoetherian. Indeed, if τ is a proper torsion theory and A is a non-zero τ-torsionfree module, then the set of all submodules of A that are not τ-dense in A has a maximal element B and then A/B is τ-cocritical.

Proposition 1.5.8 *Let A be a noetherian module which is not τ-torsion. Then there exists a submodule B of A such that A/B is τ-cocritical.*

Proof. Since A is not τ-torsion, there exists a proper submodule D of A such that A/D is τ-torsionfree. Denote $F = A/D$. Let $\mathcal{F}$ be the set of all proper submodules G of F such that F/G is not τ-torsion. Then $\mathcal{F}$ is non-empty, containing the submodule 0 of F. Since F is noetherian, the set $\mathcal{F}$ has a maximal element Q. Let H/Q be a non-zero proper submodule of F/Q. By the maximality of Q, $(F/Q)/(H/Q) \cong F/H$ is τ-torsion. But F/Q is not τ-torsion. Then F/Q is τ-torsionfree, because otherwise $t(F/Q) \neq 0$ and $(F/Q)/t(F/Q)$ would be τ-torsion. It follows that F/Q is τ-cocritical. Now let $Q = B/D$. Then $A/B \cong F/Q$ is τ-cocritical. $\square$

Now let us establish a connection between cocritical modules with respect to different hereditary torsion theories.

Proposition 1.5.9 *Let τ and σ be hereditary torsion theories such that $\tau \leq \sigma$. If A is a τ-cocritical module that is not σ-torsion, then A is σ-cocritical.*

Proof. Let $F = A/t_\sigma(A)$. Then F is σ-torsionfree, hence τ-torsionfree. Since A is τ-cocritical, we have either $F = 0$ or $F = A$. Since A is not σ-torsion, $F \neq 0$. Then A is σ-torsionfree. Moreover, every proper τ-closed submodule of A is τ-closed, hence it must be zero. Thus A is σ-cocritical. $\square$

The following particular characterization of cocritical rings with respect to the Dickson torsion theory will be used later on.

Proposition 1.5.10 *Let R be a τ_D-torsionfree ring such that every maximal left ideal of R is of the form $Rp = pR$ for some $p \in R$. Then:*

(i) $R/(\bigcap_{n=1}^{\infty} M^n)$ is τ_D-torsionfree for each maximal left ideal M of R.

(ii) R is τ_D-cocritical if and only if $\bigcap_{n=1}^{\infty} M^n = 0$ for each maximal left ideal M of R.

Proof. If M is a maximal left ideal of R, denote $I = \bigcap_{n=1}^{\infty} M^n$.

(i) Assume the contrary and let $0 \neq a+I \in \mathrm{Soc}(R/I)$. Then there exists a maximal left ideal Rq of R such that $qa \in I$. Suppose that $q \notin Rp = M$. We claim that $Rq+Rp^n = R$ for every $n \in \mathbb{N}$. Indeed, if $Rq+Rp^n \subset R$ for some n, then there exists a maximal left ideal N of R such that $Rq+Rp^n \subseteq N \neq M$. Thus $p^n \in N$, whence $p \in N$ and then $N = M$, a contradiction. It follows that $1 = rq + sp^n$ for some $r, s \in R$, whence $a = rqa + sp^n a$. Since $qa \in I$, we have $qa = bp^n$ for some $b \in R$. Then

$$a = rqa + sp^n a = rbp^n + sp^n a = (rb + sc)p^n$$

for some $c \in R$, hence $a \in I$, a contradiction. Therefore $q \in Rp$, whence we have $q = p$. Then for every $n \in \mathbb{N}$, $pa = p^{n+1}c \in I$. Since R is τ_D-torsionfree, we have $a = p^n c \in I$, a contradiction. Consequently, $\mathrm{Soc}(R/I) = 0$, that is, R/I is τ_D-torsionfree.

(ii) For the direct implication apply (i).

Conversely, suppose that $I = 0$ for each maximal left ideal M of R. Let $0 \neq J \subset R$ be a left ideal of R. Then $J \subseteq M$ for some maximal left ideal M of R. It follows that $J \cap M^{k+1} \subset J \cap M^k$ for some $k \in \mathbb{N}$. Let $a \in (J \cap M^k) \setminus (J \cap M^{k+1})$. Then $a = p^k c$ for some $c \in R \setminus M$. It follows that there exists $1 \leq l \leq k$ such that $p^l b = a$ for some $b \notin J$ and $pb \in J$. Then $(J : b) = M$ and the set of all b's modulo J is a simple submodule of R/J. Hence R is τ_D-cocritical. $\square$

In the sequel we consider a class of modules including the class of simple modules as well as the class of τ-cocritical modules. But first let us give another property of τ-cocritical modules.

Lemma 1.5.11 *Let $f : A \to B$ be a non-zero homomorphism from a τ-cocritical module A to a τ-torsionfree module B. Then f is a monomorphism.*

Proof. If $\mathrm{Ker} f \neq 0$, then $A/\mathrm{Ker} f \cong \mathrm{Im} f \subseteq B$ is τ-torsionfree. But this contradicts the fact that A is τ-cocritical. $\square$

Throughout the rest of this section we denote by $\mathcal{N}$ the class of non-zero modules having the following property: a non-zero module A belongs to $\mathcal{N}$ if and only if every non-zero endomorphism $f \in \mathrm{End}_R(A)$ is a monomorphism.

It is clear by Lemma 1.5.11 that every τ-cocritical module is in $\mathcal{N}$.

For the rest of this section we will suppose the ring R to be commutative. Recall that a module A is said to be *faithful* if $\mathrm{Ann}_R A = 0$.

Theorem 1.5.12 *Let $A \in \mathcal{N}$. Then:*

(i) $\mathrm{Ann}_R a = \mathrm{Ann}_R A$ *for every* $0 \neq a \in A$.

(ii) $\mathrm{Ann}_R A \in \mathrm{Spec}(R)$.

(iii) If A is faithful, then R is a commutative domain.

(iv) A is a torsionfree $R/\mathrm{Ann}_R A$*-module.*

(v) If A is uniform, then A is isomorphic to a submodule of the module $\mathrm{Ann}_{E(R/\mathrm{Ann}_R A)}(\mathrm{Ann}_R A)$.

Proof. (i) Let $r \in R$ be such that $r \notin \mathrm{Ann}_R A$ and let $0 \neq a \in A$. Thus there exists $b \in A$ such that $rb \neq 0$. Then the endomorphism $g : A \to A$ defined by $g(x) = rx$ is a monomorphism. Thus $g(a) = ra \neq 0$, i.e. $r \notin \mathrm{Ann}_R a$. Hence $\mathrm{Ann}_R a \subseteq \mathrm{Ann}_R A$. Clearly $\mathrm{Ann}_R A \subseteq \mathrm{Ann}_R a$. Thus $\mathrm{Ann}_R A = \mathrm{Ann}_R a$.

(ii) Let $r, s \in R$ be such that $rs \in \mathrm{Ann}_R A$ and let $0 \neq a \in A$. If $s \notin \mathrm{Ann}_R A = \mathrm{Ann}_R a$, we have $sa \neq 0$. But then $r \in \mathrm{Ann}_R(sa) = \mathrm{Ann}_R A$. Hence $\mathrm{Ann}_R A \in \mathrm{Spec}(R)$.

(iii) By (i), $0 = \mathrm{Ann}_R A \in \mathrm{Spec}(R)$, hence R is a domain.

(iv) Since $\mathrm{Ann}_R A \in \mathrm{Spec}(R)$, $R/\mathrm{Ann}_R A$ is a commutative domain. Also A has a natural structure of $R/\mathrm{Ann}_R A$-module. Denote $\overline{r} = r + \mathrm{Ann}_R A$ for every $r \in R$. Let $0 \neq a \in A$ and $0 \neq \overline{r} \in R/\mathrm{Ann}_R A$. If $\overline{r}a = 0$, then $ra = 0$, hence $r \in \mathrm{Ann}_R a = \mathrm{Ann}_R A$, i.e. $\overline{r} = 0$, a contradiction. Therefore $\overline{r}a \neq 0$. Thus A is a torsionfree $R/\mathrm{Ann}_R A$-module.

(v) Denote $p = \mathrm{Ann}_R A = \mathrm{Ann}_R a$ for every $a \in A$. Then $Ra \cong R/p$ for every $a \in A$. Since A is uniform, we have $E(A) \cong E(Ra) \cong E(R/p)$. Hence A is isomorphic to a submodule of $\mathrm{Ann}_{E(R/\mathrm{Ann}_R A)} p$. $\square$

Remark. Theorem 1.5.12 will be frequently used for a τ-cocritical module A.

Corollary 1.5.13 *Let A be a τ-cocritical module and $p = \mathrm{Ann}_R A$. Then:*
(i) R/p is τ-cocritical.
(ii) If $0 \neq B \leq \mathrm{Ann}_{E(A)} p$, then B is τ-cocritical.

Proof. (i) If $0 \neq a \in A$, then $R/\mathrm{Ann}_R A = R/\mathrm{Ann}_R a \cong Ra$ by Theorem 1.5.12. Since A is τ-cocritical, it follows by Proposition 1.5.3 that $R/\mathrm{Ann}_R A$ is τ-cocritical.

(ii) Clearly, $B \subseteq E(A)$ is τ-torsionfree. Now let D be a non-zero proper submodule of B. Let $b \in B \setminus D$. Since $\mathrm{Ann}_R b = p$, we have $Rb \cong R/p$, hence by (i), Rb is τ-cocritical. Since $Rb \cap D \neq 0$, it follows that $Rb/(Rb \cap D) \cong (Rb + D)/D$ is τ-torsion, whence B/D cannot be τ-torsionfree. Now by Proposition 1.5.2, B is τ-cocritical. $\square$

We return now to the context of the class $\mathcal{N}$ and give a characterization of uniform modules in $\mathcal{N}$. But first let us recall the following lemma.

Lemma 1.5.14 [101, Lemma 2.31] *Let R be commutative and let $p \in \mathrm{Spec}(R)$. Then the collection of all annihilators of non-zero elements of the module $E(R/p)$ has a unique maximal member, namely p itself.*

Theorem 1.5.15 *Let A be a non-zero module. Then the following statements are equivalent:*
(i) A is uniform and $A \in \mathcal{N}$.
(ii) There exist $p \in \mathrm{Spec}(R)$ and $0 \neq B \trianglelefteq \mathrm{Ann}_{E(R/p)} p$ such that $A \cong B$.

Proof. $(i) \Longrightarrow (ii)$ Let $p = \mathrm{Ann}_R A$ and apply Theorem 1.5.12.

$(ii) \Longrightarrow (i)$ For every $0 \neq a \in E(R/p)$ we have $\mathrm{Ann}_R a \subseteq p$ by Lemma 1.5.14. Hence $\mathrm{Ann}_R a = p$ for every $0 \neq a \in \mathrm{Ann}_{E(R/p)} p$. Then we have $\mathrm{Ann}_R B = \mathrm{Ann}_R a = p$ for every $0 \neq a \in B$. Since $E(R/p)$ is indecomposable injective, B is uniform. Let $0 \neq f \in \mathrm{End}_R(B)$. Then there exists $0 \neq a \in B$ such that $f(a) \neq 0$. Suppose that f is not a monomorphism. Then there exists $0 \neq b \in B$ such that $f(b) = 0$. Since B is uniform, there exist $r, s \in R$ such that $0 \neq ra = sb \in Ra \cap Rb$. Then $rf(a) = f(ra) = f(sb) = sf(b) = 0$, i.e. $r \in \mathrm{Ann}_R f(a) = p$. Hence $ra = 0$, a contradiction. Now f is a monomorphism, so that $B \in \mathcal{N}$. Thus A is uniform and $A \in \mathcal{N}$. $\square$

Corollary 1.5.16 *(i) For every $p \in \mathrm{Spec}(R)$, $R/p \in \mathcal{N}$.*

(ii) Every non-zero cyclic submodule of a module in $\mathcal{N}$ belongs to $\mathcal{N}$.

Let us see some properties of the endomorphism ring of a module in $\mathcal{N}$.

Theorem 1.5.17 *Let $A \in \mathcal{N}$. Then:*

(i) A is indecomposable, $\mathrm{End}_R(A)$ is a domain and A is a torsionfree right $\mathrm{End}_R(A)$-module.

(ii) If A is injective, then $A \cong E(R/\mathrm{Ann}_R A)$ and $\mathrm{End}_R(A)$ is a division ring.

(iii) If A is faithful and not injective, then $\mathrm{End}_R(A)$ is not a division ring.

Proof. (i) Suppose that A is not indecomposable, say $A = B \oplus C$ for some non-zero submodules B and C of A. Let $p : A \to B$ be the canonical projection and $i : B \to A$ the canonical injection. Then ip is a non-zero endomorphism of A that is not a monomorphism, hence $A \notin \mathcal{N}$, a contradiction. Therefore A is indecomposable. Now let $0 \neq f, g \in \mathrm{End}_R(A)$. Then f and g are monomorphisms, hence $fg \neq 0$. Therefore $\mathrm{End}_R(A)$ is a domain. Finally, if $0 \neq f \in \mathrm{End}_R(A)$ and $0 \neq a \in A$, then $af = f(a) \neq 0$, because f is a monomorphism. Hence A is a torsionfree right $\mathrm{End}_R(A)$-module.

(ii) Let $0 \neq a \in A$. By Theorem 1.5.12, $p = \mathrm{Ann}_R A = \mathrm{Ann}_R a \in \mathrm{Spec}(R)$. But $Ra \cong R/\mathrm{Ann}_R a = R/p$, hence $E(R/p) \cong E(Ra) \subseteq A$. By (i), A is indecomposable, hence $A \cong E(R/p)$. Let $0 \neq f \in \mathrm{End}_R(A)$. Then f is a monomorphism. Since A is indecomposable injective, f is an isomorphism. Therefore $\mathrm{End}_R(A)$ is a division ring.

(iii) Suppose that every non-zero $f \in \mathrm{End}_R(A)$ is an isomorphism. By Theorem 1.5.12, R is a commutative domain and A is a torsionfree module.

It follows that R is isomorphic to a subring of the ring $\mathrm{End}_R(A)$, hence $rA = A$ for every non-zero element $r \in R$. Therefore A is torsionfree and divisible. It follows that A is injective, a contradiction. Hence there exists $0 \neq f \in \mathrm{End}_R(A)$ which is not an isomorphism. $\square$

Example 1.5.18 Let R be a commutative domain. Then $\mathrm{Ann}_R E(R) = 0 \in \mathrm{Spec}(R)$. By Theorem 1.5.15, every non-zero submodule of $E(R)$ belongs to the class $\mathcal{N}$. Hence $E(R) \in \mathcal{N}$. Since $E(R)$ is indecomposable injective, every non-zero endomorphism $f \in \mathrm{End}_R(E(R))$ is an isomorphism. If A is a non-zero proper submodule of $E(R)$, then A is not injective because $E(R)$ is indecomposable. By Theorem 1.5.17, $\mathrm{End}_R(A)$ is not a division ring.

Proposition 1.5.19 *Let $A \in \mathcal{N}$ be quasi-injective.*

(i) If $0 \neq B \leq A$, then $B \in \mathcal{N}$.

(ii) If $p = \mathrm{Ann}_R A$ and $A \leq B \leq \mathrm{Ann}_{E(A)} p$, then $B \in \mathcal{N}$.

Proof. (i) Denote by $i : B \to A$ the inclusion homomorphism and let $0 \neq f \in \mathrm{End}_R(B)$. Since A is quasi-injective, there exists $h \in \mathrm{End}_R(A)$ such that $hi = if$. It follows that $h \neq 0$ and thus h is a monomorphism. Therefore f is a monomorphism. Hence $B \in \mathcal{N}$.

(ii) We have $\mathrm{Ann}_R a = p$ for every $0 \neq a \in \mathrm{Ann}_{E(A)} p$. Let $0 \neq f \in \mathrm{End}_R(B)$. Then there exists $0 \neq b \in B$ such that $f(b) \neq 0$. Since $A \trianglelefteq B$, there exists $r \in R$ such that $0 \neq rb \in A \cap Rb$. Therefore $r \notin p$ and $f(rb) = rf(b) \neq 0$, i.e. $f|_A \neq 0$. But f extends to an endomorphism $g \in \mathrm{End}_R(E(A))$. Since A is quasi-injective, we have $g(A) \subseteq A$, hence $f(A) \subseteq A$. Let $h \in \mathrm{End}_R(A)$ be defined by $h(a) = f(a)$ for every $a \in A$. Since $h(b) = f(b) \neq 0$, it follows that h is a monomorphism. Suppose now that f is not a monomorphism. Then there exists $0 \neq c \in B$ such that $f(c) = 0$. Also there exists $s \in R$ such that $0 \neq sc \in A \cap Rc$. We have $h(sc) = f(sc) = sf(c) = 0$, a contradiction. Hence f is a monomorphism. $\square$

Remark. Clearly, Theorem 1.5.17 (i) and Proposition 1.5.19 (i) hold for an arbitrary ring R.

1.6 τ-simple and τ-semisimple modules

We introduce now the torsion-theoretic generalizations of the notions of simple and semisimple module.

Definition 1.6.1 A non-zero module A is called *τ-simple* if it is not τ-torsion and its only τ-closed submodules are $t(A)$ and A.

The connections between τ-simple and τ-cocritical modules are given in the following lemma, whose proof is immediate.

Lemma 1.6.2 *Let A be a non-zero module. Then:*

(i) A is τ-cocritical if and only if A is τ-simple and τ-torsionfree.

(ii) A is τ-simple if and only if $A/t(A)$ is τ-cocritical.

Note that there are τ-simple modules that are not τ-cocritical and there are torsion theories τ and rings R such that there is no τ-simple module.

Example 1.6.3 (1) [65] Let p be a prime and let $n \in \mathbb{N}^*$. Then $\mathbb{Z}_{p^n}$ is clearly a τ_G-simple $\mathbb{Z}$-module that is not τ_G-cocritical.

(2) [3] Let R be an infinite direct product of copies of a field. Then there is no τ_D-simple module.

Proposition 1.6.4 *Let A be a module and $B \leq A$. Then:*

(i) B is τ-simple if and only if the τ-closure B' of B in A is τ-simple.

(ii) If A is τ-torsionfree, then B is τ-cocritical if and only if its τ-closure in A is τ-cocritical.

Proof. (i) First assume that B is τ-simple. Then it is not τ-torsion, hence B' is not τ-torsion. Let C be a proper submodule of B'. Since $C/(B \cap C) \cong (B + C)/B \subseteq B'/B$, $B \cap C$ is τ-dense in C.

If $B \cap C$ is τ-torsion, then C is τ-torsion as an extension of $B \cap C$ by $C/(B \cap C)$, hence $C \subseteq t(B')$. Now by Proposition 1.4.3, C is τ-closed in B' if and only if $C = t(B')$.

If $B \cap C$ is not τ-torsion, then $B \cap C$ is not τ-closed in B because B is τ-simple. Hence $B \cap C$ has to be τ-dense in B, that implies that $(B+C)/C$ is τ-torsion. Also, $B'/(B+C)$ is τ-torsion as a homomorphic image of B'/B. Then B'/C is τ-torsion as an extension of $(B+C)/C$ by $B'/(B+C)$. Hence C is τ-dense in B'.

Therefore B' is τ-simple.

Conversely, assume that B' is τ-simple. Then it is not τ-torsion, hence B is not τ-torsion, because otherwise, since B and B'/B are both τ-torsion, one deduces that B' is τ-torsion. Now let C be a proper submodule of B. If $C \subseteq t(B)$, then by Proposition 1.4.3 C is τ-closed in B if and only if $C = t(B)$. If C is not τ-torsion, then B'/C is τ-torsion because B' is τ-simple. It follows that C is τ-dense in B. Therefore B is τ-simple.

(ii) It follows by Lemma 1.6.2 and by (i). $\square$

Proposition 1.6.5 *Let A be a module, let B be a τ-simple submodule of A and let C be a τ-closed submodule of A. Then either $B \cap C$ is τ-torsion or $B \subseteq C$.*

Proof. Suppose that $B \cap C$ is not τ-torsion. Since B is τ-simple, it follows that $B \cap C$ is τ-dense in C. Then C is τ-dense in $B + C$. But C is τ-closed in A, so C is τ-closed in $B + C$. It follows that $B + C = C$, hence $B \subseteq C$. $\square$

Let us introduce now some notions related to τ-simple modules.

Definition 1.6.6 The *τ-socle* of a module A, denoted by $\mathrm{Soc}_\tau(A)$, is defined as the τ-closure of the sum of all τ-simple submodules of A.

A module A is called *τ-semisimple* if $\mathrm{Soc}_\tau(A) = A$.

A module A is called *τ-semiartinian* if every non-zero factor module of A has a non-zero τ-socle.

We collect in the following lemma some basic properties of the above notions.

Lemma 1.6.7 *(i) The sum of all τ-simple submodules of a module equals the sum of all its τ-closed τ-simple submodules.*

(ii) If A is a τ-torsionfree module, then there exists a maximal independent family $(A_i)_{i\in I}$ of τ-cocritical submodules of A such that $\mathrm{Soc}_\tau(A)$ is the τ-closure of $\bigoplus_{i\in I} A_i$ in A.

(iii) If $(S_i)_{i\in I}$ is the family of all τ-simple submodules of a module A, then

$$t(A) \subseteq \sum_{i\in I} S_i \subseteq \mathrm{Soc}_\tau(A)\,.$$

(iv) Every τ-torsion module is τ-semisimple.

(v) A module A is τ-semisimple if and only if $A/t(A)$ is τ-semisimple.

(vi) R is τ-semiartinian if and only if $\mathrm{Soc}_\tau(A) \trianglelefteq A$ for every module A.

Proof. (i) By Proposition 1.6.4.

(ii) By Proposition 1.5.5.

$(iii) - (vi)$ Straightforward. □

Proposition 1.6.8 *Let A be a module. Then the following statements are equivalent:*

(i) A is τ-semisimple.

(ii) The lattice of τ-closed submodules of A is complemented and every τ-closed submodule of A that is not τ-torsion contains a τ-simple submodule.

(iii) For every proper τ-closed submodule B of A, there exists a τ-simple submodule S of A such that $B \cap S = t(S)$.

Proof. $(i) \Longrightarrow (ii)$ Let B be a τ-closed submodule of A. We may assume without loss of generality that $B \neq t(A)$, because otherwise a complement of B in A is exactly A.

Denote by U the sum of all τ-simple submodules of A. Then by Lemma 1.6.7, $U = \sum_{i\in I} A_i$, where each A_i is a τ-closed τ-simple submodule of A. For every $\emptyset \neq J \subseteq I$, denote $A_J = \sum_{j\in J} A_j$ and if $J = \emptyset$, then put $A_\emptyset = t(A)$. For each A_J, denote by A'_J its τ-closure in A. Consider the family $\mathcal{M}$ of all subsets $J \subseteq I$ such that $B \cap A'_J = t(A)$. Then $\mathcal{M} \neq \emptyset$, because $\emptyset \in \mathcal{M}$. Now take a chain $(J_k)_{k \in K}$ of elements of $\mathcal{M}$ and in what follows denote $J = \bigcup_{k \in K} J_k$. We have to show that $A_J \in \mathcal{M}$. Since B and A_J are τ-closed in A, we have $t(A) \subseteq B \cap A'_J$. Now let $a \in B \cap A'_J$. Then $(A_J : a)$ is τ-dense in R. For every $r \in (A_J : a)$, $ra \in A_J$, hence $ra \in A_F$ for some finite subset $F \subseteq J$. It follows that $ra \in B \cap A'_{J_k} = t(A)$ for some $k \in K$, hence $(A_J : a)a \subseteq t(A)$. Then $a \in t(A)$ and consequently we have $B \cap A'_J = t(A)$. Thus $A_J \in \mathcal{M}$. Now by Zorn's Lemma, $\mathcal{M}$ has a maximal element, say L.

We are going to prove that A'_L is a τ-closed complement of B. We have just seen that the first condition for that holds, namely $B \cap A'_L = t(A)$. Let us suppose that the second condition does not hold, that is, we assume that $B + A'_L$ is not τ-dense in A. Then $U \nsubseteq B + A'_L$, so that $A_i \nsubseteq B + A'_L$ for some $i \in I$. Denote $W = L \cup \{i\}$. We will show that $A_W \in \mathcal{M}$. Since A_i is τ-closed in A, we have $t(A_i) = t(A)$. By Proposition 1.6.5, it follows that

$$A_i \cap (B + A'_J) = t(A_i) = t(A)\,.$$

We have $t(A) \subseteq B \cap A'_W$. Now let $a \in B \cap A'_W$. Then $(A_W : a)$ is a τ-dense left ideal of R. For every $r \in (A_W : a)$, we have $ra \in A_W$, so that $ra = a_i + x$ for some $a_i \in A_i$ and $x \in A_L$, whence $a_i \in A_i \cap (B + A'_L) = t(A) \subseteq A_L$. It follows that $ra \in A_L$, so that $ra \in B \cap A_L = t(A)$. As above, this means that $a \in t(A)$ and consequently $B \cap A'_W = t(A)$. But this contradicts the maximality of L. Therefore $B + A'_L$ is τ-dense in A and consequently the lattice of τ-closed submodules of A is complemented.

For the second part, let B be a τ-closed submodule of A that is not τ-torsion. Then B has a τ-closed complement C, hence we have $B \cap C = t(A)$ and $B + C$ is τ-dense in A. It follows that $A_i \cap C = t(A)$ for some $i \in I$, because otherwise the inclusions $A_i \subseteq C$ for each i imply $A \subseteq C$ and $B = t(A)$, a contradiction. Then $(A_i + C)/C \cong A_i/t(A_i)$ is a τ-closed τ-cocritical submodule of A/C. Also, $(B \cap (A_i + C))/C$ is a τ-cocritical submodule of B/C. Now it follows that $B \cap (A_i + C)$ is a τ-simple submodule of B.

(ii) $\Longrightarrow$ *(iii)* Let B be a proper τ-closed submodule of A. Then B has a τ-closed complement C, hence we have $B \cap C = t(A)$ and $B + C$ is τ-dense in A. Note that C is not τ-torsion, because otherwise $C \subseteq B$. Hence C has a τ-simple submodule S. Then we have $B \cap S \subseteq B \cap C = t(A)$, whence

$$t(S) \subseteq B \cap S \subseteq S \cap t(A) = t(S)\,.$$

Thus $B \cap S = t(S)$.

(iii) $\Longrightarrow$ *(i)* Suppose that A is not τ-semisimple. Then there exists a τ-simple submodule S such that $S \cap \mathrm{Soc}_\tau(A) = t(S)$. Since $S \subseteq \mathrm{Soc}_\tau(A)$, we deduce that $S = t(S)$, a contradiction. □

It is well-known that semisimple rings are characterized by the fact that every module is semisimple. Also, every semisimple module is semiartinian. The torsion-theoretic versions of these properties hold as well.

Theorem 1.6.9 *The following statements are equivalent:*

(i) R is τ-semisimple.

(ii) Every module is τ-semisimple.

Proof. *(i)* $\Longrightarrow$ *(ii)* Let $(S_i)_{i\in I}$ be the family of all τ-simple left ideals of R. Then $\sum_{i\in I} S_i$ is τ-dense in R. Now let A be a module. We may suppose that A is not τ-torsion, because otherwise it is clearly τ-semisimple. Let B be a proper τ-closed submodule of A and let $a \in A \setminus B$. Then $(B : a)$ is not τ-dense in A, hence $S_i \nsubseteq (B : a)$, that is, $S_i a \nsubseteq B$. Then $S_i a$ is not τ-torsion, so that $S_i a \cong S_i$ and thus $S_i a$ is τ-simple. It follows that $S_i a \cap B = 0$, because otherwise $S_i a/(S_i a \cap B)$ would be τ-torsion and $(S_i a + B)/B$ would be τ-torsionfree. Now A is τ-semisimple by Proposition 1.6.8.

(ii) $\Longrightarrow$ *(i)* Obvious. □

Corollary 1.6.10 *If R is τ-semisimple, then R is τ-semiartinian.*

Proof. If R is τ-semisimple, then by Theorem 1.6.9 every module A is τ-semisimple, that is, we have $\mathrm{Soc}_\tau(A) = A$ for every module A. Now by Lemma 1.6.7, R is clearly τ-semiartinian. □

1.7 τ-complemented modules

This section contains some essential properties of τ-complemented modules, that will be useful in the later stages.

Definition 1.7.1 A module A is called *τ-complemented* if every submodule of A is τ-dense in a direct summand of A.

Example 1.7.2 Every semisimple, uniform, τ-torsion or τ-cocritical module is clearly τ-complemented.

We have the following basic characterization of τ-complemented modules.

Proposition 1.7.3 *A module A is τ-complemented if and only if every τ-closed submodule of A is a direct summand of A.*

Proof. Suppose first that A is τ-complemented. Let B be a τ-closed submodule of A. By hypothesis, B is τ-dense in a direct summand C of A. Then $C/B \subseteq A/B$ is τ-torsionfree, whence $B = C$.

Conversely, assume that every τ-closed submodule of A is a direct summand of A. Let B be a submodule of A. Also, let $C/B = t(A/B)$. Since C is τ-closed in A, C is a direct summand of A. Thus B is τ-dense in the direct summand C of A, showing that A is τ-complemented. $\square$

Let us now recall the definition of an extending module and see some similarities between τ-complemented modules and extending modules. A module A is called *extending* if every submodule of A is essential in a direct summand of A. For instance, every uniform or quasi-injective module is extending.

We have an immediate characterization of extending modules, similar to the one for τ-complemented modules. Recall that a submodule of a module A is called *closed* if it does not have any proper essential extension in A.

Proposition 1.7.4 [40, p.55] *A module A is extending if and only if every closed submodule of A is a direct summand.*

Extending modules and τ-complemented modules are also connected in the sense of the following proposition.

Lemma 1.7.5 *(i) Every τ-torsionfree τ-complemented module is extending.*

(ii) Every extending module is τ_G-complemented.

Proof. (i) This is immediate noting that every τ-dense submodule of a τ-torsionfree module is essential (see Proposition 1.4.2).

(ii) Clear, since every essential submodule is τ_G-dense. □

Other examples of τ-complemented modules can be obtain as follows.

Proposition 1.7.6 *(i) The class of τ-complemented modules is closed under homomorphic images and direct summands.*

(ii) Let A be a τ-complemented module and let S be a semisimple module. Then $A \oplus S$ is τ-complemented.

(iii) Let A be a τ-complemented module and let T be a τ-torsion module. Then $A \oplus T$ is τ-complemented.

Proof. (i) Let A be a τ-complemented module and let $B \leq A$. Also let $C/B \leq A/B$. Since A is τ-complemented, C is τ-dense in a direct summand D of A. Then C/B is clearly τ-dense in the direct summand D/B of A/B. The last part is now clear.

(ii) Let $B \leq A \oplus S$. Then we have

$$A + B = A \oplus (S \cap (A + B))$$

and, since $S \cap (A + B)$ is a direct summand of S, it follows that $A + B$ is a direct summand of $A \oplus S$. Now since A is τ-complemented, $A \cap B$ is τ-dense in direct summand C of A. Also write $A = C \oplus D$ for some $D \leq A$. Then

$$(B + C)/B \cong C/(B \cap C) = C/(A \cap B \cap C) = C/(A \cap B)\,,$$

hence B is τ-dense in $B+C$. Moreover, we have

$$(B+C)\cap D=(B+C)\cap A\cap D=((A\cap B)+C)\cap D=C\cap D=0\,,$$

whence $A+B=(B+C)\oplus D$. It follows that B is τ-dense in the direct summand $B+C$ of $A\oplus S$. Therefore $A\oplus S$ is τ-complemented.

(iii) We may assume that A is τ-complemented τ-torsionfree. Let D be a τ-closed submodule of $A\oplus T$. Then by Proposition 1.4.3, $t(D)=t(A\oplus T)=T$. We also have $D=(A\cap D)\oplus T$. Since $A/(A\cap D)\cong(D+A)/D=(A\oplus T)/D$ is τ-torsionfree, the τ-closed submodule $A\cap D$ is a direct summand of the τ-complemented module A, say $A=(A\cap D)\oplus C$. But then

$$A\oplus T=(A\cap D)\oplus C\oplus T=D\oplus C\,,$$

hence D is a direct summand of $A\oplus T$. Thus $A\oplus T$ is τ-complemented. $\square$

In general, the class of τ-complemented modules is not closed under submodules or direct sums, as we can see in the next example.

Example 1.7.7 [104] Denote by $\mathbb{Q}_{(2)}$ the localization of $\mathbb{Z}$ at the prime ideal $2\mathbb{Z}$. Then $\mathbb{Q}_{(2)}\oplus\mathbb{Q}_{(2)}$ is a τ_G-torsionfree abelian group. It is also extending [64], hence it is τ_G-complemented by Lemma 1.7.5. But its submodule $\mathbb{Q}_{(2)}\oplus\mathbb{Z}$ is not extending [64], hence it is not τ_G-complemented by Lemma 1.7.5. Furthermore, $\mathbb{Z}$ and $\mathbb{Q}_{(2)}$ are uniform abelian groups, so that they are clearly τ_G-complemented, whereas we have just seen that $\mathbb{Q}_{(2)}\oplus\mathbb{Z}$ is not τ_G-complemented.

Theorem 1.7.8 *The following statements are equivalent for a module A:*

(i) A is τ-complemented.

(ii) $A=t(A)\oplus B$, where B is a (τ-torsionfree) τ-complemented submodule of A.

Proof. $(i)\Longrightarrow(ii)$ Since A is τ-complemented, $t(A)$ is τ-dense in a direct summand D of A. Then we must have $t(A)=D$, whence $A=t(A)\oplus B$ for some τ-torsionfree submodule B of A. Moreover, B is τ-complemented by Proposition 1.7.6 (i).

$(ii) \Longrightarrow (i)$ By Proposition 1.7.6 (iii). $\square$

The following result will be useful in the process of establishing direct sum decompositions for τ-complemented modules.

Proposition 1.7.9 *Let A be a τ-complemented module with finite uniform dimension. Then every submodule of A has ACC on τ-closed submodules.*

Proof. Denote by n the uniform dimension of A, let $B \leq A$ and consider a properly ascending chain $B_1 \subset B_2 \subset \dots$ of τ-closed submodules of B. Since A is τ-complemented, B_{n+1} is τ-dense in a direct summand C_1 of A. Write $A = C_1 \oplus D_1$ for some submodule D_1 of A. If $D_1 = 0$, then B_{n+1} is τ-dense in A, hence B_{n+1} is τ-dense in B, whence it follows that $B_{n+1} = B$, a contradiction. Thus $D_1 \neq 0$. By Proposition 1.7.6, C_1 is τ-complemented, hence B_n is τ-dense in a direct summand C_2 of C_1. Write $C_1 = C_2 \oplus D_2$ for some submodule D_2 of C_1. If $D_2 = 0$, then B_n is τ-dense in C_1, whence $B_{n+1} = B_n$, a contradiction. Hence $D_2 \neq 0$. Continuing the procedure, we get a direct sum $D_1 \oplus D_2 \oplus \cdots \oplus D_{n+1}$ of non-zero submodules of A, a contradiction. $\square$

Theorem 1.7.10 *The following statements are equivalent for a τ-complemented module A:*

(i) A is a direct sum of a τ-torsion and τ-cocritical modules.

(ii) R has ACC on left ideals of the form $\mathrm{Ann}_R x$, where $x \in A/t(A)$.

Proof. $(i) \Longrightarrow (ii)$ Suppose that $A = T \oplus (\bigoplus_{j \in J} C_j)$ for some τ-torsion module T and some τ-cocritical modules C_j $(j \in J)$. Then $C = \bigoplus_{j \in J} C_j$ is τ-torsionfree and $T = t(A)$. Now consider a properly ascending chain $\mathrm{Ann}_R(c_1) \subset \mathrm{Ann}_R(c_2) \subset \dots$ of left ideals, where each $c_i \in C \cong A/t(A)$. Since

$$Rc_1/\mathrm{Ann}_R(c_i)c_1 \cong R/\mathrm{Ann}_R(c_i) \cong Rc_i \subseteq C\,,$$

it follows that $\mathrm{Ann}_R(c_2)c_1 \subset \mathrm{Ann}_R(c_3)c_1 \subset \dots$ is a properly ascending chain of τ-closed submodules of Rc_1. Also, there exists a finite set $K \subseteq J$ such that $Rc_1 \subseteq \bigoplus_{j\in K} C_j$. But each C_j is uniform, hence $\bigoplus_{j\in K} C_j$ is a τ-complemented module with finite uniform dimension, which is a contradiction by Proposition 1.7.9.

$(ii) \implies (i)$ Assume (ii). By Theorem 1.7.8 and Lemma 1.7.5, we can write $A = t(A) \oplus B$ for some extending submodule B of A. Then $B = \bigoplus_{i\in I} B_i$ for some uniform submodules B_i of B [89, Lemma 3]. By Proposition 1.7.6, each B_i is τ-complemented, hence τ-cocritical. □

1.8 The torsion theories τ_n

In this section we come back to the torsion theories τ_n previously defined, in order to establish a few properties that will be used later on. These torsion theories will be the usual framework to detail results on τ-injectivity.

Throughout this section we will assume the ring R to be commutative.

Let us recall the definition of the torsion theories τ_n. For a positive integer n, let $\mathcal{A}_n$ be the class consisting of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing an ideal $p \in \mathrm{Spec}(R)$ with $\dim p \leq n$. In order to ensure that our study is not vacuous, we assume that $\dim R \geq n$. The class $\mathcal{A}_n$ is closed under submodules and homomorphic images, hence the torsion theory generated by the class $\mathcal{A}_n$ is hereditary. Denote by τ_n this hereditary torsion theory, which can be also seen as being generated by all modules of Krull dimension at most n. Also denote by $\mathcal{T}_n$ and $\mathcal{F}_n$ the τ_n-torsion class, respectively the τ_n-torsionfree class of τ_n.

Note that:

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n \subseteq \dots$$

$$\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \dots \subseteq \mathcal{T}_n \subseteq \dots$$

$$\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots \supseteq \mathcal{F}_n \supseteq \dots$$

Therefore we have

$$\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \ldots$$

If $\dim R = m$, then the above sequences end for $n = m$.

Note also that τ_0 is the hereditary torsion theory generated by the class $\mathcal{A}_0$ consisting of all simple modules, i.e. the Dickson torsion theory τ_D. Recall that this torsion theory is defined for a noncommutative ring R as well.

Recall now that every $p \in \mathrm{Spec}(R)$ is either τ-dense or τ-closed in R. For the torsion theories τ_n we may analyze this by the dimension of the ideal p.

Proposition 1.8.1 *Let* $p \in \mathrm{Spec}(R)$*. Then:*

(i) p *is* τ_n*-dense in* R *if and only if* $\dim p \leq n$*.*

(ii) p *is* τ_n*-closed in* R *if and only if* $\dim p \geq n+1$*.*

Proof. (i) Obvious.

(ii) If p is τ_n-closed in R, then by (i) we have $\dim p \geq n+1$. Now let $\dim p \geq n+1$. In order to prove that R/p is τ_n-torsionfree, we will show that $\mathrm{Hom}_R(A, R/p) = 0$ for every $A \in \mathcal{A}_n$. Let $A \in \mathcal{A}_n$ and let $f \in \mathrm{Hom}_R(A, R/p)$. Without loss of generality, we may assume that $A = U/V$, where U, V are ideals of R containing an ideal $q \in \mathrm{Spec}(R)$ with $\dim q \leq n$. Suppose that $f \neq 0$. Then there exist $r \in U \setminus V$ and $s \in R \setminus p$ such that $f(r+V) = s+p$. If $V \setminus p = \emptyset$, then $V \subseteq p$, hence $q \subseteq p$, so that $\dim p \leq \dim q$, a contradiction. Now let $v \in V \setminus p$. Then $vf(r+V) = vs+p$, hence $f(V) = vs+p$. Since $f(V) = p$, it follows that $vs \in p$, whence $v \in p$ or $s \in p$, a contradiction. Hence $f = 0$ and thus R/p is τ_n-torsionfree. □

The next proposition answers the question whether and when τ_n equals the extreme torsion theories, namely the trivial one and the improper one.

Proposition 1.8.2 *(i)* τ_n *cannot be the trivial torsion theory* ξ *on R-Mod.*

(ii) Let R *be a domain. Then* τ_n *coincides with the improper torsion theory* χ *on R-Mod if and only if* $\dim R = n$*.*

Proof. (i) Note that $\mathcal{A}_n$ contains at least all modules isomorphic to R/M for some maximal ideal M of R.

(ii) Suppose first that $\tau_n = \chi$. If $\dim R \geq n+1$, then R is τ_n-torsionfree by Proposition 1.8.1, a contradiction. Hence $\dim R = n$.

Suppose now that $\dim R = n$. Since $0 \in \mathrm{Spec}(R)$, τ_n is generated by the class $\mathcal{A}_n$ consisting of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing an ideal $p \in \mathrm{Spec}(R)$ with $\dim p \leq n$. Let A be a non-zero module and let $0 \neq a \in A$. Then $Ra \cong R/\mathrm{Ann}_R a$, hence A contains the submodule $Ra \in \mathcal{A}_n$. Hence A is τ_n-torsion. Thus $\tau_n = \chi$. □

We continue with a couple of results on τ_n-cocritical modules.

Proposition 1.8.3 *Let A be a τ_n-cocritical module. Then:*

(i) $\mathrm{Ann}_R A \in \mathrm{Spec}(R)$ *and* $\dim \mathrm{Ann}_R A = n+1$.

(ii) For every natural number $k \neq n$, A is not τ_k-cocritical.

Proof. (i) Denote $p = \mathrm{Ann}_R A$. By Theorem 1.5.12 and Corollary 1.5.13, $p \in \mathrm{Spec}(R)$ and R/p is τ_n-cocritical, hence R/p is τ_n-torsionfree. By Proposition 1.8.1, $\dim p \geq n+1$. Suppose that $\dim p > n+1$. Then there exists $q \in \mathrm{Spec}(R)$ with $\dim q = n+1$ and $p \subset q$. Moreover, again by Proposition 1.8.1, R/q is τ_n-torsionfree. On the other hand, $R/q \cong (R/p)/(q/p)$ is τ_n-torsion, a contradiction. Hence $\dim p = n+1$.

(ii) If A is τ_n-cocritical and τ_k-cocritical, then by (i) we have $p = \mathrm{Ann}_R A \in \mathrm{Spec}(R)$ and $\dim p = n+1 = k+1$, hence $k = n$. □

Throughout we will prove a few results under the hypothesis on $p \in \mathrm{Spec}(R)$ to be *N-prime*, i.e. R/p to be noetherian.

Corollary 1.8.4 *Let p be an N-prime ideal of R. Then R/p is τ_n-cocritical if and only if* $\dim p = n+1$.

Proof. The "only if" part follows by Proposition 1.8.3. Suppose now that $\dim p = n+1$. Then R/p is τ_n-torsionfree. Since the R-module R/p is noetherian, by Proposition 1.5.8 there exists an ideal q of R such that $p \subseteq q$ and R/q is τ_n-cocritical. By Proposition 1.8.3, $q = \mathrm{Ann}_R(R/q) \in \mathrm{Spec}(R)$ and $\dim q = n+1$. Then $p = q$, hence R/p is τ_n-cocritical. $\square$

Example 1.8.5 (1) Let R be a principal ideal domain. Then R is noetherian, $\dim R \leq 1$ and $0 \in \mathrm{Spec}(R)$. By Corollary 1.8.4, R is τ_0-cocritical. In particular, the ring $\mathbb{Z}$ is τ_0-cocritical.

(2) Let $R = K[X_1, \ldots, X_m]$ be the polynomial ring over a field K, where $m \geq 2$. Let $p = (X_1, \ldots, X_{m-n-1})$, where $n < m-1$. Then $p \in \mathrm{Spec}(R)$ and $\dim p = n+1$. By Corollary 1.8.4, $R/p \cong K[X_{m-n}, \ldots, X_m]$ is τ_n-cocritical.

References: T. Albu [3], T. Albu, C. Năstăsescu [4], K. Aoyama [7], L. Bican, L. Salce [10], P. Bland [13], J.L. Bueso, P. Jara [14], M.-C. Chamard [17], S. Crivei [23], [31], S.E. Dickson [39], N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer [40], P. Gabriel [45], J.L. García [47], J.S. Golan [48], [49], A. Goldie [52], [53], O. Goldman [54], A. Hudry [57], J.P. Jans [62], H. Katayama [65], J. Lambek [68], [69], W.G. Lau [70], J. N. Manocha [72], J.-M. Maranda [73], A.P. Mishina, L.A. Skornjakov [76], K. Morita [80], C. Năstăsescu [81], [82], C. Năstăsescu, C. Niţă [86], Z. Papp [92], N. Popescu [94], [95], P.F. Smith, A.M. Viola-Prioli, J.E. Viola-Prioli [104], [105], B. Stenström [107], M. Teply [110], [112], C.L. Walker, E.A. Walker [115], J. Zelmanowitz [119].

Notes on Chapter 1

This chapter mainly contains standard material on torsion theories. Their study began in early 1960's, often in the context of an abelian category and not only R-Mod. Out of an extensive literature, we should mention the influential work of P. Gabriel (1962), J.-M. Maranda (1964), S.E. Dickson (1966), and C.L. Walker and E.A. Walker (1972), that has been completed and followed by many other important papers. Since we need only selected topics in torsion theories, we are far from presenting a complete picture of them, so that we do not insist on their general history. Nevertheless, out of the special notions that we will use, τ-noetherian modules and τ-semisimple modules were first studied by C. Năstăsescu and C. Niţă (1965) and respectively by W.G. Lau (1980). Also, τ-complemented modules were introduced by J.S. Golan (1986) under the name of τ-direct modules and afterwards reconsidered by P.F. Smith, A.M. Viola-Prioli and J.E. Viola-Prioli (1997).

Chapter 2

τ-injective modules

In this chapter we introduce injective modules relative to a hereditary torsion theory τ and we study their main properties. We give characterization theorems for injective modules, including some in terms of a generating class of τ. We show that every module has a τ-injective hull, unique up to an isomorphism. The class of τ-injective modules is also studied in terms of closedness properties. Moreover, the relationship between τ-injectivity and usual injectivity is analyzed. Finally, we discuss a relative injectivity generalizing injectivity with respect to the Dickson torsion theory.

2.1 General properties

Let us begin with a basic characterization theorem, that will serve to define τ-injective modules.

Theorem 2.1.1 *The following conditions are equivalent for a module A:*

(i) A is injective with respect to every monomorphism having a τ-torsion cokernel.

(ii) A is a τ-closed submodule of $E(A)$.

(iii) Any homomorphism from a τ-dense left ideal of R to A can be extended to a homomorphism from R to A.

(iv) $\mathrm{Ext}^1_R(B, A) = 0$ *for every* τ*-torsion module* B.

(v) $\mathrm{Ext}^1_R(R/I, A) = 0$ *for every* τ*-dense left ideal* I *of* R.

Proof. $(i) \Longrightarrow (ii)$ Denote by A' the τ-closure of A in $E(A)$. Then 1_A extends by hypothesis to a homomorphism $h : A' \to A$. Then clearly h is surjective, but also injective, because $A \trianglelefteq A'$. Hence $A' = A$ and consequently A is τ-closed in $E(A)$.

$(ii) \Longrightarrow (iii)$ Let I be a τ-dense left ideal of R and let $g : I \to A$ be a homomorphism. Denote by $i : I \to R$ and $j : A \to E(A)$ the inclusion homomorphisms. Then there exists a homomorphism $h : R \to E(A)$ such that $hi = jg$. If we denote $a = h(1)$, we have

$$(Ra + A)/A \cong R/(A : a)\,.$$

Since $I \subseteq (A : a)$, $(A : a)$ is τ-dense in R, so that $(Ra + A)/A \subseteq E(A)/A$ is τ-torsion. Then by hypothesis it follows that $a \in A$, whence $\mathrm{Im}h \subseteq A$. Thus $h : R \to A$ extends g.

$(iii) \Longrightarrow (i)$ Let C be a module, B a τ-dense submodule of C and $g : B \to A$ a homomorphism. Consider the set $\mathcal{M}$ of all pairs (M, φ), where $B \subseteq M \subseteq C$ and $\varphi : M \to A$ is a homomorphism that extends g. Define on $\mathcal{M}$ a partial order by

$$(M_1, \varphi_1) \leq (M_2, \varphi_2) \Longleftrightarrow M_1 \subseteq M_2 \text{ and } \varphi_2|_{M_1} = \varphi_1\,.$$

Clearly, $\mathcal{M} \neq \emptyset$ and it is inductive. By Zorn's Lemma, $\mathcal{M}$ has a maximal element (M_0, φ_0). We will show that $M_0 = C$. Suppose that there exists $c \in C \setminus M_0$ and denote $I = (M_0 : c)$. Then I is a τ-dense left ideal of R, because $(B : c) \subseteq I$. By hypothesis, the homomorphism $\alpha : I \to A$ defined by $\alpha(x) = \varphi_0(xc)$ can be extended to a homomorphism $\beta : R \to A$. But then the homomorphism

$$\gamma : M_0 + Rc \to A\,, \quad \gamma(m_0 + rc) = \varphi_0(m_0) + \beta(r)$$

extends φ_0, that contradicts the maximality of (M_0, φ_0). Hence $M_0 = C$.

$(ii) \Longrightarrow (iv)$ For every module B, the short exact sequence $0 \to A \to E(A) \to E(A)/A \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(B, E(A)/A) \to \mathrm{Ext}^1_R(B, A) \to \mathrm{Ext}^1_R(B, E(A))$$

The first term is zero because B is τ-torsion and $E(A)/A$ is τ-torsionfree, whereas the last term is again zero by the injectivity of $E(A)$. Hence $\mathrm{Ext}^1_R(B, A) = 0$.

$(iv) \Longrightarrow (v)$ Clear.

$(v) \Longrightarrow (iii)$ Clear. □

Definition 2.1.2 A module satisfying the equivalent conditions of Theorem 2.1.1 is called *τ-injective*.

Actually, when checking τ-injectivity, we can restrict ourselves to τ-dense essential left ideals of R. Thus we have the following proposition.

Proposition 2.1.3 *A module A is τ-injective if and only if any homomorphism from a τ-dense essential left ideal of R to A can be extended to a homomorphism from R to A.*

Proof. The direct implication is obvious. For the converse, let I be a τ-dense left ideal of R and let $f : I \to A$ be a homomorphism. Consider the set $\mathcal{M}$ consisting of all pairs (J, g), where J is a left ideal of R that contains I and $g : J \to A$ is a homomorphism that extends f. Use Zorn's Lemma to obtain a maximal element of $\mathcal{M}$, say (J_0, g_0). Suppose that J_0 is not essential in R. Then there exists a non-zero left ideal K of R such that $J_0 \cap K = 0$. Then the homomorphism $h : J_0 + K \to A$ defined by $h(j + K) = g_0(j)$ clearly extends g_0, a contradiction. Hence $J_0 \trianglelefteq R$. Now the conclusion follows. □

In the view of the following result, let us give first a definition.

Definition 2.1.4 Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. A submodule B of a module A is called $\mathcal{A}$-*dense* if $A/B \in \mathcal{A}$.

The following proposition shows that in order to check τ-injectivity in this case, it is enough to consider $\mathcal{A}$-dense left ideals of R.

Proposition 2.1.5 *Let $\mathcal{A}$ be a class of modules closed under submodules and homomorphic images and let τ be the hereditary torsion theory generated by $\mathcal{A}$. Then the following statements are equivalent for a module A:*

(i) A is τ-injective.

(ii) A is injective with respect to every monomorphism having the cokernel in $\mathcal{A}$.

(iii) Any homomorphism from an $\mathcal{A}$-dense left ideal of R to A can be extended to a homomorphism from R to A.

(iv) $\mathrm{Ext}^1_R(B, A) = 0$ for every module $B \in \mathcal{A}$.

(v) $\mathrm{Ext}^1_R(R/I, A) = 0$ for every $\mathcal{A}$-dense left ideal I of R.

Proof. The equivalences $(ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv) \Longleftrightarrow (v)$ follow in a similar way as for Theorem 2.1.1 and the implication $(i) \Longrightarrow (iv)$ is obvious.

$(iv) \Longrightarrow (i)$ Suppose that A is not τ-closed in $E(A)$. Then there exists $C \leq E(A)$ such that $A \subset C$ and $C/A \in \mathcal{A}$. Now by hypothesis, the exact sequence $0 \to A \to C \to C/A \to 0$ splits, hence A is a direct summand of C. But this is a contradiction, because $A \trianglelefteq C$. Hence A is τ-closed in $E(A)$ and consequently A is τ-injective by Theorem 2.1.1. $\square$

Let us give now a characterization of τ-injective modules in terms of projectivity of some τ-torsion modules with respect to certain short exact sequences of modules.

Proposition 2.1.6 *Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. Then the following statements are equivalent for a module A:*

(i) A is τ-injective.

(ii) Every module of the class $\mathcal{A}$ is projective with respect to the exact sequence of modules $0 \to A \to E(A) \to E(A)/A \to 0$.

Proof. Since $E(A)$ is injective, for every module $B \in \mathcal{A}$ we have $\mathrm{Ext}^1_R(B,A)=0$ if and only if the induced sequence

$$0 \to \mathrm{Hom}_R(B,A) \to \mathrm{Hom}_R(B,E(A)) \to \mathrm{Hom}_R(B,E(A)/A) \to 0$$

is exact, that is, B is projective with respect to the initial exact sequence. □

Certain properties of injective modules over noetherian rings can be generalized to τ-injective modules over rings R such that every $\mathcal{A}$-dense left ideal of R is finitely generated. For instance, we give the following proposition.

Proposition 2.1.7 *Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. Let R be a ring such that every $\mathcal{A}$-dense left ideal of R is finitely generated. Then every module has a maximal τ-injective submodule.*

Proof. Let A be a non-zero module. Denote by $\mathcal{B}$ the set of all τ-injective submodules of A. Then $\mathcal{B} \neq \emptyset$, because $0 \in \mathcal{B}$. Let $(B_j)_{j\in J}$ be a chain in $\mathcal{B}$ and denote $B = \bigcup_{j\in J} B_j$. Let I be an $\mathcal{A}$-dense left ideal of R and let $f : I \to B$ be a homomorphism. If I is generated by $r_1,\dots,r_n$, then $f(I)$ is generated by $f(r_1),\dots,f(r_n)$, hence there exists $k \in J$ such that $f(r_1),\dots,f(r_n) \in B_k$, i.e. $\mathrm{Im} f \subseteq B_k$. Since B_k is τ-injective and $B_k \subseteq B$, there exists an homomorphism $g : R \to B$ that extends f. Hence B is τ-injective. By Zorn's Lemma, $\mathcal{A}$ has a maximal element, which is a maximal τ-injective submodule of A. □

We give now a characterization of stable torsion theories related to τ-injectivity (see also Proposition 1.2.8).

Proposition 2.1.8 *τ is stable if and only if $t(A)$ is a direct summand of every τ-injective module A.*

Proof. First, let A be a τ-injective module. Then A is τ-closed in $E(A)$, whence $t(E(A)) \subseteq A$. By hypothesis, $t(A) = t(E(A))$ is a direct summand of $E(A)$, so that it is injective. Hence $t(A)$ is a direct summand of A.

Conversely, let A be a τ-torsion module. By hypothesis, $t(E(A))$ is a direct summand of $E(A)$. Since $A \subseteq t(E(A))$ and $A \trianglelefteq E(A)$, we must have $t(E(A)) = E(A)$. Hence $E(A)$ is τ-torsion and, consequently, τ is stable. $\square$

Let us now give some properties for τ-torsion or τ-torsionfree τ-injective modules.

Proposition 2.1.9 *(i) Every τ-torsion τ-injective module is quasi-injective.*

(ii) If τ is stable, then every τ-torsion τ-injective module is injective.

Proof. (i) Let A be a τ-torsion τ-injective module. If $B \leq A$, then B is τ-dense in A, so that every homomorphism $B \to A$ extends to an endomorphism of A by the τ-injectivity of A.

(ii) Let A be a τ-torsion τ-injective module. By Proposition 2.1.8, we have $A = t(E(A)) = E(A)$. $\square$

Proposition 2.1.10 *The following statements are equivalent for a module A:*

(i) A is τ-torsionfree τ-injective.

(ii) For every module B and every τ-dense submodule C of B, every homomorphism $C \to A$ uniquely extends to a homomorphism $B \to A$.

Proof. $(i) \Longrightarrow (ii)$ Let B be a module and let C be a τ-dense submodule of B. Then the exact sequence $0 \to C \to B \to B/C \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}_R(B/C, A) \to \mathrm{Hom}_R(B, A) \to \mathrm{Hom}_R(C, A) \to \mathrm{Ext}^1_R(B/C, A)$$

By hypothesis we have $\mathrm{Hom}_R(B/C, A) = 0$ and $\mathrm{Ext}^1_R(B/C, A) = 0$, hence $\mathrm{Hom}_R(B, A) \cong \mathrm{Hom}_R(C, A)$, that gives the requested uniqueness.

$(ii) \Longrightarrow (i)$ Assuming (ii), A is clearly τ-injective. By hypothesis we have $\mathrm{Hom}_R(B, A) \cong \mathrm{Hom}_R(C, A)$ for every module B and every τ-dense submodule C of B. If B is any τ-torsion module and $C = 0$, then from the above exact sequence we get $\mathrm{Hom}_R(B, A) = 0$. Thus A is τ-torsionfree. $\square$

Proposition 2.1.11 *Let A be a module and $B \leq A$.*

(i) If A is τ-torsionfree and B is τ-injective, then B is τ-closed in A.

(ii) If A is τ-injective and B is τ-closed in A, then B is τ-injective.

(iii) If B is τ-injective and $B \trianglelefteq A$, then B is τ-closed in A.

Proof. (i) Let T be a τ-torsion module. The exact sequence $0 \to B \to A \to A/B \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(T, A) \to \mathrm{Hom}_R(T, A/B) \to \mathrm{Ext}^1_R(T, B)$$

Since A is τ-torsionfree and B is τ-injective, the first and the last term are zero, hence we have $\mathrm{Hom}_R(T, A/B) = 0$. Thus A/B is τ-torsionfree, that is, B is τ-closed in A.

(ii) Let T be a τ-torsion module. The exact sequence $0 \to B \to A \to A/B \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(T, A/B) \to \mathrm{Ext}^1_R(T, B) \to \mathrm{Ext}^1_R(T, A)$$

Since A/B is τ-torsionfree and A is τ-injective, the first and the last term are zero, hence we have $\mathrm{Ext}^1_R(T, B) = 0$. Thus B is τ-injective.

(iii) Since B is τ-injective, $E(A)/B = E(B)/B$ is τ-torsionfree. Then so is A/B, that is, B is τ-closed in A. $\square$

Let us now see when the τ-injectivity of a module B assures the τ-injectivity of $\mathrm{Hom}_R(A, B)$.

Theorem 2.1.12 *Let R be commutative and let A and B be modules.*

(i) If B is τ-injective and $\mathrm{Tor}^R_1(R/I, A) = 0$ for every τ-dense ideal I of R, then $\mathrm{Hom}_R(A, B)$ is τ-injective.

(ii) If B is τ-torsionfree τ-injective, then so is $\mathrm{Hom}_R(A, B)$.

Proof. (i) Let I be a τ-dense ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence

$$0 \longrightarrow \mathrm{Tor}^R_1(R/I, A) \longrightarrow I \otimes_R A \xrightarrow{f} R \otimes_R A \xrightarrow{g} R/I \otimes_R A \longrightarrow 0$$

Since $R \otimes_R A \cong A$ and $R/I \otimes_R A \cong A/IA$, we have $\mathrm{Im} f = \mathrm{Ker} g \cong IA$. Thus we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, A) \to I \otimes_R A \to IA \to 0$$

Using the hypothesis, we have $I \otimes_R A \cong IA$. Then we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(A,B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(R \otimes_R A, B) \arrow[r] \arrow[d] & \mathrm{Hom}_R(R, \mathrm{Hom}_R(A,B)) \arrow[d] \\
\mathrm{Hom}_R(IA,B) \arrow[r] & \mathrm{Hom}_R(I \otimes_R A, B) \arrow[r] & \mathrm{Hom}_R(I, \mathrm{Hom}_R(A,B))
\end{tikzcd}$$

where the horizontal arrows are isomorphisms and the first two vertical arrows are epimorphisms. It follows that the third vertical arrow is an epimorphism, that shows that $\mathrm{Hom}_R(A,B)$ is τ-injective.

(ii) Since B is τ-torsionfree, for every τ-torsion module T we have

$$\mathrm{Hom}_R(T, \mathrm{Hom}_R(A,B)) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(T,B)) = 0\,.$$

Hence $\mathrm{Hom}_R(A,B)$ is τ-torsionfree. Let I be a τ-dense ideal of R. It is enough to show that

$$\mathrm{Hom}_R(R, \mathrm{Hom}_R(A,B)) \cong \mathrm{Hom}_R(I, \mathrm{Hom}_R(A,B))\,.$$

By the τ-injectivity of B, the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, A) \to I \otimes_R A \to IA \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}_R(IA,B) \to \mathrm{Hom}_R(I \otimes_R A, B) \to \mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I,A), B) \to 0$$

We claim that $\mathrm{Tor}_1^R(R/I, A)$ is τ-torsion. To this end, consider an exact sequence $0 \to K \to P \to B \to 0$ with P projective. It induces an exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, A) \to R/I \otimes_R K \to W \to 0$$

Now for every τ-torsionfree module F we have the exact sequence

$$\mathrm{Hom}_R(R/I \otimes_R K, E(F)) \to \mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, A), E(F)) \to \mathrm{Ext}_R^1(W, E(F))$$

The last term is clearly zero and

$$\mathrm{Hom}_R(R/I \otimes_R K, E(F)) \cong \mathrm{Hom}_R(K, \mathrm{Hom}_R(R/I, E(F))) = 0\,,$$

whence we get $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, A), E(F)) = 0$. By Proposition 1.2.13, $\mathrm{Tor}_1^R(R/I, A)$ is τ-torsion.

Now $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, A), B) = 0$, whence we obtain the isomorphisms

$$\mathrm{Hom}_R(IA, B) \cong \mathrm{Hom}_R(I \otimes_R A, B) \cong \mathrm{Hom}_R(I, \mathrm{Hom}_R(A, B))\,.$$

On the other hand, again by the τ-injectivity of B, the exact sequence $0 \to IA \to A \to A/IA \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}_R(A/IA, B) \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(IA, B) \to 0$$

Since A/IA is τ-torsion, the last Hom is zero, hence we have the isomorphism $\mathrm{Hom}_R(IA, B) \cong \mathrm{Hom}_R(A, B)$. Therefore we get the isomorphisms

$$\mathrm{Hom}_R(I, \mathrm{Hom}_R(A, B)) \cong \mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_R(R \otimes_R A, B)$$

$$\cong \mathrm{Hom}_R(R, \mathrm{Hom}_R(A, B))\,,$$

that finish the proof. □

2.2 τ-injective hulls

Now we introduce the torsion-theoretic version of the notion of injective hull of a module.

Definition 2.2.1 The τ-closure of a module A in $E(A)$ is called a *τ-injective hull* of A and is denoted by $E_\tau(A)$.

The following result summarizes some first properties of τ-injective hulls, that will be often used.

Lemma 2.2.2 *Let A be a module. Then:*

(i) $E_\tau(A)$ is an essential τ-injective submodule of $E(A)$ and it is the minimal such submodule of $E(A)$.

(ii) $E_\tau(A)/A = t(E(A)/A)$.

(iii) If D is a τ-injective module, then $D = E_\tau(A)$ if and only if A is a τ-dense essential submodule of D.

(iv) If A is τ-torsion (respectively τ-torsionfree or τ-cocritical), then $E_\tau(A)$ has the same property.

Proof. (i) By Theorem 2.1.1.

(ii) By Proposition 1.4.5.

(iii) By (i) and (ii).

(iv) If A is τ-torsion, then by (i), $E_\tau(A)/A$ is τ-torsion, whence it follows that $E_\tau(A)$ is τ-torsion. If A is τ-torsionfree, then by Proposition 1.2.11, $E_\tau(A) \subseteq E(A)$ is τ-torsionfree. The τ-injective hull of a τ-cocritical module is τ-cocritical by Proposition 1.6.4. $\square$

Theorem 2.2.3 *Every module has a τ-injective hull, unique up to an isomorphism.*

Proof. The existence is clear by definition. Let A be a module and suppose that E_1 and E_2 are τ-injective hulls of A. Denote by $i : A \to E_1$ and $j : A \to E_2$ the inclusion homomorphisms. By the τ-injectivity of E_2, there exists a homomorphism $f : E_1 \to E_2$ such that $fi = j$. Since i is an essential monomorphism, it follows that f is a monomorphism. By the τ-injectivity of $f(E_1)$, the exact sequence $0 \to f(E_1) \to E_2 \to E_2/f(E_1) \to 0$ splits, say $E_2 = f(E_1) \oplus B$. Then $j(M) \cap B = 0$. Since $j(M) \trianglelefteq E_2$, we have $B = 0$, hence f is an epimorphism. Thus $E_1 \cong E_2$. $\square$

Now we can characterize τ-injective hulls in terms of their elements.

Theorem 2.2.4 *Let A be a module. Then:*

(i) $E_\tau(A) = \{x \in E(A) \mid (A : x)$ is τ-dense in $R\}$.

(ii) If E is an injective module that cogenerates τ, then

$$E_\tau(A) = \{x \in E(A) \mid f(x) = 0 \text{ for every } f : E(A) \to E \text{ with } f(A) = 0\}\,.$$

Proof. (i) Denote

$$D = \{x \in E(A) \mid (A : x) \text{ is } \tau\text{-dense in } R\}\,.$$

We will prove that A is a τ-dense essential submodule of D and D is τ-injective. It is easy to check that $A \subseteq D$ and D is a submodule of $E(A)$. Moreover, $A \trianglelefteq D$ and A is τ-dense in D by Proposition 1.4.5. Also, D is the maximal submodule of $E(A)$ that contains A as a τ-closed submodule.

Now let T be a τ-torsion module. The exact sequence $0 \to D \to E(A) \to E(A)/D \to 0$ induces the exact sequence

$$\mathrm{Hom}_R(T, E(A)/D) \to \mathrm{Ext}^1_R(T, D) \to \mathrm{Ext}^1_R(T, E(A))$$

Since $E(A)/D$ is τ-torsionfree and $E(A)$ is injective, the first and the last term are zero, hence we have $\mathrm{Ext}^1_R(T, D) = 0$. Thus D is τ-injective. Now it follows that $D = E_\tau(A)$.

(ii) First, let $x \in E_\tau(A)$. Let $f : E(A) \to E$ be a homomorphism with $f(A) = 0$. Let I be a τ-dense left ideal of R such that $Ix \subseteq A$. Then $If(x) = f(Ix) = 0$ and, since E is τ-torsionfree, it follows that $f(x) = 0$.

Now let $x \in E(A)$ be such that $f(x) = 0$ for every homomorphism $f : E(A) \to E$ with $f(A) = 0$. In order to prove that $x \in E_\tau(A)$, it suffices by (i) to show that the left ideal $I = (A : x)$ is τ-dense in R, or equivalently, to show that $\mathrm{Hom}_R(R/I, E) = 0$. Let $g : R/I \to E$ be a homomorphism and let $i : R/I \to E(A)/A$ be the monomorphism defined by $i(r + I) = rx + A$. By the injectivity of E, g extends to a homomorphism $h : E(A)/A \to E$. Denote by $p : E(A) \to E(A)/A$ the natural homomorphism. Since $(hp)(A) = 0$, by hypothesis we get $(hp)(x) = 0$, whence $g = 0$. $\square$

It is known that if B is an essential submodule of a module A, then $E(B) = E(A)$. In the torsion-theoretic case we give the following result.

Lemma 2.2.5 *Let A be a non-zero module and $B \trianglelefteq A$. Then B is τ-dense in A if and only if $E_\tau(B) = E_\tau(A)$.*

Proof. Suppose that B is τ-dense in A. We have $E(A) = E(B)$. On the other hand, $A/B \subseteq t(E(A)/B) = E_\tau(B)/B$, hence $A \subseteq E_\tau(B)$. Then $E_\tau(A) \subseteq E_\tau(B)$, hence $E_\tau(B) = E_\tau(A)$.

Assume now that $E_\tau(A) = E_\tau(B)$. Then $A/B \subseteq E_\tau(A)/B = E_\tau(B)/B$. But $E_\tau(B)/B$ is τ-torsion, hence A/B is τ-torsion. □

Proposition 2.2.6 *Let $(A_i)_{i\in I}$ be a family of modules. If one of the following conditions holds:*

(i) I is a finite set;

(ii) τ is generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images, and every $\mathcal{A}$-dense left ideal of R is finitely generated;

then

$$E_\tau(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} E_\tau(A_i)\,.$$

Proof. (i) Immediate taking into account that in this case the class of τ-injective modules is closed under direct sums (see Theorem 2.3.5).

(ii) Put $A = \bigoplus_{i\in I} A_i$. Since $A_i \trianglelefteq E_\tau(A_i)$ for every $i \in I$, it follows that $A \trianglelefteq \bigoplus_{i\in I} E_\tau(A_i) \trianglelefteq E_\tau(A)$. Since every $\mathcal{A}$-dense left ideal of R is finitely generated, the module $\bigoplus_{i\in I} E_\tau(A_i)$ is τ-injective (see Proposition 2.3.9). Hence $E_\tau(A) = \bigoplus_{i\in I} E_\tau(A_i)$. □

Proposition 2.2.7 *Let τ be a stable torsion theory and let A be a module. Then $E_\tau(A/t(A)) = t(E(A)/E(t(A)) = E(t(A))$ and $E_\tau(A) \cong E_\tau(t(A)) \oplus E_\tau(A/t(A))$.*

Proof. It is similar to the proof of Proposition 1.2.9. Thus we obtain a commutative diagram of the same type with the injective hulls replaced by the τ-injective hulls. Similarly, one shows that $E_\tau(A)/E(t(A))$ is an essential τ-injective extension of $A/t(A)$. Furthermore, since $E_\tau(A)/A$ is τ-torsion, C will be τ-torsion, hence $A/t(A)$ has to be τ-dense in $E_\tau(A)/E(t(A))$. Thus $E_\tau(A/t(A)) = E_\tau(A)/E(t(A))$ and the conclusion follows. □

Proposition 2.2.8 *Let $A = \sum_{i\in I} A_i$, where each A_i is a τ-cocritical τ-injective module. Then there exists $J \subseteq I$ such that $A = \bigoplus_{j\in J} A_j$.*

Proof. Consider a maximal independent family $(A_j)_{j\in J}$ of $(A_i)_{i\in I}$. Then for every $i \in I$, there exists $0 \neq a \in A_i \cap (\bigoplus_{j\in J} A_j)$, which implies that there exists a finite subset $K \subseteq J$ such that $A_i = E_\tau(Ra) \subseteq \bigoplus_{j\in K} A_j$. It follows that $A = \bigoplus_{j\in J} A_j$. □

Proposition 2.2.9 *The following statements are equivalent:*

(i) The essential submodules of every τ-torsionfree module are τ-dense.

(ii) The lattice of τ-closed submodules of every τ-torsionfree module is complemented.

(iii) Every τ-torsionfree τ-injective module is injective.

Proof. $(i) \Longrightarrow (iii)$ Let A be a τ-torsionfree τ-injective module. Then A is τ-closed in $E(A)$. On the other hand, $E(A)$ is τ-torsionfree and $A \trianglelefteq E(A)$, whence by hypothesis we deduce that A is τ-dense in $E(A)$. Thus $A = E(A)$ is injective.

$(iii) \Longrightarrow (i)$ Let A be a τ-torsionfree module and $B \trianglelefteq A$. By hypothesis, we deduce that $E_\tau(B) = E(B) = E(A)$. Since B is τ-dense in $E_\tau(B)$, it follows that B is τ-dense in A.

$(ii) \Longrightarrow (iii)$ Let A be a τ-torsionfree τ-injective module. Then A is τ-closed in $E(A)$. By hypothesis, A has a τ-closed complement C, hence we have $A + C = E(A)$ and $A \cap C = t(E(A)) = 0$. Now clearly A is injective.

$(iii) \implies (ii)$ Let A be a τ-torsionfree module. Note that the lattice $\mathcal{C}_\tau(A)$ is clearly isomorphic to the lattice $\mathcal{C}_\tau(E_\tau(A))$. But $E_\tau(A)$ is injective by hypothesis, hence the lattice $\mathcal{C}_\tau(E_\tau(A))$ is clearly complemented. $\square$

Proposition 2.2.10 *Let A be a τ-cocritical faithful module over a commutative ring R. Then $E_\tau(A) = E(A) \cong E(R)$.*

Proof. Let $0 \neq a \in A$. Since A is τ-cocritical, by Theorem 1.5.12 we have $\mathrm{Ann}_R a = \mathrm{Ann}_R A = 0$. Then $R \cong Ra$ is τ-cocritical, hence every τ-injective module is injective. Now $E(A) = E_\tau(A) = E_\tau(Ra) = E(Ra) \cong E(R)$. $\square$

We continue with a few results on homomorphisms between τ-injective hulls of certain modules.

Proposition 2.2.11 *Let A and B be modules with B τ-torsionfree. Then*

$$\mathrm{Hom}_R(E_\tau(A), E_\tau(B)) = \mathrm{Hom}_R(A, E_\tau(B))\,.$$

Proof. Note that $E_\tau(B)$ is τ-torsionfree and apply Proposition 2.1.10. $\square$

Proposition 2.2.12 *Let A and B be τ-cocritical modules. Then every nonzero $f \in \mathrm{Hom}_R(E_\tau(A), E_\tau(B))$ is an isomorphism.*

Proof. Let $0 \neq f \in \mathrm{Hom}_R(E_\tau(A), E_\tau(B))$. Clearly, $E_\tau(A)$ and $E_\tau(B)$ are τ-cocritical. Then by Lemma 1.5.11, f is a monomorphism. Also, $E_\tau(B)/f(E_\tau(A))$ is τ-torsion. By the τ-injectivity of $f(E_\tau(A))$, it follows that $f(E_\tau(A))$ is a direct summand of $E_\tau(B)$. On the other hand, $f(E_\tau(A))$ is τ-dense in $E_\tau(B)$, hence we have $f(E_\tau(A)) \trianglelefteq E_\tau(B)$ by Proposition 1.4.2. Then we must have $f(E_\tau(A)) = E_\tau(B)$. Thus f is an isomorphism. $\square$

Recall that R is called a *left H-ring* if whenever S_1 and S_2 are simple modules such that $\mathrm{Hom}_R(E(S_1), E(S_2)) \neq 0$, then $S_1 \cong S_2$. For instance, every commutative noetherian ring is an H-ring [101, p.110].

Proposition 2.2.13 *Let R be a left H-ring and let S_1, S_2 be simple modules. Then $S_1 \cong S_2$ if and only if* $\mathrm{Hom}_R(E_\tau(S_1), E_\tau(S_2)) \neq 0$.

Proof. The direct implication is obvious.

Conversely, assume that $\mathrm{Hom}_R(E_\tau(S_1), E_\tau(S_2)) \neq 0$. Let $f : E_\tau(S_1) \to E_\tau(S_2)$ be a non-zero homomorphism. Let $i : E_\tau(S_1) \to E(S_1)$ and $j : E_\tau(S_2) \to E(S_2)$ be the inclusion homomorphisms. By the injectivity of $E(S_2)$, there exists a non-zero homomorphism $g : E(S_1) \to E(S_2)$ such that $gi = jf$. Since R is a left H-ring, it follows that $S_1 \cong S_2$. $\square$

Proposition 2.2.14 *Let R be commutative noetherian and $p, q \in \mathrm{Spec}(R)$. Then $q \subseteq p$ if and only if* $\mathrm{Hom}_R(E_\tau(R/q), E_\tau(R/p)) \neq 0$.

Proof. Assume first that $q \subseteq p$. Then we have the following diagram with exact row:

$$\begin{tikzcd}
0 \arrow[r] & R/q \arrow[r, "i"] \arrow[d, "g"'] & E_\tau(R/q) \arrow[ddl, dashed, "f"] \\
& R/p \arrow[d, "j"'] & \\
& E_\tau(R/p) &
\end{tikzcd}$$

where i and j are inclusion homomorphisms and g is induced by the identity map of R. Since $E_\tau(R/q)$ is τ-injective and $E_\tau(R/q)/(R/q)$ is τ-torsion, there exists a homomorphism $f : E_\tau(R/q) \to E_\tau(R/p)$ such that $fi = jg$. It follows that $f \neq 0$, because $g \neq 0$.

Conversely, let $0 \neq f \in \mathrm{Hom}_R(E_\tau(R/q), E_\tau(R/p))$. Let $j : E_\tau(R/p) \to E(R/p)$ and $i : E_\tau(R/q) \to E(R/q)$ be the inclusion homomorphisms. By injectivity of $E(R/p)$, there exists $0 \neq g \in \mathrm{Hom}_R(E(R/q), E(R/p))$ such that $gi = jf$. Then it follows that $q \subseteq p$ [101, Proposition 4.21]. $\square$

Corollary 2.2.15 *Let R be commutative noetherian and let $p, q \in \mathrm{Spec}(R)$ be such that R/p and R/q are τ-cocritical. Then $q = p$ if and only if* $\mathrm{Hom}_R(E_\tau(R/q), E_\tau(R/p)) \neq 0$.

Proposition 2.2.16 *Let R be a left H-ring, S be a simple module and A be a proper submodule of $E_{\tau_D}(S)$. Then:*

(i) $\mathrm{Soc}(E_{\tau_D}(S)/A) = \bigoplus_{i\in I} S_i$*, where $S_i \cong S$ for every $i \in I$.*

(ii) There exists $B \leq E_{\tau_D}(S)$ containing A such that $\mathrm{Soc}(E_{\tau_D}(S)/B) \cong S$.

Proof. (i) Since $E_{\tau_D}(S)$ is semiartinian, we have $\mathrm{Soc}(E_{\tau_D}(S)/A) \neq 0$. Let $\mathrm{Soc}(E_{\tau_D}(S)/A) = \bigoplus_{i\in I} S_i$, where S_i is simple for every $i \in I$. Let $v : E_{\tau_D}(S) \to E_{\tau_D}(S)/A$ be the natural homomorphism and let $j \in I$. Then we have the following diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r] & S_j \arrow[r, "u"] \arrow[d, "f"'] & E_{\tau_D}(S)/A \arrow[dl, dashed, "g"] \\ & E_{\tau_D}(S_j) & \end{tikzcd}$$

where u, f are inclusion homomorphisms. Since $(E_{\tau_D}(S)/A)/S_j$ is semiartinian and $E_{\tau_D}(S_j)$ is τ_D-injective, there exists $g : E_{\tau_D}(S)/A \to E_{\tau_D}(S_j)$ such that $gu = f$. Then $0 \neq gv \in \mathrm{Hom}_R(E_{\tau_D}(S), E_{\tau_D}(S_j))$. By Proposition 2.2.13, $S_j \cong S$.

(ii) If we denote $B = \mathrm{Ker}(gv)$ with the above notations, then we have

$$0 \neq E_{\tau_D}(S)/B \cong \mathrm{Im}(gv) \subseteq E_{\tau_D}(S_j) \cong E_{\tau_D}(S)\,.$$

Therefore $\mathrm{Soc}(E_{\tau_D}(S)/B) \cong S$. $\square$

It is well known that every module is cogenerated by an injective module. We prove now that every τ_D-torsion module and every τ_D-cocritical module is cogenerated by a τ_D-injective module. But first we give the following lemma.

Lemma 2.2.17 *Let A be a non-zero module which is either semiartinian or τ_D-cocritical and let $0 \neq a \in A$. Then there exist a simple module S and a homomorphism $f : A \to E_{\tau_D}(S)$ such that $f(a) \neq 0$.*

Proof. For the proper left ideal $\mathrm{Ann}_R a$ of R, there exists a maximal left ideal M of R such that $\mathrm{Ann}_R a \subseteq M$. We may define a map $g : Ra \to R/M$

by $g(ra) = r + M$ for every $r \in R$. If $ra = 0$, then $r \in \mathrm{Ann}_R a \subset M$ and $r + M = M$, hence g is well-defined. It is easy to check that g is a homomorphism and $g(a) \neq 0$. Let $S = R/M$. Let $v : S \to E_{\tau_D}(S)$ be the inclusion homomorphism. We may suppose that $Ra \neq A$. Then A/Ra is a non-zero semiartinian module. Since $E_{\tau_D}(S)$ is τ_D-injective, there exists a homomorphism $f : A \to E_{\tau_D}(S)$ that extends vg, hence $f(a) = g(a) \neq 0$. □

Now consider all the isomorphism classes of simple modules and let $(S_i)_{i\in I}$ be a family of representatives, one for each isomorphism class.

Theorem 2.2.18 *Every semiartinian module and every τ_D-cocritical module is cogenerated by the τ_D-injective module $\prod_{i\in I} E_{\tau_D}(S_i)$.*

Proof. Denote $D = \prod_{i\in I} E_{\tau_D}(S_i)$. Let A be a non-zero module which is either semiartinian or τ_D-cocritical and let $0 \neq a \in A$. By Lemma 2.2.17, there exist $i \in I$ and a homomorphism $f_i : A \to E_{\tau_D}(S_i)$ such that $f_i(a) \neq 0$. Let $u_i : E_{\tau_D}(S_i) \to D$ be the canonical injection. Define the homomorphism $h_a : A \to D$ by $h_a = u_i f_i$. We have $h_a(a) = (u_i f_i)(a) \neq 0$. Denote $D_a = D$ for every $0 \neq a \in A$. We define

$$h : A \to \prod_{a\neq 0,\, a\in A} D_a\,, \quad h(x) = (h_a(x))_{a\in A\setminus\{0\}}$$

for every $x \in A$. It is easy to check that h is a monomorphism. Hence A can be embedded in a direct product of copies of D, i.e. A is cogenerated by the τ_D-injective module D. □

2.3 The class of τ-injective modules

Let us see first when the class of τ-injective modules coincides with R-Mod. It is well-known that semisimple rings are characterized by the fact that every module is injective. For τ-injective modules we have the following result.

Proposition 2.3.1 *The following statements are equivalent:*

(i) Every τ-dense left ideal of R is a direct summand of R.

(ii) Every module is τ-injective.

Proof. Immediate by Theorem 2.1.1. □

Remark. Clearly, if R is semisimple, then every module is τ-injective by Proposition 2.3.1. But in general the converse does not hold, as we may see in the next example.

Example 2.3.2 Let K be a field, let J be an infinite set and let $R = K^J$. Let τ be the hereditary torsion theory on R-mod whose corresponding Gabriel filter consists of those left ideals I of R such that there exists a cofinite subset H of J (i.e. $|J \setminus H|$ is finite) such that $K^H \subseteq I$. Note that every τ-dense left ideal of R is a direct summand of R. Then every module is τ-injective by Proposition 2.3.1, but clearly R is not semisimple.

Nevertheless, we can specialize the previous result for the Dickson torsion theory τ_D or for the torsion theories τ_n.

Proposition 2.3.3 *The following statements are equivalent:*

(i) R is semisimple.

(ii) Every maximal left ideal of R is τ_D-injective.

(iii) Every module is τ_D-injective.

Proof. $(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (iii)$ Let S be a simple module and let M be the left maximal ideal of R such that $S = R/M$. Since M is τ_D-injective, $R \cong M \oplus S$. Thus every simple module is projective. Now the conclusion follows by Proposition 2.1.6.

$(iii) \Longrightarrow (i)$ Let A be a non-zero module and $0 \neq a \in A$. Then Ra has a maximal submodule B. If $B = 0$, then Ra is simple. If $B \neq 0$, then by the τ_D-injectivity of B there exists a simple module S such that $Ra = B \oplus S$.

In both cases we have $\mathrm{Soc}(Ra) \neq 0$, hence $\mathrm{Soc}(A) \neq 0$. It follows that every module is semiartinian, that is, τ_D-torsion. Then every τ_D-injective module is injective, hence every module is injective. Thus R is semisimple. $\square$

Corollary 2.3.4 *The following statements are equivalent for a ring R:*

(i) R is semisimple.

(ii) For every $n \in \mathbb{N}$, every module is τ_n-injective.

(iii) There exists $n \in \mathbb{N}$ such that every module is τ_n-injective.

Proof. If there exists $n \in \mathbb{N}$ such that every module is τ_n-injective, then every module is τ_0-injective and apply Proposition 2.3.3. $\square$

In what follows let us study some closedness properties of the class of τ-injective modules.

Theorem 2.3.5 *The class of τ-injective modules is closed under direct products, finite direct sums, direct summands, extensions and τ-closed submodules.*

Proof. Let $(A_i)_{i\in I}$ be a family of τ-injective modules. Also, let A be a module, B be a τ-dense submodule of A and $f : B \to \prod_{i\in I} A_i$ be a homomorphism. For each $j \in I$, denote by $p_j : \prod_{i\in I} A_i \to A_j$ the canonical projection. For each τ-injective module A_j, there exists a homomorphism $g_j : A \to A_j$ that extends $p_j f$. Then $g : A \to \prod_{i\in I} A_i$, defined by $g(a) = (g_j(a))_{j\in I}$, extends f. Thus $\prod_{i\in I} A_i$ is τ-injective. Also, it is now clear that the class of τ-injective modules is closed under finite direct sums.

It is immediate to see that every direct summand of a τ-injective module is again τ-injective.

Now let $0 \to X \to Y \to Z \to 0$ be an exact sequence with X and Z τ-injective. For every τ-torsion module B the previous exact sequence induces the exact sequence

$$\mathrm{Ext}^1_R(B, X) \to \mathrm{Ext}^1_R(B, Y) \to \mathrm{Ext}^1_R(B, Z)$$

Since the first and the last term are both zero by hypothesis, we have $\mathrm{Ext}^1_R(B, Y) = 0$. Thus Y is τ-injective.

Every τ-closed submodule of a τ-injective module is again τ-injective by Proposition 2.1.11. $\square$

We have seen in Theorem 2.3.5 that the class of τ-injective modules is closed under finite direct sums. Under certain extra conditions, the class of τ-injective modules is closed under arbitrary direct sums. But let us see first some necessary and sufficient conditions for the classes of τ-torsion τ-injective modules and τ-torsionfree τ-injective modules to be closed under direct sums.

Theorem 2.3.6 *The following statements are equivalent:*

(i) R has ACC on τ-dense left ideals.

(ii) The class of τ-torsion τ-injective modules is closed under direct sums.

(iii) The class of τ-torsion τ-injective modules is closed under countable direct sums.

Proof. $(i) \Longrightarrow (ii)$ Let $A = \bigoplus_{j\in J} A_j$ be a direct sum of τ-torsion τ-injective modules. Then A is τ-torsion. Let I be a τ-dense left ideal of R and let $f : I \to A$ be a homomorphism. Then $\mathrm{Ker} f$ is τ-dense in I, so that it is τ-dense in R. For every $j \in J$, denote by $p_j : A \to A_j$ the canonical projection.

Let us now show that the set $F = \{j \in J \mid p_j f(I) \neq 0\}$ is finite. Suppose the contrary. Then choose an infinite countable subset $K = \{k_1, k_2, \dots\} \subseteq J$ such that $p_j f(I) = 0$ for every $j \in K$. For every $l \in \mathbb{N}^*$, denote

$$I_l = f^{-1}(\bigoplus_{l\in(J\setminus K)\cup\{k_1,\dots,k_l\}} A_l)\,.$$

But $I_1 \subseteq I_2 \subseteq \dots$ is an infinite countable chain of left ideals of R, which represents a contradiction. Thus F is finite and we have $f(I) \subseteq \bigoplus_{j\in F} A_j$.

Finally, using the previous partial result and the fact that every finite direct sum of τ-injective modules is again τ-injective, we can extend f to a homomorphism from R to A. Thus A is τ-injective.

$(ii) \Longrightarrow (iii)$ Obvious.

$(iii) \Longrightarrow (i)$ Let $I_1 \subseteq I_2 \subseteq \ldots\ldots I_j \subseteq \ldots$ $(j \in J)$ be a chain of τ-dense left ideals of R and set $I = \bigcup_{j\in J} I_j$. Then $\bigoplus_{j\in J} E_\tau(R/I_j)$ is τ-torsion τ-injective by hypothesis. It follows that the homomorphism

$$f : I \to \bigoplus_{j\in J} E_\tau(R/I_j)\,, \quad f(r) = (r + I_j)_{j\in J}$$

extends to a homomorphism $g : R \to A$. Then $g(1) \subseteq \bigoplus_{j\in J} E_\tau(R/I_j)$ has a finite number of non-zero coordinates, hence there exists an index k such that $I = I_k$. Thus R has ACC on τ-dense left ideals. $\square$

Theorem 2.3.7 *The following statements are equivalent:*

(i) τ is noetherian.

(ii) The class of τ-torsionfree τ-injective modules is closed under direct sums.

Proof. $(i) \Longrightarrow (ii)$ Let $(A_k)_{k\in K}$ be a family of τ-torsionfree τ-injective modules. Denote by $p_\lambda : \prod_{k\in K} A_k \to A_\lambda$ the canonical projection. Clearly, both the direct sum and the direct product of the modules A_k are τ-torsionfree. Let I be a τ-dense left ideal of R and let $f : I \to \bigoplus_{k\in K} A_k$ be a homomorphism. By the τ-injectivity of A_k, there exists a homomorphism $h_k : R \to A_k$ that extends $p_k f$. Define the homomorphism

$$h : R \to \prod_{k\in K} A_k\,, \quad h(r) = (h_k(r))_{k\in K}\,.$$

We claim that $\mathrm{Im}h \subseteq \bigoplus_{k\in K} A_k$. For that, it suffices to show that $\{k \in K \mid h_j \neq 0\}$ is finite. Suppose that it is infinite and consider a countable subset of indices $k_1, k_2, \ldots$. For each $l \in \mathbb{N}^*$, denote

$$I_l = \{r \in I \mid h_{k_s}(r) = 0 \text{ for every } s \geq l\}\,.$$

Then $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of left ideals of R. Moreover, their union is I, because if $r \in I$, then $h_k(r) = (p_k f)(r) \neq 0$ for finitely many indices. By hypothesis, there exists an index t such that I_t is τ-dense in R. If $s \geq t$, then $h_{k_s}(I_t) = 0$, so that h_{k_s} induces a homomorphism in $\mathrm{Hom}_R(R/I_t, \prod_{k\in K} A_k) = 0$. But then $h_{k_s} = 0$ for every $s \geq t$, a contradiction.

$(ii) \Longrightarrow (i)$ Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of left ideals of R such that their union I is τ-dense in R. For each k, let J_k denote the τ-closure of I_k in R. Suppose that there is no I_k τ-dense in R. Then each J_k is a proper left ideal of R and we have $J_1 \subseteq J_2 \subseteq \dots$. If we denote by J their union, then $I \subseteq J$, hence J is τ-dense in R. If there exists an index l such that $J_k = J_l$ for every $k \geq l$, then $J_l = J$ is τ-dense in R, a contradiction. Hence we may assume that $J_1 \subset J_2 \subset \dots$. Denote $u_k = q_k p_k$, where $p_k : J \to J/J_k$ is the natural homomorphism and $q_k : J/J_k \to R/J_k$ is the inclusion homomorphism. Denote by E the τ-torsionfree injective module that cogenerates τ (see Theorem 1.2.15). Since R/J_k is τ-torsionfree, it follows that there exists a homomorphism $v_k : R/J_k \to E$ such that $v_k u_k \neq 0$. Now consider the homomorphisms $u : J \to \bigoplus_{k\in\mathbb{N}^*} J_k$ defined by $u(j) = (u_k(j))_{k\in\mathbb{N}^*}$ and $v = \bigoplus_{k\in\mathbb{N}^*} v_k$. Then $\mathrm{Im}(vu)$ is not contain in any finite direct sum of copies of E. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "\alpha"] \arrow[d, "u"] & R \arrow[ddl, dashed, "w"] \\
& \bigoplus_{k\in\mathbb{N}^*} R/J_k \arrow[d, "v"] & \\
& \bigoplus_{k\in\mathbb{N}^*} E &
\end{tikzcd}$$

where α is the inclusion homomorphism. Since J is τ-dense in R and $\bigoplus_{k\in\mathbb{N}^*} E$ is τ-injective by hypothesis, there exists a homomorphism w that extends vu. It follows that $\mathrm{Im}(vu) \subseteq \mathrm{Im}w = Rw(1)$ is contained in a finite direct sum of copies of E, a contradiction. $\square$

Theorem 2.3.8 *The following statements are equivalent:*

(i) R has ACC on τ-dense left ideals and τ is noetherian.

(ii) The class of τ-injective modules is closed under direct sums.

(iii) The class of τ-injective modules is closed under countable direct sums.

Proof. $(i) \Longrightarrow (ii)$ Let $A = \bigoplus_{j\in J} A_j$ be a direct sum of τ-injective modules. Let I be a τ-dense left ideal of R and let $f : I \to A$ be a homomorphism. Suppose that $f(I) \nsubseteq \bigoplus_{j\in F} A_j$ for some finite $F \subseteq J$. As in the proof of Theorem 2.3.6, we construct a strictly increasing chain $I_1 \subset I_2 \subset \ldots$ of left ideals of R having the union I. Since I is τ-dense in R and τ is noetherian, there exists a τ-dense left ideal I_n of R. Now since R has ACC on τ-dense left ideals, it follows that $I_m = I_{m+1} = \ldots$ for some $m \geq n$, a contradiction. Hence there exists a finite $F \subseteq I$ such that $f(I) \subseteq \bigoplus_{j\in F} A_j$. But $\bigoplus_{j\in F} A_j$ is τ-injective, whence it follows that A is τ-injective.

$(ii) \Longrightarrow (iii)$ Obvious.

$(iii) \Longrightarrow (i)$ Let $I_1 \subseteq I_2 \subseteq \ldots\ldots I_j \subseteq \ldots$ $(j \in J)$ be a chain of τ-dense left ideals of R such that $I = \bigcup_{j\in J} I_j$ is τ-dense in R. Define the homomorphism

$$f : I \to \bigoplus_{j\in J} E_\tau(R/I_j)\,, \quad f(r) = (r + I_j)_{j\in J}\,.$$

Then there exists $x \in \bigoplus_{j\in J} E_\tau(R/I_j)$ such that $(r + I_j)_{j\in J} = xr$ for every $r \in I$. If n is a non-zero coordinate of x, then $r + I_n = 0$ for every $r \in I$, so that $I = I_n$. Thus I_n is τ-dense in R. Therefore τ is noetherian. Choosing the left ideals I_j to be τ-dense, we easily get that R has ACC on τ-dense left ideals. $\square$

In the following proposition we ask for a condition on $\mathcal{A}$-dense left ideals, where $\mathcal{A}$ is a generating class for τ.

Proposition 2.3.9 *Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. Let R be a* *ring such that every $\mathcal{A}$-dense left ideal of R is finitely generated. Then every direct sum of τ-injective modules is τ-injective.*

Proof. Let $(D_i)_{i\in I}$ be a family of τ-injective modules and put $D = \bigoplus_{i\in I} D_i$. Let M be an $\mathcal{A}$-dense left ideal of R and let $f : M \to D$ be a homomorphism. Since M is finitely generated, there exists a finite subset J of I such that $f(M) \subseteq D' = \bigoplus_{i\in J} D_i$. Consider the following diagram of modules with exact row

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "u"] \arrow[d, "g"'] & R \arrow[ld, dashed, "h'"'] \arrow[ldd, dashed, "h"] \\ & D' \arrow[d, "v"'] & \\ & D & \end{tikzcd}$$

where $g : M \to D'$ is a homomorphism defined by $g(x) = f(x)$ for each $x \in M$ and u, v are inclusion homomorphisms. Then we have $f = vg$. Since D' is a finite direct sum of τ-injective modules, D' is τ-injective, hence there exists a homomorphism $h' : R \to D'$ such that $h'u = g$. Let $h = vh'$. Then $hu = vh'u = vg = f$. Hence D is τ-injective. $\square$

Let us discuss now when the class of τ-injective modules is closed under homomorphic images.

Theorem 2.3.10 *The following statements are equivalent:*

(i) The class of τ-injective modules is closed under homomorphic images.

(ii) Every τ-torsion module has projective dimension at most 1.

(iii) Every τ-dense submodule of a projective module is projective.

(iv) Every τ-dense left ideal of R is projective.

Proof. $(i) \Longrightarrow (ii)$ Let T be a τ-torsion module and let A be any module. Then the exact sequence $0 \to A \to E(A) \to E(A)/A \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(T, E(A)/A) \to \mathrm{Ext}^2_R(T, A) \to \mathrm{Ext}^2_R(T, E(A))$$

By hypothesis we have the first Ext zero and clearly the last one is zero, hence $\mathrm{Ext}^2_R(T, A) = 0$. Thus T has projective dimension at most 1.

$(ii) \Longrightarrow (iii)$ Let P be a projective module and let B be a τ-dense submodule of P. Also, let A be any module. Then the exact sequence $0 \to B \to P \to P/B \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(P, A) \to \mathrm{Ext}^1_R(B, A) \to \mathrm{Ext}^2_R(P/B, A)$$

The first Ext is clearly zero and the last one is zero by hypothesis, hence $\mathrm{Ext}^1_R(B, A) = 0$. Thus B is projective.

$(iii) \Longrightarrow (iv)$ Obvious.

$(iv) \Longrightarrow (i)$ Let I be a τ-dense left ideal of R. Let A be a τ-injective module and let $f : A \to C$ be an epimorphism with kernel B. The exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(I, B) \to \mathrm{Ext}^2_R(R/I, B) \to \mathrm{Ext}^2(R, B)$$

By the projectivity of I and R, the first and the last Ext are zero, hence we have $\mathrm{Ext}^2_R(R/I, B) = 0$. Now the exact sequence $0 \to B \to A \to C \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(R/I, A) \to \mathrm{Ext}^1_R(R/I, C) \to \mathrm{Ext}^2_R(R/I, B)$$

The first and the last Ext are zero by the τ-injectivity of A and by what we have showed above, hence we have $\mathrm{Ext}^1_R(R/I, C) = 0$. Thus C is τ-injective and consequently the class of τ-injective modules is closed under homomorphic images. □

The previous result can be refined in the following way.

Proposition 2.3.11 *Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. Then the following statements are equivalent for a ring R:*

(i) Every $\mathcal{A}$-dense left ideal of R is projective.

(ii) The class of τ-injective modules is closed under homomorphic images.

Proof. Immediate by Theorem 2.3.10. $\square$

Corollary 2.3.12 *If every $\mathcal{A}$-dense left ideal of R is projective, then every sum of two τ-injective submodules of a module A is τ-injective.*

Proof. By Proposition 2.3.11, every factor module of a τ-injective module is τ-injective. Let B and C be two τ-injective submodules of A and define the homomorphism $f : B \oplus C \to B + C$ by $f(b, c) = b + c$. Then f is an epimorphism, hence $B + C$ is τ-injective. $\square$

Remark. In particular, if R is left hereditary, then every factor module of a τ-injective module is τ-injective and every sum of two τ-injective submodules of a module is τ-injective.

We end this section with an equivalent condition for the class of τ-injective modules to be closed under both direct sums and homomorphic images.

Theorem 2.3.13 *The following statements are equivalent:*

(i) The class of τ-injective modules is closed under direct sums and homomorphic images.

(ii) Every τ-dense left ideal of R is finitely generated projective.

Proof. $(i) \Longrightarrow (ii)$ Let I be a τ-dense left ideal of R. By Theorem 2.3.10, I is projective. Then $I = \bigoplus_{l \in L} J_l$ for some countably generated ideals J_l of R.

We claim that each $J = J_l$ is finitely generated. Let $x_1, x_2, \ldots$ be a countable set of generators for J. For each $i = 1, 2, \ldots$, denote $K_i = \sum_{j=1}^{i} Rx_j$. By hypothesis, $E = \bigoplus_{i=1}^{\infty} E(J/K_i)$ is τ-injective. Let $p_i : J \to J/K_i$ and $u_i : J/K_i \to E(J/K_i)$ be the natural epimorphism and the inclusion homomorphism respectively. Since for every $x \in J$, there exists $n \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} Rx_i$ and then $u_i p_i(x) = 0$ for every $i > n$. Thus we may define the homomorphism $f : J \to E$ by $f(x) = (u_i p_i(x))_{i \geq 1}$. By the τ-injectivity of E, f extends to a homomorphism $g : R \to E$. Then $g(R) \subseteq \bigoplus_{i=1}^{m} E(J/K_i)$ for some $m \in \mathbb{N}$, hence $f(J) \subseteq \bigoplus_{i=1}^{m} E(J/K_i)$, that is, J is generated by $x_1, \ldots, x_m$.

Now L has to be finite, because otherwise there exists a proper countably infinite subset $L' \subseteq L$ and then $J = \bigoplus_{l \in L'} J_l$ is a countably generated direct summand of I that is not finitely generated. Therefore I is finitely generated.

$(ii) \Longrightarrow (i)$ By Theorem 2.3.10, the class of τ-injective modules is closed under homomorphic images. Now let $(A_j)_{j \in J}$ be a family of τ-injective modules and denote $A = \bigoplus_{j \in J} A_j$. Let I be a τ-dense left ideal of R. By hypothesis, I is finitely generated, say by $x_1, \ldots, x_n$. Also let $f : I \to A$ be a homomorphism. Now each $f(x_i)$ is contained in a finite sum of components of A, hence $f(I)$ has the same property. This sum is clearly τ-injective and thus f extends to a homomorphism $g : R \to A$. Hence A is τ-injective, showing that the class of τ-injective modules is closed under direct sums. $\square$

2.4 τ-injectivity versus injectivity

In this section we will see several cases when injectivity and τ-injectivity are or are not the same.

Proposition 2.4.1 *Every τ_G-injective module is injective.*

Proof. Note that every essential left ideal of R is clearly τ_G-dense in R. $\square$

Hence the Goldie torsion theory is not interesting from the point of view of studying torsion-theoretic injectivity. We are going to see that there are some other torsion theories, such as the Dickson torsion theory or the torsion theories τ_n, for which τ-injectivity and usual injectivity do not coincide.

Clearly, we have the following result for an arbitrary torsion theory.

Lemma 2.4.2 *If R is τ-torsion or τ-cocritical, then every τ-injective module is injective.*

In what follows we compare usual injectivity to injectivity with respect to the Dickson torsion theory or the torsion theories τ_n.

Corollary 2.4.3 *Let R be a noetherian commutative domain such that every maximal ideal of R is principal. Then every τ_D-injective module is injective.*

Proof. Since R is a domain, clearly $\mathrm{Soc}(R) = 0$, i.e. R is τ_D-torsionfree. The condition $\bigcap_{n=1}^{\infty} M^n = 0$ for each maximal ideal M of R holds because R is a noetherian domain [101, Proposition 4.23, Corollary 1]. Hence by Proposition 1.5.10, R is τ_D-cocritical. Now use Lemma 2.4.2. $\square$

For the rest of this section the ring R will be assumed to be commutative.

Let $\mathcal{E}$ and $\mathcal{E}_n$ be the class of injective modules and τ_n-injective modules respectively. Then

$$\mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots \supseteq \mathcal{E}_n \supseteq \cdots \supseteq \mathcal{E}\,.$$

It follows that for a module A we have

$$E_{\tau_0}(A) \subseteq E_{\tau_1}(A) \subseteq \cdots \subseteq E_{\tau_n}(A) \subseteq \cdots \subseteq E(A)\,.$$

Therefore for a commutative ring τ_D-injectivity (i.e. τ_0-injectivity) is a generalization of τ_n-injectivity. On the other hand, the τ_0-injective hull of a module is the "closest" to the module among its τ_n-injective hulls.

Proposition 2.4.4 *Let R be a noetherian domain. Then the following statements are equivalent:*
(i) Every τ_n-injective module is injective.
(ii) $\dim R \leq n + 1$.

Proof. Suppose first that $\dim R \geq n + 2$. We will show that there exist τ_n-injective modules which are not injective. By Proposition 1.8.1, R is τ_n-torsionfree and by Proposition 1.8.3, R is not τ_n-cocritical. It follows that $E(R)$ is τ_n-torsionfree and $E(R)$ is not τ_n-cocritical. Then by Lemma 3.1.2, there exists a non-zero proper τ_n-injective submodule A of $E(R)$. Since $E(R)$ is indecomposable, A is not injective.

Suppose now that $\dim R \leq n+1$. If $\dim R \leq n$, then by Proposition 1.8.1, R is τ_n-torsion. If $\dim R = n+1$, then by Corollary 1.8.4, R is τ_n-cocritical. In both cases, every τ_n-injective module is injective by Lemma 2.4.2. □

Remarks. (i) Note that the hypothesis on R to be noetherian is needed only for showing that if $\dim R = n+1$, then every τ_n-injective module is injective.

(ii) The equivalence of the statements (i) and (ii) in Proposition 2.4.4 does not hold for an arbitrary hereditary torsion theory on R-Mod, where R is an arbitrary ring. We will give an example of a commutative ring R with $\dim R = 0$ and a hereditary torsion theory τ on R-Mod with the property that not every τ-injective module is injective.

Example 2.4.5 Let K be a field, let J be an infinite set and let $R = K^J$. Let τ be the hereditary torsion theory on R-Mod whose corresponding Gabriel filter consists of those left ideals I of R such that there exists a cofinite subset H of J such that $K^H \subseteq I$. We have seen in Example 2.3.2 that every module is τ-injective, but R is not semisimple. Therefore there exist τ-injective modules that are not injective. On the other hand, R is von Neumann regular, hence $\dim R = 0$ [67, Theorem 3.71].

Corollary 2.4.6 *If R is either a commutative principal ideal domain or a Dedekind domain, then every τ_n-injective module is injective.*

Proof. Every commutative principal ideal domain R is noetherian with $\dim R \leq 1$. Since every Dedekind domain R is a noetherian ring whose every non-zero prime ideal is maximal, we have $\dim R \leq 1$. Now use Proposition 2.4.4. □

In the sequel, we will give some first examples of non-injective τ_n-injective modules, other examples being included in Chapter 3.

As a consequence of Proposition 2.4.4 and the remark following it, we have the next corollary.

Corollary 2.4.7 *If* $\dim R \geq n+2$*, then there exist non-injective* τ_n*-injective modules.*

Proposition 2.4.8 *Let* R *be a unique factorization domain such that every maximal ideal of* R *is not principal. Then* R *is a* τ_0*-injective* R*-module which is not injective.*

Proof. Since R is not a field, it follows that $R \neq E(R)$, i.e. R is not injective. Consider $E(R)$ as the field of fractions of R. Suppose that $\mathrm{Soc}(E(R)/R) \neq 0$. Then there exists a simple module $S \subseteq E(R)/R$. Let M be the maximal ideal of R such that $S \cong R/M$. Then $\mathrm{Ann}_R S = M$. Let $S = R\overline{a}$, where $\overline{a} = a + R$ and $a \in E(R) \setminus R$. Since R is a unique factorization domain, there exist $b, c \in R$ such that $a = \frac{b}{c}$, where c is not invertible and the greatest common divisor of b and c is 1. Since S is simple, $\mathrm{Ann}_R \overline{a} = M$. Then for every $m \in M$ we have $m\frac{b}{c} \in R$. Hence for every $m \in M$, there exists $d \in R$ such that $m = dc \in Rc$. It follows that $M \subseteq Rc$. But c is not invertible and M is a maximal ideal of R, so we obtain $M = Rc$, i.e. M is a principal ideal. This provides a contradiction. It follows that $\mathrm{Soc}(E(R)/R) = 0$. Hence R is τ_0-closed in $E(R)$. Therefore R is a τ_0-injective R-module. □

Lemma 2.4.9 *Let* R *be a* τ_0*-injective domain with* $\dim R \geq 1$*. Then* R *has no principal maximal ideal.*

Proof. Suppose that M is a principal maximal ideal of R. Then $M \cong R$ is τ_0-dense in R and τ_0-injective, hence M is a direct summand of R, a contradiction. □

Example 2.4.10 (1) Let $R = K[[X_1, \ldots, X_m]]$ $(m \geq 2)$ be the ring of formal power series on the set of commuting indeterminates $X_1, \ldots, X_m$ over a field K. Then $M = RX_1 + \cdots + RX_m$ is the unique maximal ideal of R, hence R is a local ring. Since K is a field, R is a unique factorization domain [97, Chapter VIII, Corollary 2.2.1]. Obviously, the ideal M is not principal. By Proposition 2.4.8, R is a τ_0-injective R-module which is not injective.

(2) Let $R = K[X_1, \ldots, X_m]$ $(m \geq 2)$ be the ring of polynomials on the set of commuting indeterminates $X_1, \ldots, X_m$ over an algebraic closed field K. Then every maximal ideal of R is of the form $(X_1 - a_1, \ldots, X_m - a_m)$, where $a_1, \ldots, a_m \in K$. Therefore the unique factorization domain R does not have any principal maximal ideal. By Proposition 2.4.8, R is a τ_0-injective R-module which is not injective. Moreover, by Lemma 2.4.9, every non-maximal prime ideal of R is τ_0-injective as well. For instance, $(X_{i_1}, \ldots, X_{i_k})$ is τ_0-injective, where $k \in \{1, \ldots, m-1\}$ and $i_1, \ldots, i_k \in \{1, \ldots, m\}$.

Using Proposition 2.4.8, we are able to give examples of non-injective τ_D-injective modules over noncommutative rings as well.

Example 2.4.11 Consider the polynomial ring $R = K[X, Y]$, where K is an algebraically closed field and let Q be the field of fractions of R. By Example 2.4.10 (2), R is a τ_D-injective module that is not injective.

Consider the ring $T = \begin{pmatrix} R & 0 \\ Q & Q \end{pmatrix}$. Then T is left noetherian [82, Chapter II, Example 5.1.6] and $E(T) = M_2(Q)$ as left T-modules [67, p.79]. We have $T = A \oplus B$, where $A = \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ are left ideals of T. Also $M_2(Q) = C \oplus D$, where $C = \begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$ are indecomposable injective T-modules. Then $A \trianglelefteq C$ and $B \trianglelefteq D$. The submodules of C containing A are of the form $\begin{pmatrix} L & 0 \\ Q & 0 \end{pmatrix}$, where L is an R-submodule of Q containing R. Since R is τ_D-injective, $\mathrm{Soc}(Q/R) = 0$, hence $\mathrm{Soc}(C/A) = 0$, i.e. A is τ_D-injective. Clearly, A is not injective.

Remark. Therefore injectivity with respect to the Dickson torsion theory or to the torsion theories τ_n do not coincide in general with the usual injectivity. This is the reason for us to prefer them as the main particular torsion theories in order to strengthen results on τ-injectivity.

2.5 A relative injectivity

At this point, let us leave for the moment the context of torsion theories and discuss a special type of relative injectivity. Thus we will obtain a more general result from which a part of Proposition 2.1.5 will be recovered as a particular case.

Let $\mathcal{C}$ be a class of modules closed under isomorphisms, having also the following property:

For every module M and for every family $(M_i)_{i\in I}$ of submodules of M such that $M_i \in \mathcal{C}$ for every $i \in I$, there exists a subset J of I such that

$$\sum_{i\in I} M_i = \bigoplus_{j\in J} M_j \, .$$

Example 2.5.1 The class $\mathcal{C}$ may be considered to be the class of all simple modules or the class of all τ-cocritical τ-injective modules (see Proposition 2.2.8).

In what follows $\mathcal{C}$ will be a class of modules with the above property.

For every module M, put $\mathcal{C}_0(M) = 0$ and denote by $\mathcal{C}_1(M)$ the sum of all submodules of M which belong to the class $\mathcal{C}$. If M does not contain such submodules, take $\mathcal{C}_1(M) = 0$.

Following [1, p.1336], for every module M we define an ascending chain of submodules of M

$$0 = \mathcal{C}_0(M) \subseteq \mathcal{C}_1(M) \subseteq \cdots \subseteq \mathcal{C}_\alpha(M) \subseteq \mathcal{C}_{\alpha+1}(M) \subseteq \dots$$

where for every ordinal $\alpha \geq 0$,

$$\mathcal{C}_{\alpha+1}(M)/\mathcal{C}_\alpha(M) = \mathcal{C}_1(M/\mathcal{C}_\alpha(M))$$

and for every limit ordinal α,

$$\mathcal{C}_\alpha(M) = \bigcup_{0\leq\beta<\alpha} \mathcal{C}_\beta(M) \, .$$

The ascending chain of submodules of M defined above is called the *$\mathcal{C}$-series* of M.

Since M is a set, there exists an ordinal α such that $\mathcal{C}_\alpha(M) = \mathcal{C}_{\alpha+1}(M) = \dots$. The least ordinal with that property is called the *$\mathcal{C}$-length* of the $\mathcal{C}$-series of M and it is denoted by $l(M)$.

We will denote $c(M) = \mathcal{C}_{l(M)}(M)$. Then $l(c(M)) = l(M)$. A module M is called *$\mathcal{C}$-module* if $c(M) = M$.

Example 2.5.2 If the class $\mathcal{C}$ is the class of all simple modules, then $\mathcal{C}_1(M) = \mathrm{Soc}(M)$ and the $\mathcal{C}$-series of a module is its Loewy series [82, p.115]. In this case, the $\mathcal{C}$-modules are exactly the semiartinian modules [82, Chapter I, Theorem 11.4.10].

Let

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of modules.

Theorem 2.5.3 *The following statements are equivalent for a module D:*

(i) D is injective with respect to every exact sequence (1) *where $C \in \mathcal{C}$.*

(ii) D is injective with respect to every exact sequence (1) *where C is a $\mathcal{C}$-module.*

Proof. $(i) \implies (ii)$ Assume (i) and let (1) be a short exact sequence of modules, where C is a $\mathcal{C}$-module. We may assume without loss of generality that A is a submodule of B and u is the inclusion homomorphism. Since $B/A \cong C$, it follows that B/A is a $\mathcal{C}$-module.

Denote by $\gamma = l(B/A)$, $A_0 = A$ and $A_\gamma = B$. The $\mathcal{C}$-series of B/A is a collection $\{A_\alpha/A_0 \mid \alpha < \gamma\}$, where A_α is a submodule of B for every ordinal $\alpha < \gamma$. It follows that $\{A_\alpha \mid \alpha < \gamma\}$ is a chain of submodules of B such that $A_\beta \subseteq A_\alpha$ whenever $\beta < \alpha$.

Let $f_0 : A \to D$ be a homomorphism. Let α be an ordinal and suppose that for every $\beta < \alpha$ there exists a homomorphism $f_\beta : A_\beta \to D$ such that if $\delta < \beta$, then $f_\beta \mid_{A_\delta} = f_\delta$.

Suppose that α is a successor of an ordinal β, i.e. $\alpha = \beta + 1$. Then A_α / A_β is the sum of all submodules of B/A_β which belong to the class $\mathcal{C}$, say $A_\alpha / A_\beta = \sum_{i \in I} (M_i / A_\beta)$. Then by the definition of the class $\mathcal{C}$, there exists a subset J of I such that

$$A_\alpha / A_\beta = \bigoplus_{j \in J} (M_j / A_\beta) .$$

Then $A_\alpha = \sum_{j \in J} M_j$ and $M_h \cap (\sum_{j \neq h} M_j) = A_\beta$ for every $h \in J$.

By hypothesis, for every $j \in J$ there exists a homomorphism $g_j : M_j \to D$ such that $g_j \mid_{A_\beta} = f_\beta$. If $x \in A_\alpha$, there exist $m_{j_k} \in M_{j_k}$, where $j_k \in J$, $k = 1, \dots, n$, such that $x = m_{j_1} + \cdots + m_{j_n}$. Then we may define $f_\alpha : A_\alpha \to D$ by

$$f_\alpha(x) = g_{j_1}(m_{j_1}) + \cdots + g_{j_n}(m_{j_n}).$$

If also $x = m'_{j_1} + \cdots + m'_{j_n}$ with $m'_{j_k} \in M_{j_k}$, where each $j_k \in J$, then for every $s \in \{1, \dots, n\}$, we have

$$m_{j_s} - m'_{j_s} \in \sum_{k=1,\, k \neq s}^{n} M_{j_k}.$$

Hence there exist $a_k \in A_\beta$ such that $m_{j_k} = m'_{j_k} + a_k$ for every $k = 1, \dots, n$ and $a_1 + \cdots + a_n = 0$. It follows that

$$f_\alpha(x) = \sum_{k=1}^{n} g_{j_k}(m_{j_k}) = \sum_{k=1}^{n} (g_{j_k}(m'_{j_k}) + g_{j_k}(a_k)) =$$

$$= \sum_{k=1}^{n} g_{j_k}(m'_{j_k}) + \sum_{k=1}^{n} f_\beta(a_k) = \sum_{k=1}^{n} g_{j_k}(m'_{j_k}) ,$$

hence f_α is well-defined. It is easy to check that f_α is a homomorphism.

Now suppose that α is a limit ordinal. Then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. If $x \in A_\alpha$, then there exists an ordinal $\beta < \alpha$ such that $x \in A_\beta$ and we may define

$$f_\alpha : A_\alpha \to D\,, \quad f_\alpha(x) = f_\beta(x)\,.$$

If also $x \in A_\delta$, $\delta < \alpha$, then we have either $\delta < \beta$ or $\beta < \delta$, hence $f_\delta(x) = f_\beta(x)$, therefore f_α is well-defined. It is easy to check that f_α is a homomorphism.

By transfinite induction, there exists a homomorphism $h : B \to D$ such that $hu = f_0$, i.e. D is injective with respect to the exact sequence (1), where C is a $\mathcal{C}$-module.

$(ii) \Longrightarrow (i)$ Assume (ii) and let (1) be a short exact sequence of modules, where $C \in \mathcal{C}$. Since $\mathcal{C}_1(D) = D$, D is a $\mathcal{C}$-module. Now the result follows. $\square$

In the particular case when the class $\mathcal{C}$ is the class of all simple modules, as a consequence of Theorem 2.5.3 we obtain in a different way Proposition 2.1.5 in the particular case of the Dickson torsion theory.

Let us see a situation when the condition (i) of Theorem 2.5.3 holds.

Proposition 2.5.4 *Let $\mathcal{C}$ be the class of all τ_n-cocritical τ_n-injective modules. Then every τ_{n+1}-injective module is injective with respect to every exact sequence* (1) *where $C \in \mathcal{C}$.*

Proof. Since C is τ_n-cocritical, C cannot be τ_{n+1}-cocritical by Proposition 1.8.3. But $\tau_n \leq \tau_{n+1}$, hence by Proposition 1.5.9, C is τ_{n+1}-torsion. Now if D is a τ_{n+1}-injective module, it follows that D is injective with respect to every exact sequence (1), where $C \in \mathcal{C}$. $\square$

References: T. Albu [1], [2], J.S. Alin, S. Dickson [5], K. Aoyama [7], G. Bacella [9], P. Bland [11], J.L. Bueso, P. Jara, B. Torrecillas [15], [16], S. Crivei [22], [24], [25], [32], L. Fuchs [44], P. Gabriel [45], J.S. Golan [48], [49], J.S. Golan, M. Teply [51], O. Goldman [54], G. Helzer [56], I.D. Ion

[60], J. Lambek [68], J.-M. Maranda [73], K. Masaike, T. Horigome [74], K. Nishida [88], D.F. Sanderson [98], B. Sarath, K. Varadarajan [100], B. Stenström [107], M. Teply [111], C.L. Walker, E.A. Walker [115].

Notes on Chapter 2

As is the case for some general properties of torsion theories, it is rather difficult to trace all the results on torsion-theoretic injectivity. Different characterizations and general properties on τ-injectivity, sometimes called τ-divisibility, have been established in the 1960's. In what follows we mention some of the more significant later results. G. Helzer (1966) characterized the situation when the class of τ-injective modules is closed under both direct sums and homomorphic images. G. Helzer (1966), J.S. Golan and M. Teply (1973), and K. Masaike and T. Horigome (1980) studied when the class of τ-torsionfree τ-injective, τ-injective, respectively τ-torsion τ-injective modules is closed under direct sums. Many properties of τ-torsionfree τ-injective modules, especially used in the context of localization, were established by J.S. Alin and S. Dickson (1968). Results on the τ-injective hull of a module as well as on the τ-injectivity of $\mathrm{Hom}_R(A, B)$ for a τ-injective module B were given by K. Aoyama (1976). K. Masaike and T. Horigome (1980) showed that from any sum of τ-cocritical τ-injective modules it can be refined a direct sum of τ-cocritical τ-injective modules. This result was the author's motivation to introduce a special class of modules in the final part of the chapter.

Chapter 3

Minimal τ-injective modules

The present chapter deals with minimal τ-injective modules, which play an important part in direct sum decompositions of τ-injective modules. Partial or complete structure theorem for them will be established in several cases. In tight connection with minimal τ-injective modules, we will study the structure of the τ-injective hull of a module. Moreover, we are interested in obtaining information on the structure of the injective hull of a module by studying the existence of (minimal) τ-injective submodules contained in the injective hull. In the end we will give a few results on change of ring and direct sum decompositions for τ-injective hulls.

3.1 General properties

Definition 3.1.1 A non-zero module which is the τ-injective hull of each of its non-zero submodules is called *minimal τ-injective*.

Minimal τ-injective modules play in the torsion-theoretic context a similar part with indecomposable injective modules.

The following lemma collects some first properties of minimal τ-injective modules, that will be frequently used.

Lemma 3.1.2 *(i) Every minimal τ-injective module is uniform.*

(ii) A module A is τ-injective τ-cocritical if and only if A is τ-torsionfree minimal τ-injective.

(iii) Every minimal τ-injective module is either τ-torsion or τ-cocritical.

(iv) The endomorphism ring of a minimal τ-injective module is local.

(v) Let $(A_i)_{i\in I}$ be a family of minimal τ-injective modules. Then they are relatively injective.

Proof. (i) It is clear that every non-zero submodule of a minimal τ-injective module A is essential in A.

(ii) Suppose first that A is τ-injective τ-cocritical. Let B be a non-zero submodule of A. Clearly, $E_\tau(B) \subseteq A$. Since A is τ-cocritical, A is τ-torsionfree, B is τ-dense in A and A is uniform, hence $E(B) = E(A)$. Then $A/B \subseteq t(E(A)/B) = t(E(B)/B) = E_\tau(B)/B$, whence $A \subseteq E_\tau(B)$. Consequently, A is minimal τ-injective.

Conversely, suppose that A is τ-torsionfree minimal τ-injective. Let B be a non-zero submodule of A. Denote $C/B = t(A/B)$. Since A is minimal τ-injective, we have $E_\tau(C) = A$. Then A/B is τ-torsion as an extension of the τ-torsion module C/B by the τ-torsion module $A/C = E_\tau(C)/C$. Thus A is τ-cocritical.

(iii) Let A be a minimal τ-injective module. If A is not τ-torsion, then let $0 \neq a \in A \setminus t(A)$, whence it follows that $A = E_\tau(Ra)$ is τ-torsionfree. Then by (ii), A is τ-cocritical.

(iv) Let A be a minimal τ-injective module and let $0 \neq f \in \mathrm{End}_R(A)$. Then A is uniform. Also, $\mathrm{Ker} f \cap \mathrm{Ker}(1_A - f) = 0$, hence $\mathrm{Ker} f = 0$ or $\mathrm{Ker}(1_A - f) = 0$, that is, either f or $1_A - f$ is a monomorphism. In both cases, the image of this monomorphism is τ-injective, hence it has to be A. Now it follows that f is an automorphism.

(v) Let $i \in I$. Since A_i is a non-zero minimal τ-injective module, every proper factor module of A_i is τ-torsion. Then by the definition of τ-injectivity, A_j is A_i-injective for every $j \in I$. $\square$

Let us now consider a couple of other properties of τ-torsion or τ-torsionfree minimal τ-injective modules.

Theorem 3.1.3 *Let τ be a hereditary torsion theory generated by a class $\mathcal{A}$ of modules closed under submodules and homomorphic images. Then the following statements are equivalent for a non-zero τ-torsion module A:*

(i) A is minimal τ-injective.

(ii) $A = E_\tau(B)$, where $B \in \mathcal{A}$ and B is uniform.

Proof. $(i) \Longrightarrow (ii)$ Since A is τ-torsion, there exists a non-zero submodule B of A such that $B \in \mathcal{A}$. Then $A = E_\tau(B)$. Since A is uniform, it follows that B is uniform.

$(ii) \Longrightarrow (i)$ Let C be a non-zero submodule of A. Since A is τ-torsion, $E_\tau(C)$ is τ-dense in A. Then $E_\tau(C)$ is a direct summand of A. But A is uniform, hence $E_\tau(C) = A$. Hence A is minimal τ-injective. $\square$

In the sequel we assume R to be commutative, unless stated otherwise.

Theorem 3.1.4 *Let A be a τ-torsionfree minimal τ-injective module. Then $A \cong E_\tau(R/p)$, where $p = \mathrm{Ann}_R A \in \mathrm{Spec}(R)$.*

Proof. Note that A is τ-cocritical. By Theorem 1.5.12 and Corollary 1.5.13, $p \in \mathrm{Spec}(R)$ and R/p is τ-cocritical. Let $0 \neq a \in A$. Then again by Theorem 1.5.12 we have $Ra \cong R/\mathrm{Ann}_R a = R/p$. Since A is minimal τ-injective, $A = E_\tau(Ra) \cong E_\tau(R/p)$. $\square$

We obtain some stronger results for minimal τ_n-injective modules.

Proposition 3.1.5 *Let $p \in \mathrm{Spec}(R)$. Then:*

(i) $E_{\tau_n}(R/p)$ is τ_n-torsion minimal τ_n-injective if and only if $\dim p \leq n$.

(ii) If $E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective, then $\dim p = n+1$.

(iii) If $E_{\tau_n}(R/p)$ is τ_n-torsionfree, but not minimal τ_n-injective, then $\dim p \geq n+1$.

(iv) If $\dim p \geq n+2$, then $E_{\tau_n}(R/p)$ is τ_n-torsionfree, but not minimal τ_n-injective.

Proof. (i) Suppose that $E_{\tau_n}(R/p)$ is τ_n-torsion minimal τ_n-injective. Then R/p is τ_n-torsion, hence $\dim p \leq n$ by Proposition 1.8.1.

Assume that $\dim p \leq n$. Then $E_{\tau_n}(R/p)$ is τ_n-torsion by Proposition 1.8.1. Since R/p is uniform and $R/p \in \mathcal{A}_n$, it follows by Theorem 3.1.3 that $E_{\tau_n}(R/p)$ is minimal τ_n-injective.

(ii) Since $E_{\tau_n}(R/p)$ is τ_n-cocritical, then by Proposition 1.8.3, $q = \mathrm{Ann}_R E_{\tau_n}(R/p) \in \mathrm{Spec}(R)$ and $\dim q = n+1$. On the other hand, by Theorem 1.5.12, we have $q = \mathrm{Ann}_R a = p$ for every non-zero element $a \in R/p$. Hence $\dim p = n+1$.

(iii) Since R/p is τ_n-torsionfree, $\dim p \geq n+1$ by Proposition 1.8.1.

(iv) By Proposition 1.8.1, $E_{\tau_n}(R/p)$ is τ_n-torsionfree. Since $\dim p \neq n+1$, it follows by (ii) that $E_{\tau_n}(R/p)$ is not minimal τ_n-injective. □

For a noetherian ring R we are able to establish the form of minimal τ_n-injective modules.

Corollary 3.1.6 *Let R be noetherian and let A be a module. Then:*

(i) A is τ_n-torsion minimal τ_n-injective if and only if $A \cong E(R/p)$ for some $p \in \mathrm{Spec}(R)$ with $\dim p \leq n$.

(ii) A is τ_n-torsionfree minimal τ_n-injective if and only if $A \cong E_{\tau_n}(R/p)$ for some $p \in \mathrm{Spec}(R)$ with $\dim p = n+1$.

Proof. (i) It follows by Proposition 2.1.9, Proposition 1.8.1 and by the fact that $E(R/p)$ is indecomposable.

(ii) The "only if" part follows by Theorem 3.1.4 and Proposition 3.1.5. Conversely, suppose that $A \cong E_{\tau_n}(R/p)$ for some $p \in \mathrm{Spec}(R)$ with $\dim p = n+1$. Then by Corollary 1.8.4, R/p is τ_n-cocritical. Hence $A \cong E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective. □

We have the following characterization of minimal τ_D-injective modules. Recall that τ_D is generated by the class of simple modules.

Proposition 3.1.7 *The following statements are equivalent for a module D over a not necessarily commutative ring:*

(i) D is minimal τ_D-injective.

(ii) $D = E_{\tau_D}(A)$, where A is either τ_D-cocritical or simple.

Proof. $(i) \Longrightarrow (ii)$ Suppose that D is minimal τ_D-injective. If D is τ_D-torsion, then by Theorem 3.1.3, we have $D = E_{\tau_D}(S)$, where S is a simple module. If D is τ_D-torsionfree, then A is τ_D-cocritical.

$(ii) \Longrightarrow (i)$ Suppose that $D = E_{\tau_D}(A)$, where A is either τ_D-cocritical or simple. If A is simple, then D is minimal τ_D-injective by Theorem 3.1.3. If A is τ_D-cocritical, then $D = E_{\tau_D}(A)$ is τ_D-cocritical by Lemma 2.2.2, hence D is minimal τ_D-injective. $\square$

Proposition 3.1.8 *Let R be a not necessarily commutative ring such that every maximal left ideal of R is finitely generated projective and let D be a minimal τ_D-injective R-module. Then for every non-zero proper submodule A of D, there exists a family of simple modules $(S_i)_{i\in I}$ such that*

$$D/A = \bigoplus_{i\in I} E_{\tau_D}(S_i)\,.$$

Proof. By Propositions 2.3.11, D/A is τ_D-injective. Since D is minimal τ_D-injective, D/A is semiartinian. Then $\mathrm{Soc}(D/A) \trianglelefteq D/A$ and $(D/A)/\mathrm{Soc}(D/A)$ is semiartinian, hence $E_{\tau_D}(\mathrm{Soc}(D/A)) = E_{\tau_D}(D/A)$ by Lemma 2.2.5. Now let $\mathrm{Soc}(D/A) = \bigoplus_{i\in I} S_i$, where S_i is simple for every $i \in I$. Then by Proposition 2.2.6, we have

$$D/A = E_{\tau_D}(D/A) = E_{\tau_D}(\mathrm{Soc}(D/A)) = E_{\tau_D}(\bigoplus_{i\in I} S_i) = \bigoplus_{i\in I} E_{\tau_D}(S_i)\,. \qquad \square$$

Clearly, Proposition 3.1.8 holds for R left hereditary left noetherian. We illustrate this in the following example.

Example 3.1.9 The ring $\mathbb{Z}$ is commutative hereditary noetherian with $\dim \mathbb{Z} = 1$. Hence every τ_D-injective $\mathbb{Z}$-module is injective by Proposition 2.4.4. If $\mathcal{P}$ is the set of prime numbers, we have $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^\infty}$, where $\mathbb{Q} = E(\mathbb{Z})$ and $\mathbb{Z}_{p^\infty} = E(\mathbb{Z}_p)$ are indecomposable injective $\mathbb{Z}$-modules for every $p \in \mathcal{P}$.

Corollary 3.1.10 *Let R be a not necessarily commutative left H-ring such that every maximal left ideal of R is finitely generated projective. Let S be a simple R-module and let A be a proper submodule of $E_{\tau_D}(S)$. Then:*

(i) $E_{\tau_D}(S)/A = \bigoplus_{i\in I} E_{\tau_D}(S_i)$, where $S_i \cong S$ for every $i \in I$.

(ii) There exists a proper submodule B of $E_{\tau_D}(S)$ which contains A and $E_{\tau_D}(S)/B \cong E_{\tau_D}(S)$.

Proof. It follows by Propositions 2.2.16 and 3.1.8. $\square$

We end this section with some examples of minimal τ_D-injective modules over a noncommutative and even non-left noetherian ring.

Example 3.1.11 Consider the ring $T = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then T has the following properties:

(i) T is not left noetherian and T is not left semiartinian [82, Chapter II, Example 5.1.6].

(ii) $E(T) = M_2(\mathbb{Q})$ as left T-modules [55, p.82].

The left ideals of T are exactly the sets:

$$\left\{ \begin{pmatrix} z & q \\ 0 & u \end{pmatrix} \mid (z, q) \in H\,, u \in U \right\}$$

where H is a subgroup of $\mathbb{Z} \oplus \mathbb{Q}$ and U is an ideal of $\mathbb{Q}$ such that $U \subseteq H$ [82, Chapter II, Proposition 5.1.1]. Hence we have three types of left ideals of T:

$$\begin{pmatrix} n\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} (n \in \mathbb{N}^*)\,, \quad \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}\,, \quad \left\{ \begin{pmatrix} z & q \\ 0 & 0 \end{pmatrix} \mid (z, q) \in H \right\}.$$

Then the maximal left ideals of T are of the following two types:

$$M_p = \begin{pmatrix} p\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} (p \text{ prime}), \quad M = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}.$$

The Jacobson radical of T is $J(T) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. Clearly, the maximal left ideals of T are essential in T, hence $\mathrm{Soc}(T) \subseteq J(T)$. But $J(T)$ does not contain any simple T-submodule, because $\mathbb{Q}$ does not contain any simple subgroup. Therefore $\mathrm{Soc}(T) = 0$, i.e. T is τ_D-torsionfree.

We have $T = B \oplus C$, where $B = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Clearly, $D = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ is a T-module and $E(T) = D \oplus C$. Hence D and C are injective T-modules. Since B is essential in D, we have $E(B) = D$. The class of τ_D-torsionfree modules is closed under injective hulls and submodules, hence D and C are τ_D-torsionfree. We show that D and C are minimal τ_D-injective.

Let $X = \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{pmatrix}$. Then the T-submodules of X are of the form $X_A = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, where A is a subgroup of $\mathbb{Q}$. Since $\mathbb{Q} = E_{\tau_D}(\mathbb{Z}) = E(\mathbb{Z})$ is τ_D-cocritical, $\mathbb{Q}/A$ contains a simple subgroup, hence X/X_A contains a simple submodule. Since X is τ_D-torsionfree, it follows that X is τ_D-cocritical.

We have

$$D/X = \left\{ \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} + \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} + \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \right\} = T/M$$

Then D/X is simple. But X is essential in D, therefore $E_{\tau_D}(X) = D$. Since X is τ_D-cocritical, D is τ_D-cocritical, hence D is minimal τ_D-injective. Moreover, $E_{\tau_D}(B) = D$. Similarly, we have $E_{\tau_D}(J(T)) = C$ and C is minimal τ_D-injective.

The only τ_D-torsion minimal τ_D-injective modules are $E(T/M)$ and $E(T/M_p)$.

3.2 τ-injective hulls versus injective hulls

Let us begin with a general result characterizing when the τ-injective hull and the injective hull of a module coincide.

Theorem 3.2.1 *Let A be a τ-torsionfree module. Then $E_\tau(A) = E(A)$ if and only if for every left ideal I of R and for every homomorphism $f : I \to A$, there exist a τ-dense left ideal J of R with $I \subseteq J$ and a homomorphism $g : J \to A$ that extends f.*

Proof. First suppose that $E_\tau(A) = E(A)$. Let I be a left ideal of R and let $f : I \to A$ be a homomorphism. Then there exists a homomorphism $h : R \to E(A)$ that extends f. By hypothesis, $J = (A : h(1))$ is a τ-dense left ideal of R, $I \subseteq J$ and $h(J) \subseteq A$. Then $g = h|_J$ extends f.

For the converse, let I be a left ideal of R and let $f : I \to A$ be a homomorphism. Let $h : R \to E(A)$ be the homomorphism that extends f and denote $x = h(1)$.

We claim that $x \in E_\tau(A)$. Suppose the contrary. Let $u : (A : x) \to A$ be the homomorphism defined as the right multiplication by x. Then there exist a τ-dense left ideal I' of R with $(A : x) \subseteq I'$ and a homomorphism $v : I' \to A$ that extends u. Then $(A : x) \subset I'$, because if the equality holds, then $x \in E_\tau(A)$, a contradiction. It follows that there exists $y \in E(A)$ such that v is the right multiplication by y. Since $(A : x) \subset I' \subseteq (A : y)$, $(A : y)$ is τ-dense in R. But $E(A)$ is τ-torsionfree, whence we deduce that $\mathrm{Hom}_R(R/(A : y), E(A)) = 0$. Since $x \neq y$, it follows that $(A : y)(x - y) \neq 0$. Hence there exists $b \in (A : y)$ such that $bx \neq by$. But $Rb(x - y) \cap A \neq 0$, so there exists $r \in R$ such that $0 \neq rb(x - y) \in A$. Since $rby \in A$, we have $rbx \in A$. Then $rb(x - y) = rbx - rby = u(rb) - v(rb) = 0$, a contradiction. Therefore $x \in E_\tau(A)$.

Now let $\alpha : I \to E_\tau(A)$ be a homomorphism and denote $K = \alpha^{-1}(A)$. Then K is τ-dense in R, because α induces a monomorphism $I/K \to$

$E_\tau(A)/A$. Let $\beta = \alpha|_K$. Since $E_\tau(A)$ is τ-torsionfree τ-injective, it follows by Proposition 2.1.10 that there exists a unique homomorphism $\gamma : I \to E_\tau(A)$ that extends β. By the above arguments, there exists $x \in E_\tau(A)$ such that β is the right multiplication by x. By uniqueness, we deduce that α is the right multiplication by x. Thus α can be extended to a homomorphism $R \to E_\tau(A)$. Therefore $E_\tau(A)$ is injective, whence $E_\tau(A) = E(A)$. $\square$

Throughout the rest of this section R will be assumed to be commutative.

Theorem 3.2.2 *Let $p \in \mathrm{Spec}(R)$ be such that $E_\tau(R/p) = E(R/p)$. Then $E_\tau(R/p)$ is minimal τ-injective.*

Proof. Let A be a non-zero submodule of $E_\tau(R/p)$ and let $0 \neq a \in A$. Since $R/p \trianglelefteq E_\tau(R/p)$, there exists $r \in R$ such that $0 \neq b = ra \in Ra \cap R/p$. But $\mathrm{Ann}_R b = p$, hence $Rb \cong R/p$. It follows that $E_\tau(Rb) \cong E_\tau(R/p) = E(R/p)$, which means that $E_\tau(Rb)$ is injective. But $E(R/p)$ is indecomposable injective and $E_\tau(Rb) \trianglelefteq E(R/p)$, hence $E_\tau(Rb) = E(R/p)$. We also have

$$E_\tau(Rb) \trianglelefteq E_\tau(A) \trianglelefteq E_\tau(R/p) = E(R/p).$$

Hence $E_\tau(A) = E(R/p)$. Thus $E(R/p)$ is a minimal τ-injective module. $\square$

We deduce now a number of corollaries of Theorem 3.2.2.

Corollary 3.2.3 *The following statements are equivalent for a domain R:*

(i) R is τ-cocritical.

(ii) τ is proper and $E_\tau(R) = E(R)$.

Proof. $(i) \Longrightarrow (ii)$. It follows by Proposition 2.2.10.

$(ii) \Longrightarrow (i)$. Assume (ii). By Theorem 3.2.2, $E_\tau(R)$ is minimal τ-injective. Since $0 \in \mathrm{Spec}(R)$, R is either τ-torsion or τ-torsionfree by Lemma 1.4.7. If R is τ-torsion, then every module is τ-torsion, a contradiction. Hence R is τ-torsionfree. Therefore $E_\tau(R)$ is τ-cocritical. $\square$

Note that Corollary 3.2.3 does not hold anymore if R is not a domain, as we can see in the following example.

Example 3.2.4 Consider the ring $T = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ and use the same notations as in Example 3.1.11. Then we have $E_{\tau_D}(T) = E(T) = M_2(\mathbb{Q})$. Note that $\mathrm{Soc}(T) = 0$ and T is not τ_D-cocritical.

Let us now recall an auxiliary result.

Lemma 3.2.5 [101, Proposition 2.26, Corollary 1] *If I is a proper left ideal of a domain R, then $\mathrm{Ann}_R E(R/I) = 0$.*

Corollary 3.2.6 *Let R be a domain and let $p \in \mathrm{Spec}(R)$ be τ_n-closed in R. If $E_{\tau_n}(R/p) = E(R/p)$, then $\dim p = n+1$, $p = 0$ and every τ_n-injective module is injective.*

Proof. By Theorem 3.2.2 and Proposition 1.8.1, $E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective. By Proposition 3.1.5, $\dim p = n+1$. Since $E_{\tau_n}(R/p)$ is τ_n-cocritical, it follows by Theorem 1.5.12 that $\mathrm{Ann}_R E_{\tau_D}(R/p) = p$. Since R is a domain, $\mathrm{Ann}_R E(R/p) = 0$ by Lemma 3.2.5. It follows that $p = 0$. Now R is τ_n-cocritical, hence every τ_n-injective module is injective. □

Corollary 3.2.7 *Let $p \in \mathrm{Spec}(R)$. If one of the following conditions holds:*

(i) $\dim p \geq n+2$*;*

(ii) R is a domain with $\dim R \geq n+2$ *and* $\dim p \geq n+1$,

then $E_{\tau_n}(R/p)$ is not injective.

Proof. If $\dim p \geq n+2$, then the result follows by Theorem 3.2.2 and Proposition 3.1.5. If R is a domain with $\dim R \geq n+2$ and $\dim p \geq n+1$, then apply Corollary 3.2.6. □

We need the following preliminary result.

Proposition 3.2.8 [101, Proposition 2.27] *Let I be a two-sided ideal of R and let E be an injective R-module. Then $\mathrm{Ann}_E I$ is injective as an R/I-module. Moreover, if E is the injective hull of an R-module A, then $\mathrm{Ann}_E I$ is an injective hull of $A \cap \mathrm{Ann}_E I$ considered as an R/I-module.*

Now we are able to give the main result of this section, establishing the structure of the τ-injective hull of some R/p, where $p \in \mathrm{Spec}(R)$.

Theorem 3.2.9 *Let $p \in \mathrm{Spec}(R)$ be such that R/p is τ-cocritical. Then:*

(i) $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$.

(ii) There exists an R/p-isomorphism (and hence an R-isomorphism) between $E_\tau(R/p)$ and the field of fractions of R/p.

(iii) If p is not maximal, then $R/p \neq E_\tau(R/p)$.

Proof. (i) Denote $A = E_\tau(R/p)$. Note that A is τ-torsionfree minimal τ-injective. We have seen in the proof of Theorem 3.1.4 that $\mathrm{Ann}_R A = p$. It follows that $A \subseteq \mathrm{Ann}_{E(R/p)}p$. By Lemma 1.5.14, we have $\mathrm{Ann}_R d \subseteq p$ for every $0 \neq d \in E(R/p)$. Let $0 \neq b \in \mathrm{Ann}_{E(R/p)}p$. Then $\mathrm{Ann}_R b = p$ and $Rb \cong R/p$ is τ-cocritical. But since $E(R/p)$ is uniform, A is the maximal τ-cocritical submodule of $E(R/p)$ by Proposition 1.5.4. It follows that $Rb \subseteq A$, hence $\mathrm{Ann}_{E(R/p)}p \subseteq A$. Therefore $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$.

(ii) By (i) and Proposition 3.2.8, it follows that $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$ is an injective hull of R/p considered as an R/p-module. Since R/p is a domain, $E_\tau(R/p)$, considered as an R/p-module, is isomorphic to the field of fractions of R/p.

(iii) It follows by (ii), because R/p is not a field. $\square$

Remarks. (i) Note that the hypothesis needed in the proof is for $E_\tau(R/p)$ to be τ-torsionfree minimal τ-injective.

(ii) Consider the context of Theorem 3.2.9 and suppose that R is a domain and $p \neq 0$. Then $\mathrm{Ann}_R E(R/p) = 0$ by Lemma 3.2.5. On the other hand, by Theorem 1.5.12, $\mathrm{Ann}_R E_\tau(R/p) = p$. Hence the τ-injective hull does not coincide here with the injective hull.

Let us now see an example of determining the τ-injective hull of a τ-cocritical module R/p for some $p \in \mathrm{Spec}(R)$ by using Theorem 3.2.9.

Example 3.2.10 Consider the polynomial ring $R = K[X_1, \ldots, X_m]$ ($m \geq 2$), where K is a field, and let $p = (X_1, \ldots, X_{m-n-1})$, where $n < m-1$. Then $p \in \mathrm{Spec}(R)$ and we have the ring isomorphism

$$K[X_1, \ldots, X_m]/(X_1, \ldots, X_{m-n-1}) \cong K[X_{m-n}, \ldots, X_m]\,.$$

But $K[X_{m-n}, \ldots, X_m]$ has both a structure of R/p-module and R-module by restriction of scalars. Since R is noetherian and $\dim p = n+1$, R/p is a τ_n-cocritical R-module by Corollary 1.8.4. Then $E_{\tau_n}(R/p)$ is τ_n-torsionfree minimal τ_n-injective. Now by Theorem 3.2.9 we have the R-isomorphism

$$E_{\tau_n}(K[X_1, \ldots, X_m]/(X_1, \ldots, X_{m-n-1})) \cong K(X_{m-n}, \ldots, X_m)\,,$$

where $K(X_{m-n}, \ldots, X_m)$ is the field of fractions of $K[X_{m-n}, \ldots, X_m]$.

We continue with two important corollaries of Theorem 3.2.9.

Corollary 3.2.11 *Every τ-torsionfree minimal τ-injective module is isomorphic to the field of fractions of R/p for some $p \in \mathrm{Spec}(R)$.*

Proof. By Theorems 3.1.4 and Theorem 3.2.9. □

Recall that a module A is called *locally noetherian* if every finitely generated submodule of A is noetherian [40, p.10]. Recall also that a module A is said to be $\sum$-$\mathcal{Z}$ if any direct sum of copies of A has the property $\mathcal{Z}$. For instance, a module A is said to be $\sum$-injective if any direct sum of copies of A is injective.

Corollary 3.2.12 *Let p be an N-prime ideal of R such that R/p is τ-cocritical. Then $E_\tau(R/p)$ is locally noetherian and $\sum$-quasi-injective.*

Proof. Let A be a non-zero finitely generated submodule of $E_\tau(R/p)$. Since R/p is τ-cocritical, $E_\tau(R/p)$ is minimal τ-injective. Also $E_\tau(R/p)$ is τ-cocritical, hence A is τ-cocritical. By Theorem 3.2.9, $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$. Then $\mathrm{Ann}_R A = \mathrm{Ann}_R a = p$, for every $0 \neq a \in A$.

If $\{a_1, \ldots, a_n\}$ is a set of generators of A, then $Ra_k \simeq R/p$ are noetherian R-modules, for every $k \in \{1, \ldots, n\}$ and considering the canonical epimorphism

$$\varphi : \bigoplus_{k=1}^{n} Ra_k \to \sum_{k=1}^{n} Ra_k = A\,,$$

it follows that A is noetherian. Therefore $E_\tau(R/p)$ is locally noetherian.

Now let I be a set and denote $A_i = E_\tau(R/p)$ for every $i \in I$. Then we have seen that every A_i is minimal τ-injective and locally noetherian. By Lemma 3.1.2, A_i is A_j-injective for every $i, j \in I$. Hence $\bigoplus_{i \in I} A_i$ is A_j-injective ([40, p.10]). It follows that $\bigoplus_{i \in I} A_i$ is $\bigoplus_{i \in I} A_i$-injective. Therefore $\bigoplus_{i \in I} A_i$ is quasi-injective, i.e. $E_\tau(R/p)$ is $\sum$-quasi-injective. □

Corollary 3.2.13 *Let p be an N-prime ideal of R with* $\dim p = 1$. *Then for every non-zero finitely generated submodule A of $E_{\tau_0}(R/p)$ and for every non-zero proper submodule B of A, A/B has a composition series.*

Proof. Let A be a non-zero finitely generated submodule of $E_{\tau_0}(R/p)$ and let B be a non-zero proper submodule of A. By Corollary 3.2.12, A is noetherian and by Corollary 1.8.4, A is τ_0-cocritical. Hence A/B is noetherian and semi-artinian. Then A/B has a composition series ([82, Chapter II, Proposition 2.1.1]). □

We have seen in Lemma 3.1.2 that every minimal τ-injective module is uniform. Now we can show that the converse does not hold in general.

Example 3.2.14 Consider the polynomial ring $R = K[X_1, \ldots, X_m]$ ($m \geq 2$), where K is a field and the prime ideal $p = (X_1, \ldots, X_{m-1})$ of R. Then $\dim p = 1$ and $E_{\tau_0}(R/p)$ is a τ_0-torsionfree minimal τ_0-injective R-module by Proposition 1.8.1. But $E_{\tau_0}(R/p) = \mathrm{Ann}_{E(R/p)} p$ by Theorem 3.2.9 and R is a domain, so that $\mathrm{Ann}_R E_{\tau_0}(R/p) = p \neq 0 = \mathrm{Ann}_R E(R/p)$. Then $E_{\tau_0}(R/p) \neq E(R/p)$. Therefore $E(R/p)$ is not minimal τ_0-injective, but it is uniform since R is noetherian.

The following theorem is a partial converse of Theorem 3.2.9.

Theorem 3.2.15 *Let* $p \in \mathrm{Spec}(R)$. *If* $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$, *then* $E_\tau(R/p)$ *is minimal* τ*-injective.*

Proof. Suppose that $E_\tau(R/p)$ is not minimal τ-injective. Then there exists a non-zero proper τ-injective submodule A of $E_\tau(R/p)$. Let $0 \neq a \in A$. Then $\mathrm{Ann}_R a = p$ and

$$E_\tau(Ra) \cong E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p\,.$$

Thus $E_\tau(Ra)$ is a proper submodule of $\mathrm{Ann}_{E(R/p)}p$, hence $E_\tau(Ra)$ is a proper R/p-submodule of $\mathrm{Ann}_{E(R/p)}p$. By Proposition 3.2.8, $\mathrm{Ann}_{E(R/p)}p$ is the injective hull of R/p considered as an R/p-module. Moreover, both $\mathrm{Ann}_{E(R/p)}p$ and $E_\tau(Ra)$ are injective indecomposable R/p-modules. Then $E_\tau(Ra)$ is a direct summand of $\mathrm{Ann}_{E(R/p)}p$, a contradiction. $\square$

For $p \in \mathrm{Spec}(R)$ such that R/p is τ-cocritical, we have showed in Theorem 3.2.9 the equality between the τ-injective hull of R/p and the annihilator of p in $E(R/p)$. In what follows let us see what is the relationship between them in some other cases.

Theorem 3.2.16 *If* $p \in \mathrm{Spec}(R)$ *is* τ*-dense in* R*, then* $\mathrm{Ann}_{E(R/p)}p \subseteq E_\tau(R/p)$.

Proof. Let $0 \neq a \in \mathrm{Ann}_{E(R/p)}p$. Then $\mathrm{Ann}_R a = p$, hence $Ra \cong R/p$. Since Ra is τ-torsion, we get $Ra \subseteq E_\tau(R/p)$. Hence $\mathrm{Ann}_{E(R/p)}p \subseteq E_\tau(R/p)$. $\square$

Theorem 3.2.17 *Let* $p \in \mathrm{Spec}(R)$ *be such that* $\dim p = n+1$ *and* R/p *is* τ_n*-cocritical. Then*

$$E_{\tau_0}(R/p) \subset E_{\tau_1}(R/p) \subset \cdots \subset E_{\tau_{n-1}(R/p)} \subset E_{\tau_n}(R/p) = \mathrm{Ann}_{E(R/p)}p\,.$$

Proof. The inclusions are clear and by Theorem 3.2.9 we have $E_{\tau_n}(R/p) = \mathrm{Ann}_{E(R/p)}p$.

Suppose now that $E_{\tau_{n-1}}(R/p) = E_{\tau_n}(R/p)$. By Proposition 1.8.1, $\dim p \geq n+1$ and R/p is τ_{n-1}-torsionfree. By Proposition 1.8.3, $E_{\tau_{n-1}}(R/p)$ is not τ_{n-1}-cocritical, hence it is not minimal τ_{n-1}-injective. Therefore there exists a non-zero proper τ_{n-1}-injective submodule A of $E_{\tau_{n-1}}(R/p)$. Now let $0 \neq a \in A$. Then $Ra \cong R/p$. Hence

$$E_{\tau_{n-1}}(Ra) \cong E_{\tau_{n-1}}(R/p) = E_{\tau_n}(R/p)\,.$$

But $E_{\tau_{n-1}}(Ra) \subset E_{\tau_{n-1}}(R/p)$. This contradicts the fact that $E_{\tau_n}(R/p)$ is minimal τ_n-injective. Now the result follows inductively. $\square$

Remark. In the context of Theorem 3.2.17, we have $E_{\tau_{n-1}}(R/p) \subset E_{\tau_n}(R/p)$. Then for every $k \in \{0, \ldots, n-1\}$, $E_{\tau_k}(R/p)$ is a τ_k-injective module which is not τ_n-injective. This is also an example of a τ_k-injective module which is not injective.

Example 3.2.18 Let $R = K[X_1, \ldots, X_m]$ be the polynomial ring over a field K ($m \geq 2$). Let $0 < n < m-1$ and let $p = (X_1, \ldots, X_{m-n-1})$. Then $p \in \mathrm{Spec}(R)$ and $\dim p = n+1$. Since R is noetherian, R/p is τ_n-cocritical by Corollary 1.8.4 . Then $E_{\tau_{n-1}}(R/p)$ is a τ_{n-1}-injective module which is not τ_n-injective.

Proposition 3.2.19 *Let R be noetherian and let $0 \neq p \in \mathrm{Spec}(R)$ be τ_n-closed in R. Then $\mathrm{Ann}_{E(R/p)}p$ is τ_n-injective.*

Proof. Denote $A = \mathrm{Ann}_{E(R/p)}p$. We will show that $E(R/p)/A$ is τ_n-torsionfree. Suppose the contrary. Then there exists a non-zero submodule B of $E(R/p)/A$ such that $B \in \mathcal{A}_n$. It follows that $B \cong U/V$, where U and V are ideals of R containing an ideal $q \in \mathrm{Spec}(R)$ with $\dim q \leq n$. Then $qU \subseteq V$, hence $qB = 0$. On the other hand, there exists an element $x \in E(R/p) \setminus A$ such that $x + A \in B$. Now let $r \in q \setminus p$ and let $s \in p$ such that $sx \neq 0$. Then $rx \in A$, which implies $srx = 0$. Since multiplication by r

on $E(R/p)$ is an automorphism [118, p.83], we have $rsx \neq 0$, a contradiction. Hence A is τ_n-injective. □

For R noetherian, the following theorem offers a complete picture of the relationship between $E_{\tau_n}(R/p)$ and $\mathrm{Ann}_{E(R/p)}p$, where $0 \neq p \in \mathrm{Spec}(R)$.

Theorem 3.2.20 *Let* $p \in \mathrm{Spec}(R)$.

(i) If $\dim p \leq n$, *then* $\mathrm{Ann}_{E(R/p)}p \subseteq E_{\tau_n}(R/p)$.

(ii) Let R *be* τ_n*-noetherian. If* $\dim p = n+1$, *then* $\mathrm{Ann}_{E(R/p)}p = E_{\tau_n}(R/p)$.

(iii) Let R *be noetherian. If* $\dim p \geq n+2$, *then* $\mathrm{Ann}_{E(R/p)}p \supset E_{\tau_n}(R/p)$.

Proof. (i) By Proposition 1.8.1 and Theorem 3.2.16.

(ii) By Corollary 3.1.6 and Theorem 3.2.9.

(iii) By Proposition 3.2.19, $\mathrm{Ann}_{E(R/p)}p$ is τ_n-injective. Since $R/p \subseteq \mathrm{Ann}_{E(R/p)}p$, we have $E_{\tau_n}(R/p) \trianglelefteq \mathrm{Ann}_{E(R/p)}p$. Since $\dim p \geq n+2$, $E_{\tau_n}(R/p)$ is not minimal τ_n-injective by Proposition 3.1.5. Now by Theorem 3.2.15, it follows that $E_{\tau_n}(R/p) \neq \mathrm{Ann}_{E(R/p)}p$. □

3.3 τ-injective submodules of indecomposable injective modules

We begin our discussion on τ-injective submodules of indecomposable injective modules by considering the minimal ones.

Proposition 3.3.1 *Let* A *be an indecomposable injective module over a not necessarily commutative ring. Then* A *contains at most one minimal* τ*-injective submodule.*

Proof. Suppose that B and C are minimal τ-injective submodules of A. Since B and C are essential submodules of A, $E(B) = E(C) = A$. Since B and

C are τ-injective, it follows that B and C are τ-closed in A. Then $B \cap C$ is τ-closed in A, hence $B \cap C$ is τ-injective. Then $B \cap C = B = C$. $\square$

For the rest of this section, the ring R will be assumed to be commutative.

Let us now recall the following lemma.

Lemma 3.3.2 [101, Lemma 2.31 Corollary] *Let $p, q \in \mathrm{Spec}(R)$ be such that $E(R/p) \cong E(R/q)$. Then $p = q$.*

Proposition 3.3.3 *Let $p \in \mathrm{Spec}(R)$. Then:*

(i) If $\dim p \leq n$, then $E(R/p)$ contains a unique minimal τ_n-injective submodule, namely $E_{\tau_n}(R/p)$.

(ii) If $\dim p \geq n+2$, then $E(R/p)$ does not contain any minimal τ_n-injective module.

Proof. (i) It follows by Propositions 3.3.1 and 3.1.5.

(ii) Suppose that $E(R/p)$ does contain a minimal τ_n-injective module D. By Proposition 1.8.1, R/p is τ_n-torsionfree. It follows that D is τ_n-torsionfree. Then by Theorem 3.1.4, we have $D \cong E_{\tau_n}(R/q)$, where $q \in \mathrm{Spec}(R)$ and by Proposition 3.1.5 we have $\dim q = n+1$. Since $E(R/p)$ is indecomposable, it follows that $E(R/p) = E(D)$. But $E(D) \cong E(E_{\tau_n}(R/q)) = E(R/q)$, hence $E(R/p) \cong E(R/q)$. Then $q = p$ by Lemma 3.3.2. This provides a contradiction. $\square$

Example 3.3.4 Consider the ring $T = \begin{pmatrix} R & 0 \\ Q & Q \end{pmatrix}$, where $R = K[X, Y]$ is the polynomial ring over an algebraically closed field K and Q is the field of fractions of R. Use the notations from Example 2.4.11.

Since $\dim R = 2$ and $0 \in \mathrm{Spec}(R)$, R does not contain minimal τ_D-injective R-submodules by Proposition 3.3.3, so that A has the same property. Hence A is not minimal τ_D-injective.

Clearly, B is a simple T-module and $B \trianglelefteq D$. Then $E(B) = D$. Since T is left noetherian, $E_{\tau_D}(B) = E(B) = D$. Moreover, D is minimal τ_D-injective. Also, $C = E(A)$ is indecomposable injective and not minimal τ_D-injective.

By Proposition 3.3.3, we obtain immediately the following corollary connecting the situation when every τ_n-injective module is injective with the dimension of R, result that generalizes an implication from Proposition 2.4.4.

Corollary 3.3.5 *If every τ_n-injective module is injective, then* $\dim R \leq n+1$.

We have analyzed in Proposition 3.3.3 whether $E(R/p)$ contains or not a minimal τ_n-injective submodule depending on the dimension of $p \in \mathrm{Spec}(R)$. For an N-prime ideal p of R, and thus for a noetherian ring R, we are able to clarify the case $\dim p = n+1$ as well.

Corollary 3.3.6 *Let p be an N-prime ideal of R with* $\dim p = n+1$. *Then $E(R/p)$ contains a unique minimal τ_n-injective submodule, namely $E_{\tau_n}(R/p)$.*

Proof. By Corollary 1.8.4, R/p is τ_n-cocritical, hence $E_{\tau_n}(R/p)$ is minimal τ_n-injective. Now by Proposition 3.3.1, $E_{\tau_n}(R/p)$ is the unique minimal τ_n-injective submodule of $E(R/p)$. □

Theorem 3.3.7 *Let $p \in$* $\mathrm{Spec}(R)$ *be such that* $\dim p \geq n+2$. *Then there exist τ_n-injective modules D_k ($k \in \mathbb{N}^*$) such that*

$$\cdots \subset D_k \subset \cdots \subset D_1 \subset E_{\tau_n}(R/p)$$

and $D_k \cong E_{\tau_n}(R/p)$ for every $k \geq 1$.

Proof. By Corollary 3.2.7, $E_{\tau_n}(R/p)$ is not injective. By Proposition 3.3.3, the indecomposable injective module $E(R/p)$ does not contain any minimal τ_n-injective submodule. Now let D be a non-zero proper τ_n-injective submodule of $E_{\tau_n}(R/p)$ and let $0 \neq a \in D$. Then there exists $r \in R$ such that $0 \neq ra \in R/p$. It is known that the collection of all annihilators of non-zero elements of $E(R/p)$ has a unique maximal member, namely the ideal p. Then $\mathrm{Ann}_R(ra) = p$. But $Rra \subseteq D$, hence $\mathrm{Ann}_R(Rra) = p$. It follows that

$Rra \cong R/\mathrm{Ann}_R(ra) = R/p$. Put $D_1 = E_{\tau_n}(Rra)$. Then $D_1 \subseteq D \subset E_{\tau_n}(R/p)$ and $D_1 \cong E_{\tau_n}(R/p)$. Now repeat the argument for D_1 instead of $E_{\tau_n}(R/p)$. The requested modules D_k are obtained inductively. $\square$

Every injective submodule of an injective module is a direct summand. We can show now that there exist τ_n-injective modules with τ_n-injective submodules which are not direct summands.

Example 3.3.8 Consider $E_{\tau_n}(R/p)$ and the modules D_k from Theorem 3.3.7. Then every non-zero proper τ_n-injective submodule of $E_{\tau_n}(R/p)$ is not a direct summand, because $E_{\tau_n}(R/p)$ is uniform as a submodule of $E(R/p)$.

We continue with some results on the existence of τ-injective (but not necessarily minimal τ-injective) submodules of injective hulls of modules.

Theorem 3.3.9 *Let R be a domain and let A be a non-zero proper τ-injective submodule of $E(R)$. Then there exists a proper τ-injective submodule of $E(R)$ which strictly contains A.*

Proof. Since $0 \in \mathrm{Spec}(R)$, by Lemma 1.4.7, $E_\tau(R)$ is either τ-torsion or τ-torsionfree. First we will show that $E_\tau(R)$ is τ-torsionfree, but not τ-cocritical. Suppose that $E_\tau(R)$ is either τ-torsion or τ-cocritical. Then R is either τ-torsion or τ-cocritical. Hence every non-zero ideal of R is τ-dense. Then every τ-injective module is injective. Since $E(R)$ is indecomposable, it follows that $A = E(R)$, a contradiction. Therefore $E_\tau(R)$ is τ-torsionfree, but not τ-cocritical.

Suppose now that it does not exist any proper τ-injective submodule of $E(R)$ which strictly contains A. Then every non-zero proper submodule D of $E(R)$ strictly containing A is not τ-closed in $E(R)$, i.e. $E(R)/D$ is not τ-torsionfree. Since A is τ-injective, $E(R)/A$ is τ-torsionfree. Then by Proposition 1.5.2, $E(R)/A$ is τ-cocritical. By Theorem 1.5.12, $p = \mathrm{Ann}_R(E(R)/A) \in \mathrm{Spec}(R)$. Assume $p = 0$. Then $E(R)/A$ is faithful τ-cocritical. By Proposition 2.2.10, $E_\tau(E(R)/A) \cong E(R)$, hence $E(R)$ is minimal τ-injective, a contradiction. Therefore $p \neq 0$. Now let d be a non-zero element of p. We may assume that $E(R)$ is the field of fractions of R. Let $\frac{a}{b} \in E(R) \setminus A$. On the other hand $\frac{a}{b} = d \cdot \frac{a}{bd} \in A$, a contradiction. Therefore there exists a proper τ-injective submodule of $E(R)$ which strictly contains A. $\square$

Corollary 3.3.10 *Let R be a domain such that $E_\tau(R) \neq E(R)$. Then there exists τ-injective modules B_k ($k \in \mathbb{N}^*$) such that*

$$E_\tau(R) \subset B_1 \subset \cdots \subset B_k \subset \cdots \subset E(R)\,.$$

Recall that we have denoted by $\mathcal{A}_n$ the class of modules that generates the torsion theory τ_n.

Corollary 3.3.11 *Let R be a domain with $\dim R \geq n+2$ such that every ideal in $\mathcal{A}_n$ is finitely generated. Then $E(R) = \bigcup_{i\in I} B_i$, where $(B_i)_{i\in I}$ is a totally ordered family of τ_n-injective modules such that $E_{\tau_n}(R) \subseteq B_i \subset E(R)$ for every $i \in I$.*

Proof. By Corollary 3.2.7, $E_{\tau_n}(R) \subset E(R)$. Let $\mathcal{F}$ be the family of all τ_n-injective modules A such that $E_{\tau_n}(R) \subseteq A \subset E(R)$. Clearly, $\mathcal{F} \neq \emptyset$.

Suppose now that $E(R)$ is not a union of a totally ordered subset of $\mathcal{F}$. Let $(D_j)_{j\in J}$ be a totally ordered subset of $\mathcal{F}$ and denote $D = \bigcup_{j\in J} D_j$. Let $I \in \mathcal{A}_n$ and let $f : I \to D$ be a homomorphism. Since I is finitely generated, $f(I) \subseteq D_k$ for some $k \in J$. But D_k is τ_n-injective and I is a τ_n-dense ideal of R, hence there exists a homomorphism $g : R \to D_k$ that extends f. Thus D is τ_n-injective. We also have $D \neq E(R)$. Hence $D \in \mathcal{F}$ and D is an upper bound of $(D_j)_{j\in J}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element. On the other hand, by Theorem 3.3.9 $\mathcal{F}$ does not have a maximal element, a contradiction. Now the result follows. $\square$

Theorem 3.3.12 *Let R be a domain, let A be a non-zero τ-injective module with $\mathrm{Ann}_R A = 0$ and let B be a proper essential τ-injective submodule of A with $\mathrm{Ann}_R B = s \neq 0$. Then there exist infinitely many τ-injective submodules of A which strictly contain B.*

Proof. Suppose that it does not exist any τ-injective module C such that $B \subset C \subset A$. Then by Proposition 2.1.11, for every module D such that $B \subset D \subset A$, A/D is not τ-torsionfree. Since B is a proper essential τ-injective submodule of A, it follows that A/B is τ-torsionfree. Hence A/B is τ-cocritical. Then by Theorem 1.5.12, $p = \mathrm{Ann}_R(A/B) \in \mathrm{Spec}(R)$. Assume $p = 0$. Then A/B is faithful τ-cocritical. By Proposition 2.2.10, $E_\tau(A/B) \cong E(R)$, hence $E(R)$ is τ-torsionfree minimal τ-injective. Then R is τ-cocritical, hence every τ-injective module is injective. Then B is a non-zero proper injective submodule of A, i.e. B is a direct summand of A, a contradiction. Therefore $p \neq 0$. Now let d be a non-zero element of p and let r be a non-zero element of s. Then $dra = 0$ for every $a \in A$, a contradiction.

Therefore there exists a τ-injective module D_1 such that $B \subset D_1 \subset A$. If $\mathrm{Ann}_R D_1 = 0$, there exists a τ-injective module D_2 such that $B \subset D_2 \subset D_1$. If $\mathrm{Ann}_R D_1 \neq 0$, there exists a τ-injective module D_2 such that $D_1 \subset D_2 \subset A$. Now the result follows by complete induction. $\square$

Corollary 3.3.13 *Let R be a domain and let $0 \neq p \in \mathrm{Spec}(R)$ be such that R/p is τ-cocritical. Then there exist infinitely many τ-injective submodules of $E(R/p)$ which strictly contain $E_\tau(R/p)$.*

Proof. Since R/p is τ-cocritical, $E_\tau(R/p)$ is τ-torsionfree minimal τ-injective. Then by Theorem 3.2.9, $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)} p$. We also have $\mathrm{Ann}_R(E_\tau(R/p)) = p$ and $\mathrm{Ann}_R(E(R/p)) = 0$, hence $E_\tau(R/p) \subset E(R/p)$. The result follows now by Theorem 3.3.12. $\square$

In the sequel we are interested in studying certain particular submodules of $E(R/p)$, where R is noetherian and $0 \neq p \in \mathrm{Spec}(R)$.

Following [118, p.83], for each positive integer $m \geq 1$ denote

$$A_m = \{x \in E(R/p) \mid p^m x = 0\}.$$

Note that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq A_{m+1} \subseteq \ldots$.

If R is noetherian, then we have

$$E(R/p) = \bigcup_{m=1}^{\infty} A_m$$

[118, p.83]. In this case of a noetherian ring, we will establish several properties of the submodules A_m.

Proposition 3.3.14 *Let R be a noetherian domain and let $0 \neq p \in \mathrm{Spec}(R)$ be τ-dense in R. Then for every $m \geq 1$, A_m is not τ-injective.*

Proof. Since R is noetherian and R/p is τ-torsion, we have $E_\tau(R/p) = E(R/p)$. Then by Theorem 3.2.2, $E(R/p)$ is minimal τ-injective. On the other hand, since R is a domain, each A_m is a proper submodule of $E(R/p)$. Therefore for every $m \geq 1$, A_m is not τ-injective. $\square$

We have seen in Proposition 3.2.19 that $A_1 = \mathrm{Ann}_{E(R/p)}p$ is τ_n-injective provided R is noetherian and $0 \neq p \in \mathrm{Spec}(R)$ is τ_n-closed in R. More generally, we have the following result, whose proof is similar to that for A_1.

Theorem 3.3.15 *Let R be noetherian and let $0 \neq p \in \mathrm{Spec}(R)$ be τ_n-closed in R. Then A_m is τ_n-injective for every $m \geq 1$.*

Proof. Let $m \geq 1$. We will show that $E(R/p)/A_m$ is τ_n-torsionfree. Suppose the contrary. Then there exists a non-zero submodule B of $E(R/p)/A_m$ such that $B \in \mathcal{A}_n$. It follows that $B \cong U/V$, where U and V are ideals of R containing an ideal $q \in \mathrm{Spec}(R)$ with $\dim q \leq n$. Then $qU \subseteq V$, hence $qB = 0$. On the other hand, there exists an element $x \in E(R/p) \setminus A_m$ such that $x + A_m \in B$. Now let $r \in q \setminus p$ and let $s \in p^m$ such that $sx \neq 0$. Then $rx \in A_m$, which implies $srx = 0$. Since multiplication by r on $E(R/p)$ is an automorphism [118, p.83], we have $rsx \neq 0$, a contradiction. Hence A_m is τ_n-injective. $\square$

Corollary 3.3.16 *Let R be a noetherian domain and let $0 \neq p \in \mathrm{Spec}(R)$. Then each A_m is τ_n-injective if and only if p is τ_n-closed in R.*

Proof. By Proposition 1.8.1, Theorem 3.3.15 and Proposition 3.3.14. $\square$

Corollary 3.3.17 *Let R be a noetherian domain, let $0 \neq p \in \operatorname{Spec}(R)$ be such that $\dim p \geq n+2$ and let $m \geq 1$ be a positive integer. Then there exist τ_n-injective modules B_k ($k \in \mathbb{N}^*$) such that*

$$A_m \subset \cdots \subset B_k \subset \cdots \subset B_1 \subset A_{m+1}\,.$$

Proof. Let us show first that $A_m \neq A_{m+1}$ for each m. Suppose that there exists m such that $A_m = A_{m+1}$. Then it follows easily that $A_{m+i} = A_m$ for every $i \in \mathbb{N}$, hence $E(R/p) = A_m$. Since R is a domain, we have

$$p^m \subseteq \operatorname{Ann}_R A_m = \operatorname{Ann}_R E(R/p) = 0\,,$$

a contradiction.

Now let B be a τ_n-injective module such that $A_m \subset B \subseteq A_{m+1}$. Suppose that there does not exists a τ_n-injective module D such that $A_m \subset D \subset B$. Then B/A_m is τ_n-cocritical. By Proposition 1.8.3, $q = \operatorname{Ann}_R(B/A_m) \in \operatorname{Spec}(R)$ and $\dim q = n+1$. Now let $b \in B \setminus A_m$. Then $p^m b \neq 0$. But $qb \in A_m$, hence $p^m qb = 0$, i.e. $q \in \operatorname{Ann}_R(p^m b)$. Since $p^m b \in E(R/p)$, by Lemma 1.5.14 we have $q \subseteq p$, a contradiction. Now the result follows. $\square$

Remarks. (i) The hypothesis on R to be a domain in Corollary 3.3.17 is needed for ensuring that $A_m \subset A_{m+1}$ for each τ_D.

(ii) In the context of Corollary 3.3.17, each A_m is a proper submodule of $E(R/p)$. Since $E(R/p)$ is an indecomposable injective module, for every $m \geq 1$, A_m is not injective. Thus $E(R/p)$ is a union of τ_n-injective modules which are not injective.

3.4 Change of ring and τ-injective modules

We have seen in Proposition 3.2.8 that if I is a two-sided ideal of R and E is an injective R-module, then $\operatorname{Ann}_E I$ is an injective R/I-module. A similar result holds for τ-injective modules as well.

Theorem 3.4.1 *Let I be a non-zero proper two-sided ideal of R and let A be a τ-injective R-module. Then:*

(i) $\mathrm{Ann}_A I$ *is τ-injective as an R/I-module.*

(ii) If $I \subseteq \mathrm{Ann}_R A$, then A is τ-injective as an R/I-module.

Proof. *i*) Let J/I be a τ-dense left ideal of R/I. Then $(R/I)/(J/I) \cong R/J$ is τ-torsion, hence J is a τ-dense left ideal of R. Both J/I and R/I may be seen as R/I-modules and R-modules as well. Consider the following diagram with exact rows and commutative square:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "u"'] & R \arrow[d, "v"] \arrow[ddr, dashed, "g"] & \\
0 \arrow[r] & J/I \arrow[r, "j"] \arrow[d, "f"'] & R/I \arrow[dl, dashed, "h"] & \\
 & \mathrm{Ann}_A I \arrow[rr, "k"'] & & A
\end{tikzcd}$$

where i, j, k are inclusion homomorphisms, u, v are natural R-epimorphisms and $f : J/I \to \mathrm{Ann}_A I$ is an R/I-homomorphism. Seeing J/I and $\mathrm{Ann}_A I$ as R-modules, f is also an R-homomorphism. Since A is a τ-injective R-module and J is a τ-dense left ideal of R, there exists an R-homomorphism $g : R \to A$ such that $gi = kfu$. Note that $g(s) = kfu(s) = 0$ for every $s \in I \subseteq J$.

Now define the R/I-homomorphism

$$h : R/I \to \mathrm{Ann}_A I\,, \quad h(r + I) = g(r)$$

for every $r \in R$. If $r, s \in R$ and $r + I = s + I$, then $r - s \in I$, hence $g(r - s) = 0$, i.e. $g(r) = g(s)$. If $s \in I$ and $r \in R$, we have $sg(r) = g(sr) = 0$, which means that $g(r) \in \mathrm{Ann}_A I$. Therefore h is well-defined.

For every $r \in J$ we have

$$hj(r + I) = h(r + I) = g(r) = gi(r) = kfu(r) = kf(r + I) = f(r + I)\,.$$

Hence $hj = f$, i.e. $\mathrm{Ann}_A I$ is τ-injective as an R/I-module.

(ii) If $I \subseteq \mathrm{Ann}_R A$, then $\mathrm{Ann}_A I = A$ and the result follows by (i). □

For the rest of this section, the ring R will be assumed to be commutative.

Corollary 3.4.2 *Let R be noetherian and let $0 \neq p \in \mathrm{Spec}(R)$ be such that $\dim p \geq n+2$. Then $E_{\tau_n}(R/p)$ is τ_n-injective as an R/p-module.*

Proof. By Theorem 3.2.20, $E_{\tau_n}(R/p) \subset \mathrm{Ann}_{E(R/p)} p$ and by Lemma 1.5.14, $\mathrm{Ann}_R E_{\tau_n}(R/p) = p$. Now by Theorem 3.4.1, $E_{\tau_n}(R/p)$ is τ_n-injective as an R/p-module. □

Quite interestingly, a converse of Theorem 3.4.1 holds in the case of τ_0-injectivity provided I is an s-pure ideal of R.

A submodule B of a module A is called *s-pure* in A if $1_S \otimes i : S \otimes B \to S \otimes A$ is a monomorphism for every simple right module S, where $i : B \to A$ is the inclusion homomorphism [20].

Theorem 3.4.3 *Let I be a non-zero proper s-pure ideal of R and A an R-module such that $I \subseteq \mathrm{Ann}_R A$. Then A is τ_0-injective as an R-module if and only if A is τ_0-injective as an R/I-module.*

Proof. Suppose that A is τ_0-injective as an R-module. Since $\mathrm{Ann}_A I = A$, by Theorem 3.4.1, A is τ_0-injective as an R/I-module.

Now suppose that A is τ_0-injective as an R/I-module. It is known that I is an s-pure submodule of R if and only if $IM = I \cap M$ for every maximal ideal M of R [106, p.170].

Let M be a maximal ideal of R. Let $i : M \to R$ be the inclusion homomorphism and $f : M \to A$ an R-homomorphism.

If $r \in IM$, we have $r = \sum_{j=1}^{n} s_j t_j$, where $s_j \in I$ and $t_j \in M$ for every $j \in \{1, \ldots, n\}$. Then

$$f(r) = f(\sum_{j=1}^{n} s_j t_j) = \sum_{j=1}^{n} s_j f(t_j) = 0\,.$$

We distinguish two cases: $I \subseteq M$ and $I \not\subseteq M$.

Assume first that $I \subseteq M$. Then $IM = I$. Consider the following diagram with exact row and commutative square:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "u"] \arrow[ddl, "f"'] & R \arrow[d, "v"] \\
 & M/I \arrow[r, "j"] \arrow[dl, dashed, "g"] & R/I \arrow[dll, dashed, "h"] \\
A & &
\end{tikzcd}$$

where j is the inclusion homomorphism and u, v natural R-epimorphisms.

If $r \in I = IM$, it follows that $f(r) = 0$. Now f induces an R-homomorphism $g : M/I \to A$, defined by $g(r + I) = f(r)$ for every $r \in M$, such that $gu = f$. But g is an R/I-homomorphism as well, because $\text{Ann}_R(M/I) = \text{Ann}_R A = I$. Since A is τ_D-injective as an R/I-module, there exists an R/I-homomorphism $h : R/I \to A$ such that $hj = g$. We have $hvi = hju = gu = f$.

Now assume that $I \not\subseteq M$. Then $M + I = R$. If $r \in R$, there exists $m \in M$ such that $r + I = m + I$. Then we can define an R-homomorphism $q : R \to A$ by $q(r) = f(m)$. If $r \in R$, $r + I = m + I$ and $r + I = m' + I$, where $m, m' \in M$, then $m - m' \in I \cap M = IM$. It follows that $f(m) = f(m')$. Hence q is well-defined. We have $qi(s) = f(s)$ for every $s \in M$, i.e. $qi = f$.

Therefore A is τ_0-injective as an R-module. □

Remarks. (i) The theorem holds for every pure ideal, since every pure ideal is s-pure.

(ii) Note that the theorem does not hold if we simply replace τ_0-injective modules by injective modules.

Corollary 3.4.4 *Let R be a (von Neumann) regular ring. An R-module A is τ_0-injective if and only if A is τ_0-injective as an $R/\text{Ann}_R A$-module.*

Proof. Note that every ideal of R is pure, hence s-pure, and apply Theorem 3.4.3. □

Corollary 3.4.5 *Let I be a non-zero proper idempotent ideal of R and let A be an R-module such that $I \subseteq \mathrm{Ann}_R A$. Then A is τ_0-injective as an R-module if and only if A is τ_0-injective as an R/I-module.*

Proof. Let M be a maximal ideal of R.

If $I \subseteq M$, we have $I = I^2 \subseteq IM \subseteq I \cap M \subseteq I$. Then $IM = I \cap M$.

If $I \not\subseteq M$, then $I + M = R$, hence I and M are comaximal ideals. Then $I \cap M = IM$.

Therefore $I \cap M = IM$ for every maximal ideal M, which means that I is an s-pure ideal of R. Now the result follows by Theorem 3.4.3. □

Corollary 3.4.6 *Let $0 \neq p \in \mathrm{Spec}(R)$ be s-pure with $\dim p \geq 2$. Then:*

(i) The submodules of $A_1 = \mathrm{Ann}_{E(R/p)} p$ are τ_0-injective as R-modules if and only if they are τ_0-injective as R/p-modules.

(ii) There exist τ_0-injective R-modules B_k ($k \in \mathbb{N}^*$) such that

$$E_{\tau_0}(R/p) \subset B_1 \subset \cdots \subset B_k \subset \cdots \subset A_1 \,.$$

Proof. (i) By Theorem 3.4.3, noting that every non-zero submodule of A_1 has annihilator p.

(ii) Note that R/p is a domain with $\dim p \geq 2$, hence by Corollary 3.2.7 it follows that $E_{\tau_0}(R/p)$ is not injective. By Theorems 3.2.9 and 3.4.3, A_1 is the injective hull of R/p as an R/p-module. Now apply Corollary 3.3.10. □

References: S. Crivei [21], [24], [25], [26], [28], [31], [36], T. Izawa [61], K. Masaike, T. Horigome [74], B. Stenström [107], H. Tachikawa [108], J. Xu [118].

Notes on Chapter 3

The terminology of minimal τ-injective module was used by the author, this notion appearing in the literature either without a special name (in most of the cases) or under the name of τ-uniform τ-injective module, like in the work of J.L. Bueso, P. Jara and B. Torrecillas (1985). K. Masaike and T. Horigome (1980) showed that the endomorphism ring of a minimal τ-injective module is local. An equivalent condition for the τ-injective hull of a τ-torsionfree module to coincide with its injective hull is based on the work of H. Tachikawa (1971) and T. Izawa (1981). The chapter is completed with the author's results.

Chapter 4

τ-completely decomposable modules

This chapter develops a study of τ-completely decomposable modules, that is, direct sums of minimal τ-injective submodules. We give some classical direct sum decomposition properties and we study when τ-injective modules are τ-completely decomposable. Furthermore, we see the class of τ-completely decomposable modules as a subclass of the class of τ-complemented modules. This allows us to establish some further important properties of τ-completely decomposable modules, especially concerning their direct summands (on a generalized problem of Matlis) or their extensions.

4.1 Some τ-complete decompositions

Let us begin with the definition of the key notion of this chapter.

Definition 4.1.1 A module is called *(finitely) τ-completely decomposable* if it is a (finite) direct sum of minimal τ-injective submodules.

Example 4.1.2 (1) Taking the improper torsion theory χ on R-mod where R is left noetherian, the minimal χ-injective modules are exactly the indecomposable injective modules, therefore every injective module is χ-completely decomposable.

(2) $M_2(\mathbb{Q})$ is τ_D-completely decomposable (see Example 3.1.11).

(3) Consider the polynomial ring $R = K[X, Y]$, where K is an algebraically closed field and let Q be the field of fractions of R. Then $M_2(Q)$ is not τ_D-completely decomposable (see Examples 2.4.11 and 3.3.4).

We immediately have the following property.

Lemma 4.1.3 *The class of τ-completely decomposable modules is closed under direct sums.*

Proposition 4.1.4 *Let $A = \bigoplus_{i\in I} A_i$ be a τ-completely decomposable module and let $B = \bigoplus_{j\in J} B_j$ be a τ-completely decomposable submodule of A, where each A_i and each B_j is minimal τ-injective. If J is finite or infinite countable, then there exists a subset $K \subseteq I$ such that $B \cong \bigoplus_{k\in K} A_k$.*

Proof. For every $j \in J$, denote

$$J(j) = \{s \in J \mid B_s \cong B_j\}$$

$$I(j) = \{t \in I \mid A_t \cong A_j\}.$$

Let us show that for every $j \in J$, we have $|J(j)| \leq |I(j)|$. Let $j \in J$, let $s_1, s_2, \ldots, s_n \in J(j)$ be distinct and, for every $u \in \{1, \ldots, n\}$, let x_u be a non-zero element of B_{s_u}. We have

$$\bigoplus_{u=1}^{n} B_{s_u} = \bigoplus_{u=1}^{n} E_\tau(Rx_u) = E_\tau(\bigoplus_{u=1}^{n} Rx_u).$$

Then there exists a monomorphism $f : \bigoplus_{u=1}^{n} B_{s_u} \to \bigoplus_{v=1}^{m} A_{i_v}$ for some m such that $\bigoplus_{u=1}^{n} B_{s_u}$ is not contained in any direct sum of a proper subset of $A_{i_1}, \ldots, A_{i_m}$. For every $v \in \{1, \ldots, m\}$, denote by $p_{i_v} : A \to A_{i_v}$ the canonical projection. For every $v \in \{1, \ldots, m\}$ choose $y_v \in \bigoplus_{u=1}^{n} B_{s_u}$ such that $p_{i_v} f(y_v) \neq 0$ and $p_{i_t} f(y_v) = 0$ for every $t \neq v$. We have

$$f(Ry_1 \oplus \cdots \oplus Ry_m) = f(Ry_1) \oplus \cdots \oplus f(Ry_m) \subseteq A_{i_1} \oplus \cdots \oplus A_{i_m},$$

hence $E_\tau(\bigoplus_{v=1}^{m} Ry_v) = A_{i_1} \oplus \cdots \oplus A_{i_m}$. It follows that $\bigoplus_{u=1}^{n} B_{s_u} \cong \bigoplus_{v=1}^{m} A_{i_v}$. By Lemma 3.1.2, the endomorphism ring of a minimal τ-injective module is local, hence by Krull-Remak-Schmidt-Azumaya Theorem, it follows that $n = m$ and each A_{i_v} is isomorphic to some B_{s_u}. Therefore if $J(j)$ is finite, then $|J(j)| \leq |I(j)|$. Also, if $J(j)$ is infinite, it follows that $|I(j)|$ is infinite and the conclusion follows from the fact that $|J|$ is infinite countable. $\square$

In what follows we will study when every (τ-torsion, τ-torsionfree) τ-injective module is τ-completely decomposable.

Theorem 4.1.5 *The following statements are equivalent:*

(i) R has ACC on τ-dense left ideals.

(ii) Every τ-torsion τ-injective module is τ-completely decomposable.

Proof. $(i) \Longrightarrow (ii)$ Assume (i). Let A be a τ-torsion τ-injective module. Also, let $0 \neq x \in A$. Then $Rx \cong R/\mathrm{Ann}(x)$ is clearly τ-torsion and noetherian by hypothesis. Hence Rx contains a uniform submodule B. It follows that A has a minimal τ-injective submodule, namely $E_\tau(B)$.

Now let $(A_j)_{j\in J}$ be a maximal independent family of minimal τ-injective submodules of A. Let us show that $\bigoplus_{j\in J} A_j \trianglelefteq A$. If $0 \neq y \in A$, then $E_\tau(Ry)$ has a minimal τ-injective submodule, hence $(\bigoplus_{j\in J} A_j) \cap E_\tau(Ry) \neq 0$. But then $(\bigoplus_{j\in J} A_j) \cap Ry \neq 0$ and consequently $\bigoplus_{j\in J} A_j \trianglelefteq A$. Since $\bigoplus_{j\in J} A_j$ is τ-injective by Theorem 2.3.6, it is a direct summand of A, hence $\bigoplus_{j\in J} A_j = A$.

$(ii) \Longrightarrow (i)$ Assume (ii). Consider a complete family $(B_j)_{j\in J}$ of representatives of isomorphism classes of τ-torsion τ-injective uniform modules. Then clearly each B_j is minimal τ-injective. Let $A = E_\tau(\bigoplus_{j\in J} B_j^{(\mathbb{Z})})$. By hypothesis, we can write $A = \bigoplus_{k\in K} A_k$ as a direct sum of minimal τ-injective submodules. By Proposition 4.1.4, $C = \bigoplus_{j \in J} B_j^{(\mathbb{Z})}$ is isomorphic to a direct summand of A, hence it is τ-injective.

Now suppose that there exists a strictly increasing chain $I_1 \subset I_2 \subset \dots$ of τ-dense left ideals of R. Denote $I = \bigcup_{k \in \mathbb{N}^*} I_k$. Then for every k, $E_\tau(R/I_k)$ is a direct sum of τ-torsion minimal τ-injective modules. It follows that every I_k is an annihilator of a subset of $\bigoplus_{j \in J} B_j$. Now choose some elements $x_1, x_2, \dots$ of $\bigoplus_{j \in J} B_j$ such that for every k we have $I_k x_k = 0$ and $I_{k+1} x_{k+1} \neq 0$. Define the homomorphism $f : I \to C$ by $f(r) = (r x_k)_{k \geq 1}$. Since I is τ-dense in R and C is τ-injective, f extends to a homomorphism $g : R \to C$. But this contradicts the fact that for every $n \in \mathbb{N}^*$, there exists an element r such that $r x_n \neq 0$. Thus R has ACC on τ-dense left ideals. $\square$

We have seen in Theorem 1.4.14 that R is τ-noetherian if and only if every τ-torsionfree injective module is a direct sum of indecomposable injective modules. We can give now, for a stable torsion theory τ, a sufficient condition for τ-torsion injective (or equivalently τ-torsion τ-injective by Proposition 2.1.9) modules to be a direct sum of indecomposable injective modules.

Theorem 4.1.6 *Suppose that τ is stable and R has ACC on τ-dense left ideals. Then every τ-torsion injective module is a direct sum of indecomposable injective modules.*

Proof. Let A be a τ-torsion injective module. By Zorn's Lemma, A has a family $(A_i)_{i \in I}$ of indecomposable injective submodules that is maximal with respect to the property that their sum is direct. Denote $B = \bigoplus_{i \in I} A_i$. Then B is injective by Theorem 2.3.6 and Proposition 2.1.9, hence $A = B \oplus C$ for some $C \leq A$. Suppose that $C \neq 0$. If $c \in C$, then Rc has ACC on τ-dense submodules as a homomorphic image of R. Since Rc is τ-torsion, every submodule of Rc is τ-dense, hence Rc is noetherian. Then it has a submodule D such that $E(D)$ is indecomposable. Now the sum $B + E(D)$ is direct, because $E(D) \subseteq C$. This contradicts the maximality of B. Thus

$C = 0$ and consequently $A = \bigoplus_{i \in I} A_i$ is a direct sum of indecomposable injective modules. $\square$

Now let us see when every τ-torsionfree τ-injective module is τ-completely decomposable. We need the following lemma.

Lemma 4.1.7 *The following statements are equivalent:*

(i) R is τ-semisimple.

(ii) Every τ-torsionfree τ-injective module is τ-semisimple.

Proof. $(i) \Longrightarrow (ii)$ By Theorem 1.6.9.

$(ii) \Longrightarrow (i)$ Let A be a module. By Lemma 1.6.7 (ii) and (iii), we may assume without loss of generality that A is τ-torsionfree. Let B be a τ-closed submodule of A. Then $E_\tau(B)$ is τ-closed in the τ-torsionfree τ-injective module $E_\tau(A)$. By Proposition 1.6.8, there exists a τ-simple submodule C of $E_\tau(A)$ such that $E_\tau(A) \cap C = t(C)$. Then $C' = B \cap C$ is a τ-simple submodule of B and $B \cap C' = t(C')$. Now by Proposition 1.6.8, A is τ-semisimple. Finally, by Theorem 1.6.9, R is τ-semisimple. $\square$

Theorem 4.1.8 *The following statements are equivalent:*

(i) R is τ-noetherian and τ-semisimple.

(ii) R is τ-noetherian and every τ-torsionfree τ-injective module is injective.

(iii) Every τ-torsionfree τ-injective module is τ-completely decomposable.

Proof. $(i) \Longrightarrow (ii)$ By Lemma 4.1.7 and Propositions 1.6.8 and 2.2.9.

$(ii) \Longrightarrow (iii)$ Let A be a τ-torsionfree τ-injective module. Then A is injective by hypothesis. Since R is τ-noetherian, it follows by Theorem 1.4.14 that A is a direct sum of indecomposable injective modules, say $\bigoplus_{i \in I} A_i$. Since R is τ-semisimple, each A_i is τ-semisimple by Theorem 1.6.9, hence each A_i contains a τ-cocritical submodule C_i. But A_i is uniform, hence $C_i \trianglelefteq A_i$ and since A is τ-torsionfree, C_i is τ-dense in A_i by Proposition 1.6.8.

Now by Proposition 1.5.3 A_i is τ-cocritical. It follows that A is τ-completely decomposable.

$(iii) \Longrightarrow (i)$ By hypothesis, every τ-torsionfree τ-injective module is τ-semisimple. Then by Lemma 4.1.7, R is τ-semisimple. Again by Lemma 4.1.7 and by Proposition 1.6.8, it follows that the lattice of τ-closed submodules of every τ-torsionfree module is complemented. Now by Proposition 1.6.8, every τ-torsionfree τ-injective module is injective. Finally, by Theorem 1.4.14, R is τ-noetherian. $\square$

We will see in a forthcoming result (see Theorem 4.5.4) when every τ-injective module is τ-completely decomposable.

We continue with an equivalent condition for the τ-injective hull of a finitely generated module to be a direct sum of uniform modules.

Theorem 4.1.9 *Let A be a finitely generated module. The following statements are equivalent:*

(i) $E_\tau(A) = \bigoplus_{i=1}^n A_i$ for some uniform submodules $A_1, \dots, A_n$ of A.

(ii) There exist submodules $B_1, \dots, B_k$ of A such that $\bigcap_{j=1}^k B_j = 0$ and for each j, $C_j = \bigcap_{l=1, l\neq j}^k B_l \nsubseteq B_j$, A/B_j is uniform and $B_j + C_j$ is τ-dense in A.

Proof. $(i) \Longrightarrow (ii)$ For every $i \in \{1, \dots, n\}$, denote $C_j = A \cap A_i$ and $B_j = A \cap \sum_{j=1, j\neq i}^n A_j$. Then $B_j \neq 0$ and $\bigcap_{j=1}^k B_j = 0$. Also $C_j \neq 0$ and $C_j = \bigcap_{l=1, l\neq j}^k B_l \nsubseteq B_j$. Since A/B_j is isomorphic to a submodule of A_j, it is uniform. Furthermore, C_j is τ-dense in A_j, because

$$A_j/C_j \cong (A + A_j)/A \subseteq E_\tau(A)/A\,.$$

Then $\sum_{j=1}^n C_j$ is τ-dense in $E_\tau(A)$, hence it is τ-dense in A. Now since we have $\sum_{j=1}^n C_j \subseteq B_j + C_j$, it follows that $B_j + C_j$ is τ-dense in A.

$(ii) \Longrightarrow (i)$ Since $\bigcap_{j=1}^k B_j = 0$, there exists a canonical homomorphism $\varphi : A \to \bigoplus_{i=1}^k A/B_j$. Then $\mathrm{Im}\varphi \trianglelefteq \bigoplus_{i=1}^k A/B_j$ because

$$\mathrm{Im}\varphi \cap (A/B_j) \supseteq (B_j + C_j)/B_j \neq 0\,.$$

Also, $B_j + C_j$ is τ-dense in A, so that $\mathrm{Im}\varphi$ is τ-dense in $\bigoplus_{i=1}^{k} A/B_j$. It follows that

$$E_\tau(A) = E_\tau(\mathrm{Im}\varphi) \cong \bigoplus_{i=1}^{k} E_\tau(A/B_j)$$

and clearly each $E_\tau(A/B_j)$ is uniform. □

We will now establish conditions under which the τ-injective hull of a module, not necessarily finitely generated, is a direct sum of minimal τ-injective modules. Recall that every minimal τ-injective module is uniform, but the converse does not hold in general.

We begin with a useful preliminary theorem. Recall that an intersection $K_1 \cap \cdots \cap K_n$ of submodules of a module A is said to be *irredundant* if

$$K_i \not\supseteq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$$

for every $i \in \{1, \ldots, n\}$.

Theorem 4.1.10 *Let A be a module and let $B = B_1 \cap \cdots \cap B_n$ be an irredundant intersection of submodules of A such that $E_\tau(A/B_i)$ is a minimal τ-injective module for every $i \in \{1, \ldots, n\}$ Then*

$$E_\tau(A/B) \cong \bigoplus_{i=1}^{n} E_\tau(A/B_i)$$

and any two such direct sum decompositions are isomorphic.

Proof. Let $f : A \to \bigoplus_{i=1}^{n} E_\tau(A/B_i)$ be defined by

$$f(a) = (a + B_1, \ldots, a + B_n)\,.$$

Then f is a homomorphism with $\mathrm{Ker} f = B$. Hence f induces a monomorphism $g : A/B \to \bigoplus_{i=1}^{n} E_\tau(A/B_i)$. For each $i \in \{1, \ldots, n\}$, let $q_i : E_\tau(A/B_i) \to \bigoplus_{i=1}^{n} E_\tau(A/B_i)$ denote the canonical injection. Since the intersection $B = B_1 \cap \cdots \cap B_n$ is irredundant, for every i there exists $b_i \in$

$B_1 \cap \cdots \cap B_{i-1} \cap B_{i+1} \cap \cdots \cap B_n$ such that $b_i \notin B_i$. Then $g(b_i + B) = q_i(b_i + B_i)$ is a non-zero element of $g(A/B) \cap q_i(A/B_i)$. But $E_\tau(A/B_i)$ is minimal τ-injective, hence $q_i(E_\tau(A/B_i))$ has the same property. Then $q_i(E_\tau(A/B_i))$ is a τ-injective hull of $g(A/B) \cap q_i(A/B_i)$. Hence

$$\bigoplus_{i=1}^{n} E_\tau(A/B_i) = \bigoplus_{i=1}^{n} q_i(E_\tau(A/B_i)) =$$

$$= E_\tau(\bigoplus_{i=1}^{n}(g(A/B) \cap q_i(A/B_i))) = E_\tau(g(A/B))$$

But $E_\tau(g(A/B) \cong E_\tau(A/B)$. It follows that $E_\tau(A/B) \cong \bigoplus_{i=1}^{n} E_\tau(A/B_i)$.

Now let $B = C_1 \cap \cdots \cap C_m$ be another irredundant intersection of submodules of A such that $E_\tau(A/C_j)$ is a minimal τ-injective module for every $j \in \{1, \ldots, m\}$. We have the isomorphisms

$$E_\tau(A/B) \cong \bigoplus_{i=1}^{n} E_\tau(A/B_i) \cong \bigoplus_{i=1}^{n} E_\tau(A/C_j).$$

Now by Lemma 3.1.2 and by Krull-Remak-Schmidt-Azumaya Theorem, we have $m = n$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $E_\tau(A/B_i) \cong E_\tau(A/C_{\sigma(i)})$ for every $i \in \{1 \ldots, n\}$. $\square$

Remark. In the context of the previous theorem, since $E(A/B_i)$ is indecomposable, B_i is irreducible for every $i \in \{1, \ldots, n\}$.

Theorem 4.1.11 *Let n be a natural number, let A be a module and $B \leq A$. Then the following statements are equivalent:*

(i) $E_\tau(A/B) = \bigoplus_{i=1}^{n} E_i$, where E_i is a minimal τ-injective module for every $i \in \{1, \ldots, n\}$.

(ii) There exists an irredundant intersection $B = B_1 \cap \cdots \cap B_n$ of submodules of A such that $E_\tau(A/B_i)$ is a minimal τ-injective module for every $i \in \{1, \ldots, n\}$.

Proof. $(ii) \Longrightarrow (i)$. This is Theorem 4.1.10.

$(i) \Longrightarrow (ii)$. Let $p : A \to A/B$ and $k : A/B \to E_\tau(A/B)$ be the natural homomorphism and the inclusion homomorphism respectively. For every $i \in \{1, \ldots, n\}$, denote by $q_i : E_\tau(A/B) \to E_i$ the canonical projection and by $g_i : A \to E_i$ the combined homomorphism $g_i = q_i kp$. Also put $B_i = \mathrm{Ker} g_i$. Then $B = B_1 \cap \cdots \cap B_n$. Since $E_i \cap (A/B) \neq 0$, it follows that $B_i \neq A$. We have $A/B_i \cong g_i(A) \subseteq E_i$, hence $E_\tau(A/B_i) \cong E_i$, because E_i is a minimal τ-injective module. Suppose that the intersection $B = B_1 \cap \cdots \cap B_n$ is not irredundant. Then we can refine from it an irredundant intersection with fewer terms by omission. By Theorem 4.1.10, $E_\tau(A/B)$ is isomorphic to a direct sum of less than n minimal τ-injective modules, a contradiction. $\square$

Example 4.1.12 Consider the polynomial ring $R = K[X_1, \ldots, X_{n+2}]$, where K is a field and $n \geq 2$. Let $p = (X_1X_2, X_1X_3)$. If $p_1 = (X_1)$ and $p_2 = (X_2, X_3)$, then $p = p_1 \cap p_2$ is an irredundant intersection of the prime ideals p_1 and p_2 of R. We have $\dim p_1 = n + 1$ and $\dim p_2 = n$. Then R/p_2 is τ_n-torsion. Since R is noetherian, it follows that R/p_1 is τ_n-cocritical by Corollary 1.8.4 and $E_{\tau_n}(R/p_2) = E(R/p_2)$. Now by Corollary 3.1.6, $E_{\tau_n}(R/p_1)$ and $E_{\tau_n}(R/p_2)$ are minimal τ_n-injective. Then by Theorem 4.1.10,

$$E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1X_2, X_1X_3)) \cong$$
$$\cong E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1)) \oplus E(K[X_1, \ldots, X_{n+2}]/(X_2, X_3))\,.$$

But we have the ring isomorphisms $K[X_1, \ldots, X_{n+2}]/(X_1) \cong K[X_2, \ldots, X_{n+2}]$ and $K[X_1, \ldots, X_{n+2}]/(X_2, X_3) \cong K[X_1, X_4, \ldots, X_{n+2}]$. By Theorem 3.2.9, it follows that we have the R-isomorphism

$$E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1X_2, X_1X_3)) \cong$$
$$\cong K(X_2, \ldots, X_{n+2}) \oplus E(K[X_1, X_4, \ldots, X_{n+2}])\,,$$

where $K(X_2, \ldots, X_{n+2})$ is the field of fractions of $K[X_2, \ldots, X_{n+2}]$. Moreover, $K(X_2, \ldots, X_{n+2})$ is τ_n-torsionfree minimal τ_n-injective and $E(K[X_1, X_4, \ldots, X_{n+2}])$ is τ_n-torsion minimal τ_n-injective.

4.2 τ-complemented τ-injective modules

In this section we consider a subclass of the class of τ-complemented modules, that will be useful for establishing results on τ-completely decomposable modules. Recall that a module A is called *τ-complemented* if every submodule of A is τ-dense in a direct summand of A.

We begin with a very useful characterization of τ-complemented τ-injective modules.

Proposition 4.2.1 *The following statements are equivalent for a τ-injective module A:*

(i) A is τ-complemented.

(ii) Every τ-injective submodule of A is a direct summand.

(iii) Every τ-injective submodule of A is closed.

(iv) A has no proper essential τ-injective submodule.

Proof. $(i) \Longrightarrow (ii)$ Suppose that A is τ-complemented and let B be a τ-injective submodule of A. Then B is τ-dense in a direct summand D of A, hence D/B is τ-torsion. Since B is τ-injective, it is a direct summand of D and, consequently, of A.

$(ii) \Longrightarrow (iii)$ Clear.

$(iii) \Longrightarrow (iv)$ Clear.

$(iv) \Longrightarrow (i)$ Suppose that A has no proper essential τ-injective submodule and let B be a submodule of A. If $E_\tau(B) = A$, we are done. Assume further that $E_\tau(B)$ is a proper submodule of A. Then it is not essential in A. Let D be a complement of $E_\tau(B)$ in A. Since $E_\tau(B) \cap D = 0$, we have $E_\tau(B) \cap E_\tau(D) = 0$, whence $D = E_\tau(D)$. Then $E_\tau(B) \oplus E_\tau(D) = E_\tau(B) \oplus D \trianglelefteq A$. Since $E_\tau(B) \oplus D$ is τ-injective, it follows that $E_\tau(B) \oplus D = A$. Thus B is τ-dense in the direct summand $E_\tau(B)$ of A. Therefore A is τ-complemented. $\square$

In order to show that a τ-injective module is τ-complemented, Proposition 4.2.1 will be frequently used.

Remarks. (i) Note that every minimal τ-injective module is τ-complemented.

(ii) In general, a τ-injective submodule of a τ-injective module is not a direct summand and the notions of τ-injective module and τ-complemented module are independent, as we may see in the following examples.

Example 4.2.2 Let R be a commutative noetherian domain with $\dim R \geq 2$ and let $p \in \mathrm{Spec}(R)$ be such that $\dim p = 1$.

(1) By Corollaries 3.3.6 and 3.3.13, $E_{\tau_0}(R/p)$ is a proper τ_0-injective submodule of the indecomposable module $E(R/p)$, hence it is not a direct summand.

(2) Since R/p is τ_0-cocritical, it is τ_0-complemented, but by Theorem 3.2.9 R/p is not τ_0-injective. On the other hand, $E_{\tau_0}(R/p)$ is a proper essential submodule of $E(R/p)$ by Corollary 3.3.13. Then by Proposition 4.2.1, $E(R/p)$ is not τ_0-complemented.

Every τ-torsionfree τ-complemented module is extending by Lemma 1.7.5. We will see that every τ-complemented τ-injective module is even quasi-injective. But first let us mention another subclass of the class of quasi-injective modules. We have seen in Proposition 2.1.9 that the class of τ-torsion τ-injective modules is contained in the class of quasi-injective modules. We will see that the class of τ-complemented τ-injective modules is placed in between. Indeed, every τ-torsion module is clearly τ-complemented. On the other hand we have the following lemma.

Lemma 4.2.3 *Every τ-complemented τ-injective module is quasi-injective.*

Proof. Let A be a τ-complemented τ-injective module. Also, let B be a submodule of A, $f : B \to A$ a homomorphism and $i : B \to A$ the inclusion. Since $E_\tau(B)/B$ is τ-torsion and A is τ-injective, there exists a homomorphism

$g : E_\tau(B) \to A$ extending f. Since A is τ-complemented τ-injective, by Proposition 4.2.1 there exists a submodule D of A such that $A = E_\tau(B) \oplus D$. If $h = g \oplus 1_D : A \to A$, then $hi = f$. Thus A is quasi-injective. □

Remarks. (*i*) The class of τ-complemented τ-injective modules does not coincide with the class of τ-torsion τ-injective modules. For instance, if R is left seminoetherian, then there exist τ-cocritical τ-injective modules, i.e. there exist τ-torsionfree minimal τ-injective modules.

(*ii*) The class of τ-complemented τ-injective modules does not coincide either with the class of quasi-injective modules, as we will show in a forthcoming example (see Example 4.3.11).

It is well-known that every injective module is a direct summand of any module which contains it. By Proposition 4.2.1, we immediately have a similar property for τ-injective submodules of τ-complemented τ-injective modules.

Lemma 4.2.4 *Let A be a τ-complemented τ-injective module and let B be a τ-injective submodule of A. Then B is a direct summand of any submodule of A which contains B.*

We have seen that every minimal τ-injective module is uniform, but in general not conversely (see Lemma 3.1.2 and Example 3.2.14). Nevertheless, we have the following result for τ-complemented τ-injective modules.

Lemma 4.2.5 *The following statements are equivalent for a τ-complemented τ-injective module A:*

(i) A is minimal τ-injective.

(ii) A is uniform.

(iii) A is non-zero indecomposable.

Proof. (*i*) $\Longrightarrow$ (*ii*) $\Longrightarrow$ (*iii*) Clear.

(*iii*) $\Longrightarrow$ (*i*) Suppose that A is not minimal τ-injective. Then there exists a non-zero proper submodule B of A such that $E_\tau(B)$ is a proper submodule of A. Since A is τ-complemented τ-injective, $E_\tau(B)$ is a non-zero proper direct summand of A, a contradiction. $\square$

Proposition 4.2.6 *Let $(A_i)_{i\in I}$ be a family of τ-complemented τ-injective modules. Then $E_\tau(\bigoplus_{i\in I} A_i)$ is τ-complemented.*

Proof. We will show that $A = E_\tau(\bigoplus_{i\in I} A_i)$ has no proper essential τ-injective submodules. Then the result will follow by Proposition 4.2.1.

Let D be an essential τ-injective submodule of A and let $i \in I$. Then $D \cap A_i \neq 0$ and $E(D) = E(A)$. Since D is τ-injective, it follows that $E(A)/D$ is τ-torsionfree, hence A/D is τ-torsionfree. Then $A_i/(D \cap A_i)$ is τ-torsionfree, being isomorphic to the submodule $(D + A_i)/D$ of A/D. This means that $D \cap A_i$ is τ-closed in the τ-injective module A_i. Therefore $D \cap A_i$ is τ-injective. Now let $0 \neq a_i \in A_i$. Then there exists $r_i \in R$ such that $0 \neq r_i a_i \in D$, hence $r_i a_i \in D \cap A_i$. It follows that $D \cap A_i \trianglelefteq A_i$. By Proposition 4.2.1, $D \cap A_i = A_i$. Then $\bigoplus_{i\in I} A_i \trianglelefteq D \trianglelefteq A$. Hence $D = A$. $\square$

We have seen in Example 1.7.7 that the class of τ-complemented modules is not closed under submodules or finite direct sums. But we have the following proposition for τ-complemented τ-injective modules.

Proposition 4.2.7 *The class of τ-complemented τ-injective modules is closed under τ-injective submodules, direct summands, τ-closed submodules and finite direct sums.*

Proof. Let A be a τ-complemented τ-injective module and let B be a submodule of A. If B is τ-injective and D is a τ-injective submodule of B, then D is a direct summand of A and, consequently, a direct summand of B. Thus B is τ-complemented by Proposition 4.2.1. If B is a direct summand of A, then B is τ-injective. It also follows that B is τ-complemented. If B is τ-closed in A, then B is τ-injective. Hence B is τ-complemented τ-injective.

Now let $\bigoplus_{i\in I} A_i$ be a finite direct sum of τ-complemented τ-injective modules. Then $\bigoplus_{i\in I} A_i$ is τ-injective, hence $\bigoplus_{i\in I} A_i$ is τ-complemented by Proposition 4.2.6. $\square$

Remarks. (i) Similarly, it can be proved that the class of τ-complemented τ-injective modules is closed under arbitrary direct sums, provided τ is noetherian and R has ACC on τ-dense left ideals. In this case, direct sums of τ-injective modules are τ-injective by Theorem 2.3.8.

(ii) We have seen that every τ-complemented τ-injective module is quasi-injective. But the class of τ-complemented τ-injective modules and the class of quasi-injective modules does not coincide, because the former is closed under finite direct sums and the latter is not.

Proposition 4.2.8 *The class of τ-complemented modules is closed under τ-injective hulls.*

Proof. Let A be a τ-complemented module. Then by Theorem 1.7.8, $A = B \oplus C$, where B is a τ-torsionfree τ-complemented submodule of A and $C = t(A)$. It follows that

$$E_\tau(A) = E_\tau(B) \oplus E_\tau(C)\,.$$

Then $E_\tau(C)$ is τ-complemented τ-injective. We will show that $E_\tau(B)$ is τ-complemented. Let D be a non-zero τ-injective submodule of $E_\tau(B)$. Denoting $F = B \cap D$, we have $F \trianglelefteq D$. Since B is τ-complemented, there exist two submodules G and H of B such that $B = G \oplus H$ and F is τ-dense in G. But G is τ-torsionfree, so that $F \trianglelefteq G$ by Lemma 1.7.5. We have

$$E_\tau(B) = E_\tau(G) \oplus E_\tau(H)\,.$$

Clearly $F \cap H = 0$, hence $D \cap E_\tau(H) = 0$. Since F is τ-dense in G and G is τ-dense in $E_\tau(G)$, F is τ-dense in $E_\tau(G)$. Since $F \trianglelefteq G$, it follows that $E_\tau(F) = E_\tau(G)$. Then

$$F \oplus H \trianglelefteq D \oplus E_\tau(H) \trianglelefteq E_\tau(F) \oplus E_\tau(H) = E_\tau(B)\,.$$

But $D \oplus E_\tau(H)$ is τ-injective. Then we must have $E_\tau(B) = D \oplus E_\tau(H)$, i.e. D is a direct summand of $E_\tau(B)$. Thus $E_\tau(B)$ is τ-complemented by Proposition 4.2.1. Now apply Proposition 4.2.7. □

Corollary 4.2.9 *If $(A_i)_{i\in I}$ is a family of τ-complemented modules, then $E_\tau(\bigoplus_{i\in I} A_i)$ is τ-complemented.*

Proof. Clearly we have

$$\bigoplus_{i\in I} A_i \trianglelefteq \bigoplus_{i\in I} E_\tau(A_i) \trianglelefteq E_\tau(\bigoplus_{i\in I} A_i)\,,$$

whence it follows that

$$E_\tau(\bigoplus_{i\in I} A_i) = E_\tau(\bigoplus_{i\in I} E_\tau(A_i))\,.$$

By Proposition 4.2.8, $E_\tau(A_i)$ is τ-complemented for every $i \in I$. Now apply Proposition 4.2.6. □

Corollary 4.2.10 *Let A be a τ-complemented module and let B be a τ-injective submodule of A. Then B is a direct summand of A and B is τ-complemented.*

Proof. By Proposition 4.2.8, $E_\tau(A)$ is τ-complemented. Then by Proposition 4.2.1, B is a direct summand of $E_\tau(A)$, hence B is a direct summand of A. By Proposition 4.2.7, B is τ-complemented. □

Remark. In general, a τ-injective submodule B of a τ-injective module A is not τ-closed in A. For instance, let $A = B \oplus C$, where B is a τ-injective module and C is a non-zero τ-torsion τ-injective module. Then A/B is not τ-torsionfree, i.e. B is not τ-closed in A.

Proposition 4.2.11 *Let A be a τ-torsionfree τ-complemented τ-injective module. Then:*

(i) Any intersection of τ-injective submodules of A is τ-complemented τ-injective.

(ii) If B and C are τ-injective submodules of A, then $B+C$ is τ-complemented τ-injective.

Proof. (i) Let $(B_i)_{i\in I}$ be a family of τ-injective submodules of A and let $B=\bigcap_{i\in I} B_i$. Then B_i is τ-closed in A for every $i \in I$ by Proposition 2.1.11. But the class of τ-closed submodules of a module is closed under arbitrary intersections. Hence B is τ-closed in the τ-injective module A. Therefore B is τ-injective by Proposition 2.1.11. Now by Proposition 4.2.7, B is τ-complemented.

(ii) Consider the short exact sequence of modules

$$0 \longrightarrow B\cap C \xrightarrow{f} B\oplus C \xrightarrow{g} B+C \longrightarrow 0$$

where the homomorphisms f and g are defined by $f(b)=(b,-b)$ for every $b\in B\cap C$ and $g(b,c)=b+c$ for every $(b,c)\in B\oplus C$. The modules B and C are τ-complemented τ-injective, because A is τ-complemented τ-injective. By Proposition 4.2.7, $B\oplus C$ is τ-complemented τ-injective. Since $B\cap C$ is τ-injective, $f(B\cap C)$ is a τ-injective submodule of $B\oplus C$. It follows that

$$B\oplus C \cong f(B\cap C)\oplus(B+C)\,.$$

Therefore $B+C$ is τ-complemented τ-injective. $\square$

In the sequel we will see that the class of τ-complemented τ-injective modules is not closed under arbitrary extensions, having however some weaker properties in this sense.

For the rest of this section we will refer to a short exact sequence of modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

We may assume without loss of generality that A is a submodule of B and f is the inclusion homomorphism.

Remark. Let R be left hereditary. Let B be a non-zero τ-injective module and let A be a proper essential τ-complemented τ-injective submodule of B such that it does not exist any proper τ-injective submodule of B which strictly contains A. Then B/A is τ-injective τ-cocritical, hence B/A is minimal τ-injective, therefore B/A is τ-complemented τ-injective. But since $A \trianglelefteq B$, B is not τ-complemented. Hence the class of τ-complemented τ-injective modules is not closed under extensions.

Proposition 4.2.12 *Let A be τ-injective. If one of the following two conditions holds:*

(i) A is not essential in B and C is minimal τ-injective;

(ii) A is closed in B and C is τ-complemented τ-injective,

then the above sequence splits.

Proof. We may assume that A is a non-zero proper submodule of B. Since the class of τ-injective modules is closed under extensions, B is τ-injective. Note that each of the two conditions assumes that A is not essential in B.

Let D be a complement of A in B. Then $D \neq 0$, $A \cap D = 0$ and $A \oplus D \trianglelefteq B$. Since D is closed in B and B is τ-injective, D is τ-injective. We have $g(D) \cong D \neq 0$, hence $g(D)$ is a non-zero τ-injective submodule of C.

Assume that the first condition is satisfied. Since C is minimal τ-injective, we have $g(D) = C$. Then it follows easily that $B = A \oplus D \cong A \oplus C$, hence the sequence splits.

Assume now that the second condition is satisfied. We will also show that $g(D) = C$. Suppose the contrary and let $c \in C \setminus g(D)$. Then there exists $b \in B \setminus (A \oplus D)$ such that $g(b) = c$. Then A is a proper submodule of $Rb + A$. Since A is closed, A is maximal with the property $A \cap D = 0$. Hence $(Rb + A) \cap D \neq 0$. Then there exist $r \in R$, $a \in A$ and a non-zero element $d \in D$ such that $d = rb + a$. Hence $0 \neq g(d) = g(rb + a) = rg(b)$, i.e. $0 \neq rc \in g(D)$. It follows that $g(D) \trianglelefteq C$. Thus $g(D)$ is a proper essential submodule of the τ-complemented τ-injective module C. By Proposition

4.2.1, this is a contradiction. Therefore $g(D) = C$. As above, we have $B \cong A \oplus C$, hence the sequence splits. □

Corollary 4.2.13 *Let A and C be τ-complemented τ-injective and suppose that A is not essential in B. If either A is closed in B or C is minimal τ-injective, then B is τ-complemented τ-injective.*

Proof. By Proposition 4.2.7, $A \oplus C$ is τ-complemented τ-injective. By Proposition 4.2.12, $B \cong A \oplus C$, hence B is τ-complemented τ-injective. □

4.3 τ-completely decomposable modules versus τ-complemented modules

Let us show now that the class of τ-completely decomposable modules is a subclass of the class of τ-complemented modules.

Theorem 4.3.1 *Every τ-completely decomposable module is τ-complemented.*

Proof. Let A be a τ-completely decomposable module and let $A \xrightarrow{f} B \longrightarrow 0$ be an exact sequence of modules with B τ-torsionfree. We will show that the sequence splits. Then the result will follow by Proposition 1.7.3.

Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A. We may suppose that f is a non-zero homomorphism. Denote $f_i = f|_{A_i}$ for every $i \in I$.

Now let $i \in I$. It follows that $f_i(A_i)$ is τ-torsionfree. Since A_i is minimal τ-injective, A_i is either τ-torsion or τ-torsionfree. If A_i is τ-torsion or $f_i = 0$, then $f_i(A_i) = 0$. Suppose now that A_i is τ-torsionfree and $f_i \neq 0$. Then $f_i(A_i) \cong A_i$, because A_i is τ-cocritical. Let $J = \{j \in I \mid f(A_j) \neq 0\}$. Then $B = f(A) = \sum_{j \in J} f(A_j)$. It follows that there exists a subset $K \subseteq J$ such that $B = \bigoplus_{k \in K} f(A_k)$ by Proposition 2.2.8. Now let $g : B \to A$ be the homomorphism defined by $g = \bigoplus_{k\in K} f_k^{-1}$. Then $fg = 1_B$, i.e. the above sequence splits. $\square$

Corollary 4.3.2 *Let A be a τ-completely decomposable module and let B be a τ-injective submodule of A. Then B is a direct summand of A and B is τ-complemented.*

Proof. It follows by Corollary 4.2.10 and Theorem 4.3.1. $\square$

Now recall that every τ-torsionfree τ-complemented module is extending by Lemma 1.7.5. In order to show a stronger result for τ-torsionfree τ-completely decomposable modules we need first two propositions.

Proposition 4.3.3 *Let A be a τ-torsionfree τ-completely decomposable module and $B \leq E(A)$.*

(i) If B is minimal τ-injective, then B is a direct summand of A.

(ii) If B is τ-completely decomposable and $E(B) = E(A)$, then $B = A$.

Proof. Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a minimal τ-injective submodule of A.

(i) Suppose that B is minimal τ-injective. Since $E(A)$ is τ-torsionfree, B is τ-injective τ-cocritical. Since $A \trianglelefteq E(A)$, we have $B \cap A \neq 0$. Let x be a non-zero element of $B \cap A$. Then there exists a finite subset $J \subseteq I$ such that $Rx \subseteq B \cap (\bigoplus_{i\in J} A_i)$. But $\bigoplus_{i\in J} A_i$ is τ-injective. It follows that $E_\tau(Rx) \subseteq \bigoplus_{i\in J} A_i$. Since Rx is τ-cocritical, $E_\tau(Rx)$ is τ-injective τ-cocritical. But $B \cap E_\tau(Rx) \neq 0$, hence $B = E_\tau(Rx)$. Therefore B is a submodule of A. Now by Corollary 4.3.2, B is a direct summand of A.

(ii) Let $B = \bigoplus_{j\in J} B_j$, where each B_j is a minimal τ-injective submodule of B and suppose that $E(B) = E(A)$. Then B is τ-torsionfree. Since each B_j is a minimal τ-injective submodule of $E(A)$ and each A_i is a minimal τ-injective submodule of $E(B)$, it follows that each B_j is a submodule of A and each A_i is a submodule of B by (i). Then $A = B$. $\square$

Proposition 4.3.4 *Let A and B be τ-completely decomposable modules with B τ-torsionfree and let $f : E(A) \to E(B)$ be a homomorphism. Then $f(A) \subseteq B$.*

Proof. Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A. We may suppose that f is a non-zero homomorphism. Following the proof of Theorem 4.3.1, there exists a subset $K \subseteq I$ such that $f(A) = \sum_{k \in K} f(A_k)$, where each A_k is τ-cocritical τ-injective, i.e. minimal τ-injective. By Proposition 4.3.3 it follows that each $f(A_k)$ is a submodule of B. Therefore $f(A) \subseteq B$. □

Corollary 4.3.5 *Every τ-torsionfree τ-completely decomposable module is quasi-injective.*

By Theorem 1.7.8, we know that a module is τ-complemented if and only if it is a direct sum of a τ-torsion module and a τ-torsionfree τ-complemented module. By Theorem 4.1.5, every τ-torsion τ-injective module is τ-completely decomposable if and only if R has ACC on τ-dense left ideals. Therefore over a ring that satisfies ACC for τ-dense left ideals, the problem of a τ-complete decomposition of a τ-complemented τ-injective module A reduces to the case when A is τ-torsionfree.

Theorem 4.3.6 *Let A be a τ-torsionfree τ-complemented τ-injective module. Then $A = B \oplus C$, where B is the τ-injective hull of a τ-completely decomposable module and C is a τ-injective submodule of A that does not contain any uniform submodule.*

Proof. Note that since A is τ-torsionfree, a submodule of A is minimal τ-injective if and only if it is τ-injective τ-cocritical. If A does not contain any minimal τ-injective submodule, then $B = 0$.

Suppose now that A does contain a minimal τ-injective submodule. Let $D = \sum_{i \in I} D_i$, where $(D_i)_{i \in I}$ is the family of all minimal τ-injective submodules of A. Then there exists a subset J of I such that $D = \bigoplus_{j \in J} D_j$ by

Proposition 2.2.8. Then $E_\tau(D)$ is a τ-injective submodule of A, hence it is a direct summand of A. Therefore $A = B \oplus C$, where $B = E_\tau(D)$ and C is a τ-injective submodule of A which does not contain any minimal τ-injective, hence uniform submodule. $\square$

Theorem 4.3.7 *Let R be τ-noetherian and let A be a module. Then A is τ-complemented τ-injective if and only if $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module.*

Proof. Suppose that A is τ-complemented τ-injective. By Theorem 1.7.8, we have $A = B \oplus C$, where $B = t(A)$ and C is τ-torsionfree τ-complemented τ-injective. Hence B is τ-torsion τ-injective. We may suppose that $C \neq 0$. The hypothesis on R allow us to write $E(C) = \bigoplus_{i\in I} E_i$ as a direct sum of indecomposable injective modules by Theorem 1.4.14. For every $i \in I$, denote $D_i = C \cap E_i$. Then for every $i \in I$, we have $0 \neq D_i \trianglelefteq E_i$, hence

$$\bigoplus_{i\in I} D_i \trianglelefteq \bigoplus_{i\in I} E_i = E(C)\,.$$

It follows that $\bigoplus_{i\in I} D_i \trianglelefteq C$. By Proposition 4.2.1, $C = E_\tau(\bigoplus_{i\in I} D_i)$. Obviously, $\bigoplus_{i\in I} D_i$ is τ-torsionfree. We still have to show that each D_i is minimal τ-injective. Let $i \in I$. Then

$$E_i/D_i \cong (C + E_i)/C \subseteq E(C)/C$$

is τ-torsionfree. Then D_i is τ-closed in E_i, hence D_i is τ-injective. Now by Proposition 4.2.7, D_i is τ-complemented τ-injective. But D_i is uniform, hence by Lemma 4.2.5, D_i is minimal τ-injective.

Conversely, suppose that $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module. Then B is τ-complemented τ-injective. Now apply Propositions 4.2.6 and 4.2.7. $\square$

We have seen in Theorem 1.7.10 a partial result on the structure of τ-complemented modules. For a τ-complemented τ-injective module we have the following important theorem.

Theorem 4.3.8 *Let R be a ring that has ACC both on τ-dense and τ-closed left ideals. Then a module A is τ-complemented τ-injective if and only if A is τ-completely decomposable.*

Proof. Suppose that A is τ-complemented τ-injective. Then by Theorem 4.3.7, $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module. Since R has ACC on τ-dense left ideals, every τ-torsion τ-injective module has a τ-complete decomposition by Theorem 4.1.5. Under both hypotheses on R, direct sums of τ-injective modules are τ-injective by Theorem 2.3.8. Now the result follows.

For the converse apply Theorem 4.3.1 and again Theorem 2.3.8. □

Remark. Clearly, Theorem 4.3.8 holds for a left noetherian ring, but there also exist non-left noetherian rings that satisfy ACC both on τ-dense and τ-closed left ideals.

Example 4.3.9 [91, Example 28] Consider

$$R = \mathbb{Z} \oplus (\bigoplus_{i \in \mathbb{N}^*} \mathbb{Z}_{p_i}),$$

where p_i is the i-th prime number. Then R is a ring with the following operations:

$$(a, \ldots, x_i, \ldots) + (b, \ldots, y_i, \ldots) = (a + b, \ldots, x_i + y_i, \ldots)$$

$$(a, \ldots, x_i, \ldots) \circ (b, \ldots, y_i, \ldots) = (ab, \ldots, bx_i + ay_i, \ldots).$$

Let τ be the hereditary torsion theory generated by all R-modules $\mathbb{Z}_{p_i}$ for $i \in \mathbb{N}^*$. Then R is not left noetherian, but R has ACC both on τ-dense and τ-closed left ideals.

Let us see an example of determining the structure of τ-completely decomposable modules by using Theorem 4.3.8.

Example 4.3.10 Let R be a commutative noetherian ring with $\dim R \geq 1$. Then by Corollary 3.1.6, a module M is minimal τ_0-injective if and only if $M \cong E_{\tau_0}(R/p)$, where $p \in \mathrm{Spec}(R)$ and $\dim p \in \{0, 1\}$. If p is maximal, then R/p is τ_0-torsion and we have $E_{\tau_0}(R/p) = E(R/p)$. If $\dim p = 1$, then R/p is τ_0-cocritical. Now Theorem 4.3.8 gives the structure of τ_0-completely decomposable (τ_0-complemented τ_0-injective) modules. Thus A is a τ_0-completely decomposable if and only if

$$A \cong (\bigoplus_{i \in I} E_{\tau_0}(R/p_i)) \oplus (\bigoplus_{j \in J} E(S_j)),$$

where each $p_i \in \mathrm{Spec}(R)$ and has $\dim p_i = 1$ and each S_j is a simple module.

Finally, we are now able to show that the class of τ-completely decomposable modules is strictly included in the class of τ-complemented modules, as it may be seen in the following example.

Example 4.3.11 Consider the polynomial ring $R = K[X_1, \ldots, X_m]$ ($m \geq 2$), where K is a field and the prime ideal $p = (X_1, \ldots, X_{m-n-1})$ of R, where $n < m - 1$. We have seen in Example 3.2.10 that

$$K[X_{m-n}, \ldots, X_m] \cong K[X_1, \ldots, X_m]/(X_1, \ldots, X_{m-n-1})$$

is a τ_n-cocritical R-module and its τ_n-injective hull is R-isomorphic to its field of fractions $K(X_{m-n}, \ldots, X_m)$. Clearly, $K \cong K[X_1, \ldots, X_m]/(X_1, \ldots, X_m)$ is a τ_n-torsion R-module. Then by Theorem 1.7.8, it follows that $K \oplus K[X_{m-n}, \ldots, X_m]$ is a τ_n-complemented R-module. Since $K[X_{m-n}, \ldots, X_m]$ is not τ_n-injective, $K \oplus K[X_{m-n}, \ldots, X_m]$ cannot be τ_n-injective. Having noted that R is noetherian, Theorem 4.3.8 shows that $K \oplus K[X_{m-n}, \ldots, X_m]$ is not a τ_n-completely decomposable module.

4.4 Direct summands of τ-completely decomposable modules

Previously established results on τ-completely decomposable modules will allow us to give partial answers to a question generalizing a problem of Matlis. Among classical questions to ask on a class of modules is the following one:

If $A = \bigoplus_{i\in I} A_i$ is a direct sum of modules of a class $\mathcal{A}$, is a direct summand B of A still a direct sum of modules of the class $\mathcal{A}$?

This is apparently an open question if $\mathcal{A}$ is either the class of all modules with local endomorphism rings or the class of all indecomposable injective modules [42, p.267]. For the former, the answer is known to be yes if each A_i is countably generated [42, Corollary 2.55]. For the latter, raised by E. Matlis [75], the answer is known to be yes in several cases, such as: R left noetherian [75], I finite [101], A injective [43], A quasi-injective [63], B countably generated [43] or B injective [63].

Now consider the following condition on a module A [40, p.16]:

(C_2) *Every submodule isomorphic to a direct summand of A is a direct summand of A.*

Among the modules satisfying (C_2) we mention *continuous* modules (that can be seen as extending modules with (C_2)) and, in particular, quasi-injective modules.

If the class $\mathcal{A}$ consists of all minimal τ-injective modules, that are known to have local endomorphism rings, we will also give an affirmative answer to the question mentioned above, provided R has ACC both on τ-dense and τ-closed left ideals, A satisfies the condition (C_2), A is τ-torsionfree, A is τ-injective, I is finite, B is τ-injective or B is countably generated.

In the sequel, we will use some properties of τ-complemented modules in order to give partial answers to the following question:

Is a direct summand of a τ-completely decomposable module still a τ-

completely decomposable module?

Theorem 4.4.1 *Let R be a ring that has ACC both on τ-dense and τ-closed left ideals and let A be a τ-completely decomposable module. Then any direct summand of A is τ-completely decomposable.*

Proof. By Theorem 4.3.8, A is τ-complemented τ-injective. Then every direct summand B of A is τ-complemented τ-injective. Again by Theorem 4.3.8, it follows that B is τ-completely decomposable. $\square$

The following theorem is the main result of this section.

Theorem 4.4.2 *Let A be a τ-completely decomposable module that satisfies the condition (C_2). Then any direct summand of A is τ-completely decomposable.*

Proof. Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a minimal τ-injective submodule of A and let B be a non-zero proper direct summand of A. Since each A_i is uniform and B is not essential in A, there exists $k \in I$ such that $B \cap A_k = 0$ [40, p.38]. By Zorn's Lemma, there exists a maximal subset $J \subseteq I$ such that $B \cap (\bigoplus_{j\in J} A_j) = 0$. Let $p : A \to \bigoplus_{i\in I\setminus J} A_i$ the natural projection. Then the restriction $p|_B$ is a monomorphism, whence $p(B) \cong B$. Since A satisfies the condition (C_2), $p(B)$ is a direct summand of A. Hence $p(B)$ is a direct summand of $\bigoplus_{i\in I\setminus J} A_i$. Suppose that $p(B) \neq \bigoplus_{i\in I\setminus J} A_i$. Then there exists $h \in I \setminus J$ such that $p(B) \cap A_h = 0$ [40, p.38]. It follows that

$$B \cap (A_h \oplus (\bigoplus_{j\in J} A_j)) = 0\,,$$

which contradicts the maximality of J. Therefore $p(B) = \bigoplus_{i\in I\setminus J} A_i$. Then B is τ-completely decomposable. $\square$

Corollary 4.4.3 *Let A be a τ-completely decomposable module. If one of the following extra conditions on A holds:*

(i) A is continuous;

(ii) A is τ-torsionfree;

(iii) A is τ-injective;

(iv) A is finitely τ-completely decomposable,

then any direct summand of A is τ-completely decomposable.

Proof. If (i) holds, apply Theorem 4.4.2. If (ii) holds, then A is quasi-injective by Corollary 4.3.5 and the conclusion follows by the result for (i). If (iii) holds, by Theorem 4.3.1, A is τ-complemented and by Lemma 4.2.3, A is quasi-injective. If (iv) holds, note that the class of τ-injective modules is closed under finite direct sums and apply the result for (iii). □

In the following two results we ask for some conditions on the direct summands.

Theorem 4.4.4 *(i) Every τ-torsionfree direct summand of a τ-completely decomposable module is τ-completely decomposable.*

(ii) Every τ-injective direct summand of a τ-completely decomposable module is τ-completely decomposable.

Proof. Let $A = \bigoplus_{i \in I} A_i$, where each A_i is minimal τ-injective, and let B be a direct summand of A.

(i) Assume that B is τ-torsionfree. Let $0 \neq b \in B$. Then $E_\tau(Rb) \subseteq B$ and $E_\tau(Rb)$ is contained in a finite sum of submodules A_i. By Proposition 4.1.4, $E_\tau(Rb)$ is isomorphic to a finite direct sum of submodules A_i. Each $E_\tau(Rb)$ is τ-torsionfree minimal τ-injective, hence τ-cocritical τ-injective. Now by Proposition 2.2.8, $B = \sum_{b \in B} E_\tau(Rb)$ is τ-completely decomposable.

(ii) Assume that B is τ-injective. As for injective modules (see [116]), there exists $J \subseteq I$ such that $B = \bigoplus_{j \in J} A_j$, so that B is τ-completely decomposable. □

For completeness, we also mention the following theorem, whose proof is similar with the corresponding one given for indecomposable injective modules. In order to complete the proof, Corollary 4.3.2 is needed.

Theorem 4.4.5 *Let A be a τ-completely decomposable module. Then:*

(i) If B is a direct summand of A and C is a finitely generated submodule of B, then B contains a finitely τ-completely decomposable τ-injective hull of C.

(ii) Any countably generated direct summand of A is τ-completely decomposable.

Proof. Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a minimal τ-injective submodule of A.

(i) Since C is finitely generated, there exists a finite subset $J \subseteq I$ such that $C \subseteq \bigoplus_{j\in J} A_j$. Then $E_\tau(C) \subseteq \bigoplus_{j\in J} A_j$ and by Corollary 4.3.2, $E_\tau(C)$ is a direct summand of $\bigoplus_{j\in J} A_j$. But $\bigoplus_{j\in J} A_j$ is finitely τ-completely decomposable. Then by Corollary 4.4.3, $E_\tau(C)$ is finitely τ-completely decomposable. Now consider $p : A \to B$ the canonical projection. Then $p|_C$ is a monomorphism, hence $p|_{E_\tau(C)}$ is a monomorphism. Hence $p(E_\tau(C)) \cong E_\tau(C)$ is a finitely τ-completely decomposable τ-injective hull of C.

(ii) Let D be a countably generated direct summand of A and let $d_1, \ldots, d_n, \ldots$ be a countable generating set of D. By (i) for each natural number $n \geq 1$, there exists a finitely τ-completely decomposable module D_n such that $d_1, \ldots, d_n \in D_n$. By Corollary 4.3.2, each D_n is a direct summand of A, hence each D_n is a direct summand of D. But $D = \bigcup_{n\geq 1} D_n$. Denoting $D_0 = 0$, we have

$$D \cong \bigoplus_{n\geq 0} D_{n+1}/D_n\,,$$

that is, a direct sum of finitely τ-completely decomposable modules. Therefore D is τ-completely decomposable. □

Remark. If the Gabriel filter associated to τ consists of all left ideals of R, then τ-injective modules and minimal τ-injective modules become injective and indecomposable injective modules respectively. Thus the well-known results for indecomposable injective modules are obtained as particular cases of Theorems 4.4.1, 4.4.5 and Corollary 4.4.3.

We have seen in Corollary 4.3.5 that every τ-torsionfree τ-completely decomposable module is quasi-injective. More generally, we have the following property.

Proposition 4.4.6 *Every τ-torsionfree direct summand of a τ-completely decomposable module is quasi-injective.*

Proof. By Theorem 4.4.4, every τ-torsionfree direct summand of a τ-completely decomposable module is τ-completely decomposable. Then use Corollary 4.3.5. $\square$

We have seen that a direct summand of a τ-torsionfree τ-completely decomposable module is τ-completely decomposable. The converse is also true.

Theorem 4.4.7 *Let A be a τ-torsionfree τ-completely decomposable module. Then a submodule of A is a direct summand if and only if it is τ-completely decomposable.*

Proof. The "only if" part follows by Corollary 4.4.3.

Suppose now that B is a τ-completely decomposable submodule of A. By Theorem 4.3.1 and Proposition 4.2.6, $E_\tau(A)$ is τ-complemented. Since $E_\tau(B)$ is a τ-injective submodule of $E_\tau(A)$, there exists a non-zero submodule D of $E_\tau(A)$ such that $E_\tau(A) = E_\tau(B) \oplus D$. By Corollary 4.5.2, $D = E_\tau(C)$, where C is a τ-completely decomposable submodule of D. Hence $E_\tau(A) = E_\tau(B \oplus C)$. Since $B \oplus C$ is τ-completely decomposable, by Proposition 4.3.3 it follows that $A = B \oplus C$. Therefore B is a direct summand of A. $\square$

We will continue this section with a result that generalizes a corresponding one given for indecomposable injective modules. We need the following lemma.

Lemma 4.4.8 *Let A be a τ-completely decomposable module, B be a direct summand of A and C be a τ-injective submodule of A such that $B \cap C = 0$. Then $B \oplus C$ is a direct summand of A.*

Proof. There exists a submodule D of A such that $A = B \oplus D$. Let $p : A \to D$ be the canonical projection. Since $B \cap C = 0$ we have $p(C) \cong C$. Hence $p(C)$ is a τ-injective submodule of A. By Corollary 4.3.2, $p(C)$ is a direct summand of A, hence $p(C)$ is a direct summand of D. But $B \oplus C = B \oplus p(C)$. Therefore $B \oplus C$ is a direct summand of A. □

Let us now recall an auxiliary result.

Lemma 4.4.9 [71, Lemma 2.1] *Let X, Y, Z be submodules of a module such that $X \oplus Y = X \oplus Z$. Then there exists an isomorphism $f : Y \to Z$ such that*

$$f(B) \cap C = (X \oplus B) \cap C$$

for every submodule B of Y and for every submodule C of Z.

Theorem 4.4.10 *Let A be a τ-completely decomposable module. Then every non-zero direct summand B of A contains a minimal τ-injective direct summand.*

Proof. Let $\mathcal{P}$ be the family of all finite subsets J of I such that $(\bigoplus_{j \in J} A_j) \cap B \neq 0$. Note that $\mathcal{P}$ is non-empty, because B is a non-zero submodule of A. Denote by k the least (finite) cardinal of the elements of $\mathcal{P}$, say $k = |K|$ and take $K = \{i_1, \dots, i_k\}$. Also write $A = B \oplus C$.

Suppose first that $k = 1$. Since $(A_{i_1} \cap B) \cap (A_{i_1} \cap C) = 0$, $A_{i_1} \cap B \neq 0$ and A_{i_1} is uniform, we have $A_{i_1} \cap C = 0$. Then by Lemma 4.4.8, it follows that $A_{i_1} \oplus C$ is a direct summand of A, say $A = A_{i_1} \oplus C \oplus D$. But we also have $A = B \oplus C$. Then there exists an isomorphism $f : A_{i_1} \oplus D \to B$. Hence $B = f(A_{i_1} \oplus D) = f(A_{i_1}) \oplus f(D)$. Therefore $f(A_{i_1})$ is a minimal τ-injective direct summand of B.

Suppose now that $k > 1$. Denote

$$M = A_{i_1} \oplus \cdots \oplus A_{i_{k-1}}\,,$$

$$L = \bigoplus_{i \in I \setminus \{i_1, \ldots, i_{k-1}\}} A_i\,.$$

Clearly, $M \cap B = 0$ by the choice of k. By Lemma 4.4.8, since M is τ-injective, $M \oplus B$ is a direct summand of A, say $A = M \oplus B \oplus N$. On the other hand, we have $A = M \oplus L$. By Lemma 4.4.9, it follows that there exists an isomorphism $g : L \to B \oplus N$ such that

$$g(A_{i_k}) \cap B = (M \oplus A_{i_k}) \cap B\,.$$

But since

$$(g(A_{i_k}) \cap B) \cap (g(A_{i_k}) \cap N) = 0\,,$$

$$g(A_{i_k}) \cap B = (\bigoplus_{i \in K} A_i) \cap B \neq 0$$

and $g(A_{i_k})$ is uniform, we have $g(A_{i_k}) \cap N = 0$. Now repeat the argument used for $k = 1$. Then B will contain a minimal τ-injective direct summand of B isomorphic to $g(A_{i_k})$. □

4.5 Essential extensions of τ-completely decomposable modules

Now we are able to characterize the τ-injective hull of a τ-completely decomposable module.

Theorem 4.5.1 *A module A is the τ-injective hull of a τ-completely decomposable module if and only if A is τ-complemented τ-injective and the τ-injective hull of every non-zero cyclic submodule of A contains a uniform submodule.*

Proof. Suppose first that $A = E_\tau(\bigoplus_{i\in I} B_i)$, where each B_i is a minimal τ-injective module. By Proposition 4.2.6, A is τ-complemented. Let C be a non-zero cyclic submodule of $A = E_\tau(\bigoplus_{i\in I} B_i)$. Then there exists a non-zero element $x \in C \cap (\bigoplus_{i\in I} B_i)$. It follows that there exists a finite subset $J \subseteq I$ such that $Rx \subseteq \bigoplus_{j\in J} B_j$. But $\bigoplus_{j\in J} B_j$ is τ-complemented τ-injective. Hence $E_\tau(Rx)$ is a direct summand of the τ-completely decomposable module $\bigoplus_{j\in J} B_j$. By Corollary 4.4.3, $E_\tau(Rx)$ is τ-completely decomposable, hence there exists a minimal τ-injective, hence uniform submodule $D \subseteq E_\tau(Rx) \subseteq C$.

Suppose now that A is τ-complemented τ-injective and the τ-injective hull of every non-zero cyclic submodule of A contains a uniform submodule. Let $\mathcal{B}$ be the family of all minimal τ-injective submodules of A. Since A is τ-injective, there exists a uniform submodule B of A. Then $E_\tau(B)$ is minimal τ-injective by Proposition 4.2.7 and Lemma 4.2.5. Hence $E_\tau(B) \in \mathcal{B}$, i.e. $\mathcal{B}$ is non-empty. Then there exists a maximal collection $(B_i)_{i\in I}$ of members of $\mathcal{B}$ whose sum is direct [101, Proposition 1.7].

We will show that $\bigoplus_{i\in I} B_i \trianglelefteq A$. Let C be a non-zero submodule of A. Suppose that $(\bigoplus_{i\in I} B_i) \cap C = 0$. But A is τ-injective, hence $E_\tau(C) \subseteq A$. It follows that $(\bigoplus_{i\in I} B_i) \cap E_\tau(C) = 0$. As above, $E_\tau(C)$ contains a minimal τ-injective submodule D. Then $(\bigoplus_{i\in I} B_i) \oplus D \subseteq A$, which contradicts the maximality of the family $(B_i)_{i\in I}$. Therefore $\bigoplus_{i\in I} B_i \trianglelefteq A$. It follows that $E_\tau(\bigoplus_{i\in I} B_i) \trianglelefteq A$. By Proposition 4.2.1, $A = E_\tau(\bigoplus_{i\in I} B_i)$. $\square$

Corollary 4.5.2 *Let A be the τ-injective hull of a τ-completely decomposable module. Then any τ-injective submodule of A is the τ-injective hull of a τ-completely decomposable module.*

Proof. Let $A = E_\tau(\bigoplus_{i\in I} B_i)$, where each B_i is a minimal τ-injective module and let C be a non-zero τ-injective submodule of A. By Theorem 4.5.1, the τ-injective hull of every non-zero cyclic submodule of A, hence of C, contains a uniform submodule. But C is τ-complemented τ-injective. Then again by Theorem 4.5.1, C is the τ-injective hull of a τ-completely decomposable module. $\square$

In what follows we will deal with the following two problems:

Problem 1. *For some particular torsion theories τ, characterize the rings with the property that every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.*

Problem 2. *For some particular classes of rings, characterize the torsion theories with the property that every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.*

Concerning **Problem 1** we have the following result.

Theorem 4.5.3 *Let τ be a noetherian torsion theory. The following conditions are equivalent:*

(i) Every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.

(ii) R has ACC on τ-dense left ideals and R is τ-semiartinian.

Proof. $(i) \Longrightarrow (ii)$ Let A be a τ-torsion τ-injective module. Then every τ-injective submodule of A is a direct summand. It follows that A has an essential τ-injective submodule B of A, that is τ-completely decomposable. Since B is τ-dense in A, we have $B = A$, that is, A is τ-completely decomposable. Now by Theorem 4.1.5, R has ACC on τ-dense left ideals.

Since every τ-torsionfree minimal τ-injective module is τ-cocritical, every τ-injective module has essential τ-socle. Hence every module has an essential τ-socle, that is, R is τ-semiartinian.

$(ii) \Longrightarrow (i)$ Let A be a τ-injective module. Then $t(A)$ is τ-injective as a τ-closed submodule of A. By Theorem 4.1.5, $t(A)$ is τ-completely decomposable. We may assume that it is not essential in A. Then let B be a closed submodule of A such that $B \cap t(A) = 0$ and $B \oplus t(A) \trianglelefteq A$. It follows that B

is τ-injective. Then there is a family $(B_i)_{i\in I}$ of minimal τ-injective submodules of B such that $\mathrm{Soc}_\tau(B) = E_\tau(\bigoplus_{i\in I} B_i)$. Then $\mathrm{Soc}_\tau(B) \trianglelefteq B$, because R is τ-semiartinian. Since τ is noetherian, it follows by Theorem 2.3.7 that $\bigoplus_{i\in I} B_i$ is τ-injective. Then A is an essential extension of the τ-injective τ-completely decomposable module $t(A) \oplus (\bigoplus_{i\in I} B_i)$. $\square$

As a consequence of Theorem 4.5.3, we can characterize the situation when every τ-injective module is τ-completely decomposable.

Theorem 4.5.4 *The following statements are equivalent:*

(i) R is left noetherian, τ-semisimple and τ is stable.

(ii) R has ACC both on τ-dense and τ-closed left ideals, R is τ-semisimple and τ is stable.

(iii) Every τ-injective module is τ-completely decomposable.

Proof. $(i) \Longrightarrow (ii)$ Clear.

$(ii) \Longrightarrow (iii)$ Let A be a τ-injective module. Since R is τ-noetherian, it follows that τ is noetherian. Since R is τ-semisimple, R is τ-semiartinian by Corollary 1.6.10. Hence by Theorem 4.5.3, A is an essential extension of a τ-injective τ-completely decomposable module, say $B = \bigoplus_{i\in I} A_i$. Denote $J = \{i \in I \mid A_i \text{ is } \tau\text{-torsion}\}$ and $K = \{i \in I \mid A_i \text{ is } \tau\text{-torsionfree}\}$. Then $I = J \cup K$. Denote $C = \bigoplus_{j\in J} A_j$ and $D = \bigoplus_{k\in K} A_k$. Then $C \trianglelefteq t(A)$, but C is also τ-injective and τ-dense in $t(A)$, hence $C = t(A)$. By the stability of τ it follows that $(C \oplus D)/C \trianglelefteq A/C = A/t(A)$. Since R is τ-semisimple, by Proposition 1.6.8 the lattice $\mathcal{C}_\tau(R)$ is complemented. Again by Proposition 1.6.8, $(C \oplus D)/C$ is τ-dense in $A/t(A)$. But $(C \oplus D)/C$ is also τ-injective, so that we have $(C \oplus D)/C = A/t(A)$, whence $A = C \oplus D$. Thus A is τ-completely decomposable.

$(iii) \Longrightarrow (i)$ By hypothesis, every injective module A is τ-completely decomposable, say $A = \bigoplus_{i\in I} A_i$. But then each A_i is uniform and injective. Therefore R is left noetherian. By Theorem 4.1.8, R is τ-semisimple. By hypothesis, it follows that every indecomposable injective module is minimal

τ-injective, that is, it is either τ-torsion or τ-torsionfree by Lemma 3.1.2. Now by Proposition 1.2.10, τ is stable. $\square$

Remark. Note that by Example 4.3.9, there exist rings R that have ACC both on τ-dense and τ-closed left ideals, without being left noetherian.

Let us now deal with **Problem 2**. Thus we will determine those torsion theories on R-Mod, where R is a commutative noetherian ring that is not a domain, having the property that every τ-injective module is an essential extension of a τ-complemented τ-injective module or equivalently of a τ-completely decomposable module (see Theorem 4.3.8).

For the rest of this section we will assume the ring R to be commutative.

Let $\mathcal{P}$ be a non-empty set of minimal prime ideals of R. Denote by $\mathcal{A}_{\mathcal{P}}$ the class of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing a non-zero prime ideal $q \notin \mathcal{P}$. Denote by $\tau_{\mathcal{P}}$ the hereditary torsion theory generated by $\mathcal{A}_{\mathcal{P}}$.

Proposition 4.5.5 *Let R be noetherian, let τ be the torsion theory $\tau_{\mathcal{P}}$ defined above and let $0 \neq p \in \operatorname{Spec}(R)$. Then R/p is τ-cocritical if and only if $p \in \mathcal{P}$.*

Proof. Suppose first that R/p is τ-cocritical. Then by Theorem 1.5.12, $p \in \operatorname{Spec}(R)$. If $p \notin \mathcal{P}$, then R/p is τ-torsion by the definition of τ, a contradiction. Hence $p \in \mathcal{P}$.

Suppose now that $p \in \mathcal{P}$. Assume that R/p is τ-torsion. Then R/p contains a non-zero submodule A isomorphic to U/V, where U and V are ideals of R containing a non-zero prime ideal $q \notin \mathcal{P}$. Let $0 \neq a \in A$. Since $p \in \operatorname{Spec}(R)$, $\operatorname{Ann}_R a = p$. Let $r \in q \setminus p$. Then $ra = 0$, whence $r \in p$, a contradiction. Therefore R/p is not τ-torsion. Now by Lemma 1.4.7, R/p is τ-torsionfree. Clearly R/p is a noetherian R-module, hence by Proposition 1.5.8 there exists an ideal q of R such that $p \subseteq q$ and R/q is τ-cocritical. By

Theorem 1.5.12, $q = \mathrm{Ann}_R(R/q) \in \mathrm{Spec}(R)$. If $q \neq p$, then $q \notin \mathcal{P}$, hence R/q is τ-torsion, a contradiction. Therefore $q = p$ and R/p is τ-cocritical. $\square$

Theorem 4.5.6 *Let R be commutative noetherian that is not a domain. Then the following statements are equivalent:*

(i) Every τ-injective module is an essential extension of a τ-complemented τ-injective module.

(ii) τ is the improper torsion theory χ or $\tau = \tau_{\mathcal{P}}$ for some non-empty set $\mathcal{P}$ of minimal prime ideals of R.

Proof. By the hypotheses and Theorem 4.3.8, τ-completely decomposable modules and τ-complemented τ-injective modules coincide.

$(i) \implies (ii)$ Suppose that τ is proper. Then there exists a τ-cocritical module A by Example 1.5.7. By Theorem 3.1.4, $E_\tau(A) \cong E_\tau(R/p)$, where $p \in \mathrm{Spec}(R)$. Since R is not a domain, $p \neq 0$.

We show first that p is a minimal prime ideal. Suppose the contrary. Then there exists $q \in \mathrm{Spec}(R)$ such that $q \subset p$. Since R/p is τ-torsionfree, R/q is τ-torsionfree. Moreover, R/q cannot be τ-cocritical, because otherwise $R/p \cong (R/q)/(p/q)$ would be τ-torsion. On the other hand, $E_\tau(R/q)$ is an essential extension of a τ-complemented τ-injective module B. Since $E_\tau(R/q)$ is uniform, B is uniform. Now by Lemma 4.2.5, B is minimal τ-injective. Furthermore, B is also τ-torsionfree and, consequently, τ-cocritical τ-injective. Since $B \trianglelefteq E_\tau(R/q)$, there exists a non-zero element $b \in B \cap R/q$. We have $\mathrm{Ann}_R B = \mathrm{Ann}_R b = q$ and R/q is τ-cocritical, a contradiction. Therefore p is minimal.

Denote by $\mathcal{P}$ the set of all minimal prime ideals s of R such that $E_\tau(R/s)$ is τ-cocritical. Note that $\mathcal{P}$ is non-empty, since $p \in \mathcal{P}$.

Let us now show that τ-torsion and $\tau_{\mathcal{P}}$-torsion modules coincide.

Let M be a τ-torsion module. By the hypotheses on R, every torsion theory is stable, hence we have

$$E_\tau(M) = E(M) = \bigoplus_{i \in I} E(R/p_i)\,,$$

where each $p_i \notin \mathcal{P}$ is a (non-zero) prime ideal of R. Then $R/p_i \in \mathcal{A}_{\mathcal{P}}$, whence it follows immediately that M is $\tau_{\mathcal{P}}$-torsion. Hence every τ-torsion module is $\tau_{\mathcal{P}}$-torsion.

Now let $N \in \mathcal{A}_{\mathcal{P}}$. Then $N \cong U/V$, where U and V are ideals of R containing a (non-zero) prime ideal $q' \notin \mathcal{P}$. Suppose that R/q' is τ-torsionfree. By hypothesis, $E_\tau(R/q')$ is an essential extension of a τ-complemented τ-injective module C. Repeating the above arguments, it follows that R/q' is τ-cocritical, which contradicts the choice of q'. Then R/q' is τ-torsion. Hence R/V and, consequently, $N \cong U/V$ is τ-torsion. Thus every $\tau_{\mathcal{P}}$-torsion module is τ-torsion. Therefore $\tau = \tau_{\mathcal{P}}$.

$(ii) \Longrightarrow (i)$ Suppose first that $\tau = \chi$, i.e. every module is τ-torsion. Then every module is τ-complemented and the result follows.

Suppose now that $\tau = \tau_{\mathcal{P}}$, for some non-empty set $\mathcal{P}$ of minimal prime ideals of R. Let A be a τ-injective module. By the stability of τ and by Proposition 2.1.8, we may write $A = t(A) \oplus C$, where C is τ-torsionfree τ-injective. Clearly, $t(A)$ is τ-complemented τ-injective, hence τ-completely decomposable. By the hypotheses on R, we have $E(C) = \bigoplus_{i\in I} E(R/p_i)$, where each p_i is a (non-zero) prime ideal of R. Then $E(R/p_i)$ is τ-torsionfree, hence $p_i \in \mathcal{P}$ for every $i \in I$. Now let $i \in I$. By Proposition 4.5.5, R/p_i is τ-cocritical, whence $E_\tau(R/p_i)$ is minimal τ-injective. Thus $\bigoplus_{i\in I} E_\tau(R/p_i)$ is τ-completely decomposable.

We have $C \cap E_\tau(R/p_i) \neq 0$. Then

$$E_\tau(R/p_i)/(C \cap E_\tau(R/p_i)$$

is both τ-torsion, because $E_\tau(R/p_i)$ is τ-cocritical, and τ-torsionfree, because

$$E_\tau(R/p_i)/(C \cap E_\tau(R/p_i)) \cong (C + E_\tau(R/p_i))/C \subseteq E(C)/C\,.$$

Hence $E_\tau(R/p_i) \subseteq C$ and thus $\bigoplus_{i\in I} E_\tau(R/p_i) \trianglelefteq C$. Now A is an essential extension of a τ-completely decomposable module, namely $t(A) \oplus (\bigoplus_{i\in I} E_\tau(R/p_i))$. $\square$

Remark. The hypothesis on R not to be a domain is essential in Theorem 4.5.6. Indeed, suppose that R is a domain and consider $\tau = \tau_{\mathcal{P}}$ for some non-empty set $\mathcal{P}$ of minimal prime ideals of R. Clearly $E_\tau(R)$ is τ-torsionfree and $\mathrm{Ann}_R x = 0$ for every $0 \neq x \in E_\tau(R)$. Suppose that $E_\tau(R)$ is an essential extension of a τ-completely decomposable (or equivalently τ-complemented τ-injective) module A. Then, since A is uniform, A has to be minimal τ-injective, hence τ-cocritical. It follows that $\mathrm{Ann}_R A = p$, where $p \in \mathcal{P}$, because R/p is τ-cocritical by Proposition 4.5.5. Hence $\mathrm{Ann}_R A \neq 0$, a contradiction. Thus $E_\tau(R)$ is a τ-injective module that is not an essential extension of any τ-completely decomposable module.

Corollary 4.5.7 *Let R be commutative noetherian that is not a domain. Consider the set $\mathcal{P}$ of all minimal prime ideals of R and put $\tau = \tau_{\mathcal{P}}$. Then every τ-injective module A is isomorphic to an essential extension of*

$$(\bigoplus_{i\in I} E_\tau(R/p_i)) \oplus (\bigoplus_{j\in J} E(R/q_j)),$$

where each $p_i, q_j \in \mathrm{Spec}(R)$. Moreover, each $p_i \in \mathcal{P}$ and each $q_j \notin \mathcal{P}$.

Proof. By Theorem 4.5.6, A is an essential extension of a τ-complemented τ-injective module B. By Theorem 4.3.8, B is τ-completely decomposable, i.e. B is a direct sum of minimal τ-injective modules. By Theorem 3.1.3, a τ-torsion minimal τ-injective module A is of the form $A = E_\tau(B)$, where $B \in \mathcal{A}_{\mathcal{P}}$ and B is uniform. Since R is commutative noetherian, τ is stable and thus $E_\tau(B) = E(B)$ by Proposition 2.1.9. Since B is uniform, $A = E(B)$ is isomorphic to $E(R/p)$ for some $p \in \mathrm{Spec}(R)$. By Theorem 3.1.4, a τ-torsionfree minimal τ-injective module is isomorphic to $E_\tau(R/q)$ for some $q \in \mathrm{Spec}(R)$. Therefore every τ-injective module A is isomorphic to an essential extension of

$$(\bigoplus_{i\in I} E_\tau(R/p_i)) \oplus (\bigoplus_{j\in J} E(R/q_j)),$$

where each p_i and each $q_j \in \mathrm{Spec}(R)$. Moreover, each $E_\tau(R/p_i))$ is τ-cocritical, hence each R/p_i is τ-cocritical. Then by Proposition 4.5.5 and by Lemma 1.4.7, each $p_i \in \mathcal{P}$ and each $q_j \notin \mathcal{P}$. $\square$

References: J.L. Bueso, P. Jara, B. Torrecillas [15], [16], S. Crivei [33], [34], K. Masaike, T. Horigome [74], S. Mohamed, B. Müller, S. Singh [78], S. Mohamed, S. Singh [79], P.F. Smith, A.M. Viola-Prioli, J.E. Viola-Prioli [104], [105], A.M. Viola-Prioli, J.E. Viola-Prioli [114], J. Zelmanowitz [119].

Notes on Chapter 4

The name of τ-completely decomposable module was coined by K. Masaike and T. Horigome (1980), overtaking the terminology of completely decomposable module, used by C. Faith and E. Walker [43] for a direct sum of indecomposable injective modules. They characterized the rings for which every τ-torsion τ-injective module is τ-completely decomposable and first studied direct summands and extensions of τ-completely decomposable modules. J.L. Bueso, P. Jara and B. Torrecillas (1985) characterized the rings for which every τ-torsionfree τ-injective module is τ-completely decomposable and refined the result of K. Masaike and T. Horigome (1980) on when every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module. S. Mohamed and S. Singh (1981) established a decomposition theorem of the τ-injective hull of a finitely generated module into a direct sum of uniform submodules. The author's contribution is the use of τ-complemented modules in order to give a solution in several cases to a generalized Matlis' problem on the τ-complete decomposability of direct summands of τ-completely decomposable modules, and to determine torsion theories such that every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.

Chapter 5

τ-quasi-injective modules

τ-quasi-injective modules generalize quasi-injective modules in the relative case of a torsion theory. Several properties similar to the case of quasi-injective modules can be established, including the existence and uniqueness up to an isomorphism of the τ-quasi-injective hull. In the final part, we will use some of their properties to discuss relationships between certain conditions on τ-injectivity and τ-quasi-injectivity for modules in the context of τ-natural classes, that is, classes of modules closed under isomorphic copies, submodules, direct sums and τ-injective hulls.

5.1 General properties

Definition 5.1.1 A module A is said to be *τ-quasi-injective* if whenever B is a τ-dense submodule of A, every homomorphism $B \to A$ extends to an endomorphism of A.

Lemma 5.1.2 (i) *A module A is τ-quasi-injective if and only if* $\mathrm{Ext}^1_R(A/B, A) = 0$ *for every τ-dense submodule B of A.*

(ii) *Every quasi-injective module and every τ-injective module is τ-quasi-injective.*

(*iii*) *R is τ-quasi-injective if and only if it is τ-injective.*

(*iv*) *Every τ-torsion τ-quasi-injective module is quasi-injective.*

Proof. Immediate. □

Now let us give a characterization of τ-quasi-injective modules similar to the well-known characterization of quasi-injective modules, that are fully invariant submodules of their injective hulls.

Theorem 5.1.3 *Let A be a module. Then A is τ-quasi-injective if and only if A is a fully invariant submodule of $E_\tau(A)$.*

Proof. We may suppose that $A \neq 0$. Denote $K = \mathrm{End}_R(E_\tau(A))$.

Assume first that A is τ-quasi-injective and let $f \in K$. Denote $g = f|_A$ and $B = g^{-1}(A)$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "u"'] & A \arrow[r, "j"] \arrow[dl, dashed, "v"'] \arrow[ddl, "g"] & E_\tau(A) \arrow[r] \arrow[ddll, "f"] & E_\tau(A)/A \\
 & A \arrow[d, "k"'] & & & \\
 & E_\tau(A) & & &
\end{tikzcd}$$

where i, j, k are inclusion homomorphisms and $u : B \to A$ is defined by $u(b) = g(b)$ for every $b \in B$.

We will show that B is a τ-dense submodule of A. The homomorphism g induces a monomorphism $w : A/B \to E_\tau(A)/A$, defined by $w(a + B) = g(a) + A$ for every $a \in A$. Then A/B is τ-torsion, because $E_\tau(A)/A$ is τ-torsion. Hence B is a τ-dense submodule of A.

Since A is τ-quasi-injective, there exists $v \in \mathrm{End}_R(A)$ such that $vi = u$. By the τ-injectivity of $E_\tau(A)$, there exists $h \in K$ such that $hj = kv$. Thus $h(A) \subseteq A$. Assume $(h - f)(A) \neq 0$. Then $(h - f)(A) \cap A \neq 0$ and there exist $x, y \in A$, $y \neq 0$ such that $y = (h - f)(x)$. It follows that

$(h-f)(x) = v(x) - f(x) = y$, hence $f(x) = v(x) - y \in A$. Then $x \in B$ and $y = v(x) - f(x) = 0$, contradiction. Therefore $(h-f)(A) = 0$, i.e. $f(A) = h(A) \subseteq A$. Hence A is a fully invariant submodule of $E_\tau(A)$.

Suppose now that A is a fully invariant submodule of $E_\tau(A)$. Let B be a τ-dense submodule of A and let $g : B \to A$ be a homomorphism. The module $E_\tau(A)/B$ is τ-torsion because $E_\tau(A)/A$ and A/B are τ-torsion. Then g extends to $h \in K$ because $E_\tau(A)$ is τ-injective. Since $h(A) \subseteq A$, g extends to an endomorphism of A. Therefore A is τ-quasi-injective. $\square$

The proof of the following corollary is immediate by Theorem 5.1.3. It might be also obtained as a particular case of a forthcoming theorem.

Corollary 5.1.4 *If every τ-injective module is injective, then every τ-quasi-injective module is quasi-injective.*

Another relationship between τ-injective and τ-quasi-injective modules can be given.

Proposition 5.1.5 *The following statements are equivalent:*

(i) Every module is τ-injective.

(ii) Every module is τ-quasi-injective.

Proof. $(i) \Longrightarrow (ii)$ Obvious.

$(ii) \Longrightarrow (i)$ Let A be a module. Also, let I be a τ-dense left ideal of R and let $f : I \to A$ be a homomorphism. Then $A \oplus I$ is τ-dense in $A \oplus R$. Thus the homomorphism $g : A \oplus I \to A \oplus R$ defined by $g(a, r) = (f(r), 0)$ can be extended to a homomorphism $h : A \oplus R \to A$ by the τ-quasi-injectivity of A. Now $h|_R$ extends f and thus A is τ-injective. $\square$

By Theorem 5.1.3 and in a similar way as for quasi-injective modules, one may prove immediately the following proposition.

Proposition 5.1.6 *The class of τ-quasi-injective modules is closed under direct summands and finite direct sums of copies.*

Proposition 5.1.7 *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence of modules and let* $h : B \to A \oplus D$ *be a monomorphism, where* D *is a module. If* $(hf)(A)$ *is a* τ*-dense submodule of* $A \oplus D$ *and* $A \oplus D$ *is* τ*-quasi-injective, then the above sequence splits.*

Proof. Let $\alpha : A \to A \oplus D$ be the canonical injection. Since $(A \oplus D)/(hf)(A)$ is τ-torsion and $A \oplus D$ is τ-quasi-injective, there exists an endomorphism $\theta : A \oplus D \to A \oplus D$ such that $\theta hf = \alpha$. Let $p : A \oplus D \to A$ be the canonical projection and define $\gamma : B \to A$ by $\gamma = p\theta h$. Then clearly $\gamma f = 1_A$, hence the above sequence splits. □

Corollary 5.1.8 *Let* $f : A \to B$ *be a monomorphism of modules. If* B *is* τ*-torsion and* $A \oplus B$ *is* τ*-quasi-injective, then* $A \oplus B$ *is* τ*-injective if and only if* B *is* τ*-injective.*

Proof. The "only if" part is obvious. For the "if" part, in the Proposition 5.1.7, let $h : B \to A \oplus B$ be the canonical injection. Since B is τ-torsion, A and $B/f(A)$ are τ-torsion. Hence $(A \oplus B)/(hf)(A) \cong (A \oplus B)/f(A)$ is τ-torsion. By Proposition 5.1.7, $f(A)$ is a direct summand of B, hence A is τ-injective. Therefore $A \oplus B$ is τ-injective. □

Proposition 5.1.9 *A module* A *is* τ*-injective if and only if* $A \oplus E_\tau(A)$ *is* τ*-quasi-injective.*

Proof. The "only if" part is obvious. Suppose now that $A \oplus E_\tau(A)$ is τ-quasi-injective. Also assume that A is not τ-injective. Consider the exact sequence of modules:

$$0 \longrightarrow A \xrightarrow{i} E_\tau(A) \xrightarrow{p} E_\tau(A)/A \longrightarrow 0 \qquad (1)$$

where i is the inclusion homomorphism and p is the natural homomorphism. Let $\alpha_1 : A \to A \oplus E_\tau(A)$ be the canonical injection, $\beta : A \to A \oplus A$ defined by $\beta(a) = (0, a)$ for every $a \in A$, $j = 1_A \oplus i : A \oplus A \to A \oplus E_\tau(A)$ and

$\sigma : A \oplus A \to A \oplus E_\tau(A)$ defined by $\sigma(a_1, a_2) = (a_2, a_1)$. Now consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\alpha_1"'] & A \oplus A \arrow[r, "j"] \arrow[dl, "\sigma"] & A \oplus E_\tau(A) \arrow[dll, dashed, "\gamma"] \\
A \oplus E_\tau(A) & &
\end{tikzcd}$$

Since $A \oplus A$ is τ-dense in $E_\tau(A \oplus A) = E_\tau(A) \oplus E_\tau(A)$, it follows that $A \oplus A$ is τ-dense in $A \oplus E_\tau(A)$. But $A \oplus E_\tau(A)$ is τ-quasi-injective, hence there exists a homomorphism $\gamma : A \oplus E_\tau(A) \to A \oplus E_\tau(A)$ such that $\gamma j = \sigma$. Then $\gamma j \beta = \sigma\beta = \alpha_1$. Let $\alpha_2 : E_\tau(A) \to A \oplus E_\tau(A)$ be the canonical injection and let $\pi : A \oplus E_\tau(A) \to A$ be the canonical projection. Note that $\alpha_2 i = j\beta$. Now take $\delta = \pi\gamma\alpha_2$. Then we have $\delta i = \pi\gamma\alpha_2 i = \pi\gamma j\beta = \pi\alpha_1 = 1_A$, hence the sequence (1) splits. But this contradicts the fact that $A \trianglelefteq E_\tau(A)$. Therefore A is τ-injective. $\square$

Lemma 5.1.10 *Let A be a τ-quasi-injective module. If $(E_\tau(A))^{(I)}$ is τ-injective, then $A^{(I)}$ is τ-quasi-injective for every set I.*

Proof. It is known that if B is a fully invariant submodule of a module A, then $B^{(I)}$ is a fully invariant submodule of $A^{(I)}$ for every set I. Now apply Theorem 5.1.3. $\square$

We have seen that every quasi-injective module is τ-quasi-injective. The converse does not hold, as the following example shows.

Example 5.1.11 Let R be a unique factorization domain such that every maximal ideal of R is not principal. Then by Proposition 2.4.8 we know that R is a non-injective τ_D-injective R-module. Hence R is τ_D-quasi-injective. Since R is quasi-injective if and only if R is injective, it follows that R is not quasi-injective.

In what follows let us discuss some further properties on τ-quasi-injectivity for some particular torsion theories, namely τ_n.

Clearly, if R is left τ-cocritical, then every τ-quasi-injective module is quasi-injective. For the torsion theories τ_n we will see a couple of cases when quasi-injectivity and τ_n-quasi-injectivity are the same.

Proposition 5.1.12 *Let R be commutative. Then every τ_n-quasi-injective module is quasi-injective provided R has one of the following properties:*

(i) R is semiartinian.

(ii) R is a noetherian domain with $\dim R \leq n+1$.

Proof. If (i) holds, clearly every τ_n-injective module is injective.

If (ii) holds, then by Proposition 2.4.4, every τ_n-injective module is injective. Now the result follows again by Corollary 5.1.4. $\square$

We will end this section with a few results on quasi-injective modules with respect to the Dickson torsion theory.

In the sequel, starting with a τ_D-quasi-injective module that is not τ_D-injective, we will construct some other such modules. We need here the Loewy series of a module (see Example 2.5.2).

Proposition 5.1.13 *Let A be a τ_D-quasi-injective module which is not τ_D-injective and denote $M = E_{\tau_D}(A)$. Consider the Loewy series of M/A*

$$0 = S_0(M/A) \subseteq S_1(M/A) \subseteq \cdots \subseteq S_\alpha(M/A) \subseteq S_{\alpha+1}(M/A) \subseteq \ldots$$

where, for each ordinal $\alpha \geq 0$,

$$S_{\alpha+1}(M/A)/S_\alpha(M/A) = \mathrm{Soc}((M/A)/S_\alpha(M/A))$$

and if α is a limit ordinal, then

$$S_\alpha(M/A) = \bigcup_{0 \leq \beta < \alpha} S_\beta(M/A)\,.$$

For every ordinal $\alpha \geq 0$, let $M_\alpha \leq M$ be such that

$$S_\alpha(M/A) = M_\alpha/A\,.$$

Then every non-zero proper submodule M_α of M is τ_D-quasi-injective, but not τ_D-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that M_α is a proper submodule of M and let $f \in \mathrm{End}_R(M)$. Since A is τ_D-quasi-injective, $f(A) \subseteq A$ by Theorem 5.1.3. Then f induces an endomorphism $f^* \in \mathrm{End}_R(M/A)$. Since $M_\alpha/A = S_\alpha(M/A)$ is fully invariant [40, 3.11, p.25], $f^*(M_\alpha/A) \subseteq M_\alpha/A$, therefore $f(M_\alpha) \subseteq M_\alpha$, i.e. M_α is τ_D-quasi-injective. On the other hand, M_α is a proper submodule of $E_{\tau_D}(A) = M$, hence M_α is not τ_D-injective. □

Proposition 5.1.14 *Let S be a simple module which is not τ_D-injective and denote $M = E_{\tau_D}(S)$. Consider the Loewy series of M*

$$0 = S_0(M) \subseteq S_1(M) \subseteq \cdots \subseteq S_\alpha(M) \subseteq S_{\alpha+1}(M) \subseteq \ldots$$

where, for each ordinal $\alpha \geq 0$,

$$S_{\alpha+1}(M)/S_\alpha(M) = \mathrm{Soc}(M/S_\alpha(M))$$

and if α is a limit ordinal, then

$$S_\alpha(M) = \bigcup_{0 \leq \beta < \alpha} S_\beta(M)\,.$$

Then every non-zero proper submodule $S_\alpha(M)$ of M is quasi-injective, but not τ_D-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that $S_\alpha(M)$ is a proper submodule of M. Then $S_\alpha(M)$ is a fully invariant submodule of M [40, 3.11, p.25], therefore τ_D-quasi-injective by Theorem 5.1.3. Also $S_\alpha(M)$ is semiartinian as a submodule of the semiartinian module M. It follows that $S_\alpha(M)$ is quasi-injective. Since $M = E_{\tau_D}(S)$ is minimal τ_D-injective, $S_\alpha(M)$ is not τ_D-injective. □

5.2 τ-quasi-injective hulls

In this section, we introduce the notion of τ-quasi-injective hull and show that every module has such a hull, unique up to an isomorphism.

Definition 5.2.1 A *τ-quasi-injective hull* of a module A is defined as a τ-quasi-injective module Q such that A is a τ-dense essential submodule of Q and is denoted by $Q_\tau(A)$.

Throughout this section, for every module A, we denote $S = \mathrm{End}_R(E_\tau(A))$ and

$$SA = \{\sum_{i=1}^{n} f_i(a_i) \mid f_i \in S, a_i \in A, i \in \{1,\dots,n\}, n \in \mathbb{N}^*\}.$$

Proposition 5.2.2 *Let A be a module. Then:*

(i) A is τ-quasi-injective if and only if $SA = A$.

(ii) SA is a τ-quasi-injective module.

(iii) SA is the intersection of all τ-quasi-injective submodules of $E_\tau(A)$ containing A.

Proof. (i) Suppose first that A is τ-quasi-injective. Then by Theorem 5.1.3, for every $f \in S$, we have $f(A) \subseteq A$. Then $SA \subseteq A$ and consequently $SA = A$.

Conversely, suppose that $SA = A$. Let B a τ-dense submodule of A and let $f : B \to A$ a homomorphism. Then f extends to a homomorphism $g : A \to E_\tau(A)$ and g extends to an endomorphism $h \in S$. Since $SA = A$, we have $h(A) \subseteq A$, so that $h|_A : A \to A$ extends f. Thus A is τ-quasi-injective.

(ii) We have $A \subseteq SA \subseteq E_\tau(A)$, hence A is a τ-dense essential submodule of SA. Then $E_\tau(A) = E_\tau(SA)$, whence

$$S = \mathrm{End}_R(E_\tau(A)) = \mathrm{End}_R(E_\tau(SA)).$$

It follows that $S(SA) = SA$. By (i), SA is τ-quasi-injective.

(iii) Denote by $\mathcal{B}$ the set of all τ-quasi-injective submodules of $E_\tau(A)$ containing A. For every $B \in \mathcal{B}$, we have $A \subseteq B \subseteq E_\tau(A)$, whence it follows that A is a τ-dense essential submodule of B. But then $E_\tau(A) = E_\tau(B)$, so that $S = \mathrm{End}_R(E_\tau(A)) = \mathrm{End}_R(E_\tau(B))$.

Since $A \subseteq \bigcap_{B\in\mathcal{B}} B$, we have $SA \subseteq S(\bigcap_{B\in\mathcal{B}} B)$. Let $f(b)$ be a generator of $S(\bigcap_{B\in\mathcal{B}} B)$ for some $f \in S$ and $b \in \bigcap_{B\in\mathcal{B}} B$. Then $f(b) \in SB$ for every $B \in \mathcal{B}$. But B is τ-quasi-injective, hence by (ii) it follows that $f(b) \in \bigcap_{B\in\mathcal{B}} B$. Thus $S(\bigcap_{B\in\mathcal{B}} B) \subseteq \bigcap_{B\in\mathcal{B}} B$, whence $SA \subseteq \bigcap_{B\in\mathcal{B}} B$.

For the converse inclusion, we know by (ii) that SA is τ-quasi-injective, hence $SA \in \mathcal{B}$. It follows that $\bigcap_{B\in\mathcal{B}} B \subseteq SA$. $\square$

Lemma 5.2.3 *Let A be a module, $i : A \to Q_\tau(A)$ be the inclusion homomorphism and $f : A \to Q$ be a τ-quasi-injective τ-dense extension of A. Then there exists a monomorphism $\alpha : Q_\tau(A) \to Q$ such that $\alpha i = f$.*

Proof. There exists a monomorphism $g : E_\tau(A) \to E_\tau(Q)$ such that

$$f(A) \subseteq g(Q_\tau(A)) \subseteq g(E_\tau(A)) \subseteq E_\tau(Q)\,.$$

Denote $T = \mathrm{End}_R(g(E_\tau(A)))$ and $U = \mathrm{End}_R(E_\tau(Q))$. Since $g(E_\tau(A))$ is a τ-injective hull of the τ-quasi-injective module $g(Q_\tau(A))$, we have $Tg(Q_\tau(A)) \subseteq g(Q_\tau(A))$ (see the notation preceding Proposition 5.2.2). Since $f(A) \subseteq Q \subseteq E_\tau(Q)$, $f(A)$ is τ-dense in Q. But $f(A) \subseteq g(E_\tau(A))$, whence it follows that $g(E_\tau(A))$ is τ-dense in $E_\tau(Q)$. Hence every homomorphism $h \in T$ extends to a homomorphism $h' \in U$. Since $UQ \subseteq Q$, we have $h(Q) = h'(Q) \subseteq Q$, whence $TQ \subseteq Q$. Then

$$T(g(Q_\tau(A)) \cap Q) \subseteq g(Q_\tau(A)) \cap Q\,.$$

Since

$$f(A) \subseteq g(Q_\tau(A)) \cap Q \subseteq g(E_\tau(A))\,,$$

$g(E_\tau(A))$ is a τ-injective hull of $g(Q_\tau(A)) \cap Q$, so that $g(Q_\tau(A)) \cap Q$ is τ-quasi-injective. It follows that

$$A \subseteq g^{-1}(g(Q_\tau(A)) \cap Q) \subseteq E_\tau(A)$$

and $g^{-1}(g(Q_\tau(A))$ is τ-quasi-injective. Also,

$$g^{-1}(g(Q_\tau(A) \cap Q) \subseteq g^{-1}(g(Q_\tau(A))) = Q_\tau(A)\,,$$

whence we have

$$g^{-1}(g(Q_\tau(A) \cap Q) = Q_\tau(A)\,,$$

because $Q_\tau(A)$ is the least τ-quasi-injective submodule of $E_\tau(A)$ containing A. Then $g(Q_\tau(A)) \cap Q = g(Q_\tau(A))$, that is, $g(Q_\tau(A)) \subseteq Q$. It follows that $\alpha = g|_{Q_\tau(A)} : Q_\tau(A) \to Q$ is a monomorphism such that $\alpha i = f$. $\square$

We have seen that every module has a τ-injective hull, unique up to an isomorphism. Now we give the following result.

Theorem 5.2.4 *Every module A has a τ-quasi-injective hull unique up to an isomorphism.*

Moreover, $Q_\tau(A) = SA$, that is, the intersection of all τ-quasi-injective submodules of $E_\tau(A)$ containing A.

Proof. We have $A \subseteq SA \subseteq E_\tau(A)$, hence A is a τ-dense essential submodule of SA. By Proposition 5.2.2, SA is τ-quasi-injective. Hence SA is a τ-quasi-injective hull of A.

Denote $Q_\tau(A) = SA$ and let $i : A \to Q_\tau(A)$ be the inclusion homomorphism. Suppose that there exists another τ-quasi-injective hull of A, say Q. Let $j : A \to Q$ be the inclusion homomorphism. Then by Lemma 5.2.3, there exist homomorphisms $\alpha : Q_\tau(A) \to Q$ and $\beta : Q \to Q_\tau(A)$ such that $\alpha i = j$ and $\beta j = i$.

We claim that $\beta\alpha = 1_{Q_\tau(A)}$. If not, since $A \trianglelefteq Q_\tau(A)$, we have

$$(\beta\alpha - 1_{Q_\tau(A)})(Q_\tau(A)) \cap A \neq 0\,,$$

say it contains an element $a \neq 0$. Since $\beta\alpha = \beta\alpha i = \beta j = i$ and $\beta\alpha - 1_{Q_\tau(A)}$ is injective, it follows that $a = 0$, a contradiction. Hence $\beta\alpha = 1_{Q_\tau(A)}$. Similarly, $\alpha\beta = 1_Q$. Therefore $Q_\tau(A)$ is unique up to an isomorphism. $\square$

5.3 Direct sums and τ-natural classes of modules

Throughout this section, we will denote by $\mathcal{K}$ a class of modules closed under isomorphic copies.

Recall that a class $\mathcal{K}$ is called a *natural class* if $\mathcal{K}$ is closed under submodules, direct sums and injective hulls. For instance, R-Mod, any hereditary torsionfree class of modules and any stable hereditary torsion class of modules are examples of natural classes.

In the context of torsion theories, we introduce the following definition.

Definition 5.3.1 *The class $\mathcal{K}$ is called a τ-natural class if $\mathcal{K}$ is closed under submodules, direct sums and τ-injective hulls.*

Remark. Clearly, every natural class is a τ-natural class. If τ is the improper torsion theory, then every τ-natural class becomes a natural class.

Example 5.3.2 (1) R-Mod, any hereditary torsionfree class of modules and any stable hereditary torsion class of modules are natural classes, hence τ-natural classes.

(2) Let σ be a hereditary torsion theory such that $\tau \leq \sigma$. Then the class of all σ-torsion modules is a τ-natural class, that is a natural class if and only if σ is stable.

Following [90], denote by $H_{\mathcal{K}}(R)$ the set of left ideals I of R such that $R/I \in \mathcal{K}$ and consider the following generalized conditions, where $\mathcal{K}$ is a natural class or a τ-natural class:

$\mathcal{C}_1(\mathcal{K})$: Every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.

$\mathcal{C}_2(\mathcal{K})$: Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates.

$\mathcal{C}_3(\mathcal{K})$: $H_{\mathcal{K}}(R)$ has ACC on τ-dense left ideals.

$\mathcal{C}_4(\mathcal{K})$: Every direct sum of τ-quasi-injective modules in $\mathcal{K}$ is τ-quasi-injective.

$\mathcal{C}_5(\mathcal{K})$: Every τ-quasi-injective module in $\mathcal{K}$ is τ-injective.

$\mathcal{C}_6(\mathcal{K})$: Every τ-injective module in $\mathcal{K}$ is $\sum$-τ-injective.

$\mathcal{C}_7(\mathcal{K})$: Every τ-quasi-injective module in $\mathcal{K}$ is $\sum$-τ-quasi-injective.

$\mathcal{C}_8(\mathcal{K})$: Every τ-injective module in $\mathcal{K}$ is injective.

$\mathcal{C}_9(\mathcal{K})$: Every τ-quasi-injective module in $\mathcal{K}$ is quasi-injective.

They have been extensively studied when τ is the improper torsion theory on R-Mod and $\mathcal{K}$ is the σ-torsionfree class for a hereditary torsion theory σ or the σ-torsion class for a stable hereditary torsion theory σ.

We intend to establish here certain connections between the above conditions for an arbitrary hereditary torsion theory τ.

Theorem 5.3.3 *Let $\mathcal{K}$ be a τ-natural class. Then $\mathcal{C}_2(\mathcal{K}) \Longrightarrow \mathcal{C}_3(\mathcal{K})$.*

Proof. Let $I_1 \subseteq I_2 \subseteq \dots$ an ascending chain of τ-dense left ideals in $H_{\mathcal{K}}(R)$. Then each $R/I_j \in \mathcal{K}$. Since $\mathcal{K}$ is closed under τ-dense submodules, each $I_{j+1}/I_j \in \mathcal{K}$. By hypothesis, the above chain terminates, hence $\mathcal{C}_3(\mathcal{K})$ holds. □

Theorem 5.3.4 *Let $\mathcal{K}$ be a τ-natural class. Then $\mathcal{C}_1(\mathcal{K}) \Longrightarrow \mathcal{C}_2(\mathcal{K})$.*

Proof. Suppose that $I_1 \subset I_2 \subset \dots$ is a strictly ascending chain of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$. By hypothesis,

$$E = \bigoplus_j E_\tau(I_{j+1}/I_j) \in \mathcal{K}$$

is τ-injective. Let $I = \bigcup_{j=1}^{\infty} I_j$, let $p_j : I_{j+1} \to I_{j+1}/I_j$ be the natural homomorphism and let $\alpha_j : I_{j+1}/I_j \to E_\tau(I_{j+1}/I_j)$ be the inclusion homomorphism for each j. By the τ-injectivity of $E_\tau(I_{j+1}/I_j)$, it follows that there exists a homomorphism $\beta_j : R \to E_\tau(I_{j+1}/I_j)$ that extends $\alpha_j p_j$. Hence we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_{j+1} \arrow[r] \arrow[d, "p_j"'] & R \arrow[d, dashed, "\beta_j"] \\ & I_{j+1}/I_j \arrow[r, "\alpha_j"'] & E_\tau(I_{j+1}/I_j) \end{tikzcd}$$

We may define

$$f : I \to E\,, \quad f(x) = (\beta_j(x))_j\,.$$

It is easy to check that f is a well-defined homomorphism. Since I is τ-dense and E is τ-injective, there exists a homomorphism g that extends f. Since $g(1) \subseteq \sum_{j=1}^{n} E_\tau(I_{j+1}/I_j)$ for some n, we have

$$f(I) = g(I) \subseteq \sum_{j=1}^{n} E_\tau(I_{j+1}/I_j)\,.$$

It follows that $\beta_j(x) = 0$ for every $x \in I$ and every $j > n$. If $x \in I_{n+1}$, then $0 = \beta_{n+1}(x) = x + I_n$. Hence $I_{n+1} = I_n$, a contradiction. Therefore $\mathcal{C}_2(\mathcal{K})$ holds. $\square$

Remark. Note that in the proof of Theorem 5.3.4 each I_{j+1}/I_j is τ-torsion. Hence we used only the fact that every direct sum of τ-torsion τ-injective modules in $\mathcal{K}$ is τ-injective.

Proposition 5.3.5 *Let $\mathcal{K}$ be a τ-natural class. Suppose that every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R whose union is τ-dense in R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates. Then $\mathcal{C}_1(\mathcal{K})$ holds.*

Proof. It is sufficient to prove that every countable direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective (see Theorem 2.3.8). Let $A = \bigoplus_{i=1}^{\infty} A_i$ be a direct sum of τ-injective modules in $\mathcal{K}$. Also let I be a τ-dense left ideal of R and $f : I \to A$ a homomorphism. For each n denote

$$I_n = \{x \in I \mid f(x) \in \bigoplus_{i=1}^{n} A_i\}\,.$$

Clearly $I_1 \subseteq I_2 \subseteq \dots$ and $\bigcup_{j=1}^{\infty} I_j = I$. We may consider the monomorphism

$$\alpha_n : I_{n+1}/I_n \to (\bigoplus_{i=1}^{n+1} A_i)/(\bigoplus_{i=1}^{n} A_i)$$

defined by

$$\alpha_n(x + I_n) = f(x) + (\bigoplus_{i=1}^{n} A_i)\,.$$

Since the codomain of α_n is isomorphic to $A_{n+1} \in \mathcal{K}$, we have $I_{n+1}/I_n \in \mathcal{K}$. By hypothesis, there exists k such that $I_{k+j} = I_k$ for each j. Then $f(I) \subseteq \bigoplus_{i=1}^{k} A_i$. Since $\bigoplus_{i=1}^{k} A_i$ is τ-injective, there exists a homomorphism $g : R \to \bigoplus_{i=1}^{k} A_i \subseteq A$ that extends f. Then A is τ-injective and thus $\mathcal{C}_1(\mathcal{K})$ holds. $\square$

Corollary 5.3.6 *Let $\mathcal{K}$ be a τ-natural class. If τ is noetherian, then $\mathcal{C}_1(\mathcal{K}) \Longleftrightarrow \mathcal{C}_2(\mathcal{K})$.*

Proof. The direct implication follows by Theorem 5.3.4. For the converse, let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of left ideals of R whose union is τ-dense in R such that each $I_{j+1}/I_j \in \mathcal{K}$. Since τ is noetherian, there exists k such that I_k is τ-dense in R. Then I_n is τ-dense in R for every $n \geq k$. By $\mathcal{C}_2(\mathcal{K})$, the chain $I_k \subseteq I_{k+1} \subseteq \dots$ terminates, hence the chain $I_1 \subseteq I_2 \subseteq \dots I_k \subseteq I_{k+1} \subseteq \dots$ terminates. Now use Proposition 5.3.5. $\square$

Theorem 5.3.7 *Let $\mathcal{K}$ be a τ-natural class. Then $\mathcal{C}_4(\mathcal{K}) \Longleftrightarrow \mathcal{C}_1(\mathcal{K})+\mathcal{C}_5(\mathcal{K})$.*

Proof. Suppose first that $\mathcal{C}_4(\mathcal{K})$ holds. Let $A \in \mathcal{K}$ be a τ-quasi-injective module. Since $\mathcal{K}$ is a τ-natural class, $E_\tau(A) \in \mathcal{K}$. By hypothesis, $A \oplus E_\tau(A)$ is τ-quasi-injective. Now by Proposition 5.1.9, A is τ-injective. Therefore $\mathcal{C}_5(\mathcal{K})$ holds. Now let $A = \bigoplus_{i \in I} A_i$, where each A_i is a τ-injective module in $\mathcal{K}$. Hence each A_i is a τ-quasi-injective module in $\mathcal{K}$. By $\mathcal{C}_4(\mathcal{K})$, A is a τ-quasi-injective module in $\mathcal{K}$. Since $\mathcal{C}_5(\mathcal{K})$ holds as well, A is a τ-injective module in $\mathcal{K}$. Therefore $\mathcal{C}_1(\mathcal{K})$ holds.

Conversely, suppose that $\mathcal{C}_1(\mathcal{K})$ and $\mathcal{C}_5(\mathcal{K})$ hold. Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a τ-quasi-injective module in $\mathcal{K}$. By $\mathcal{C}_5(\mathcal{K})$, each A_i is a τ-injective module in $\mathcal{K}$. Now by $\mathcal{C}_1(\mathcal{K})$, A is τ-injective module in $\mathcal{K}$, hence A is τ-quasi-injective module in $\mathcal{K}$. Therefore $\mathcal{C}_4(\mathcal{K})$ holds. $\square$

We need the following lemma.

Lemma 5.3.8 [90, Lemma 7] *Let A be a module and $a_1, \dots, a_n \in A$. If all homomorphic images of $Ra_1, \dots, Ra_n$ which are submodules of $E(A)$ have finite uniform dimension, then $E(Ra_1) + \cdots + E(Ra_n)$ has finite uniform dimension.*

Recall the following definition, motivated by torsion theory context. For a natural class $\mathcal{K}$, a non-zero module A is said to be $\mathcal{K}$-*cocritical* if $A \in \mathcal{K}$ and for every non-zero proper submodule B of A, $A/B \notin \mathcal{K}$.

For instance, if $\mathcal{K}$ is a torsionfree class for a hereditary torsion theory τ, then a $\mathcal{K}$-cocritical module means a τ-cocritical module. We will consider the same definition for a τ-natural class as well.

The next two lemmas on $\mathcal{K}$-cocritical modules will be useful.

Lemma 5.3.9 *Let A be a $\mathcal{K}$-cocritical module. Then A is uniform and any non-zero homomorphism from a submodule of A to a module of $\mathcal{K}$ is a monomorphism. In particular, the class of $\mathcal{K}$-cocritical modules is closed under non-zero submodules.*

Proof. Let B be a non-zero submodule of A. There exists a submodule C of A maximal with respect to the property $B \cap C = 0$. Then C is closed in A and B is isomorphic to an essential submodule of A/C. But $B \in \mathcal{K}$, so that $A/C \in \mathcal{K}$. Since A is $\mathcal{K}$-cocritical, we have $C = 0$. Thus $B \trianglelefteq A$ and A is uniform. Now let $f : B \to D$ be a non-zero homomorphism for some $D \in \mathcal{K}$. Suppose that f is not a monomorphism. Then we have $0 \neq B/\mathrm{Ker}f \in \mathcal{K}$. Let $B'/\mathrm{Ker}f$ be a submodule of $A/\mathrm{Ker}f$ maximal with respect to the property

$(B/\mathrm{Ker}f) \cap (B'/\mathrm{Ker}f) = 0$. Then $B/\mathrm{Ker}f$ is isomorphic to an essential submodule of A/B', whence $A/B' \in \mathcal{K}$. But then we have $B' = 0$ or $B' = A$, a contradiction. $\square$

Denote by $H_{\mathcal{K}}(R)$ the set of left ideals I of R such that $R/I \in \mathcal{K}$ and consider the following condition:

$(*)$ *For every ascending chain* $I_1 \subseteq I_2 \subseteq \dots$ *of left ideals in* $H_{\mathcal{K}}(R)$,

$$\bigcup_{j=1}^{\infty} I_j \in H_{\mathcal{K}}(R)\,.$$

Lemma 5.3.10 *If the condition $(*)$ holds, then every cyclic module in $\mathcal{K}$ has a $\mathcal{K}$-cocritical homomorphic image.*

Proof. Let $A \in \mathcal{K}$ be a cyclic module. We may assume that $A = R/I$ for some $I \in H_{\mathcal{K}}(R)$. Let $\mathcal{A}$ be the set of all left ideals J of R that contain I and $0 \neq R/J \in \mathcal{K}$. By Zorn's Lemma, $\mathcal{A}$ has a maximal element, say J. Then R/J is a $\mathcal{K}$-cocritical homomorphic image of A. $\square$

Now we are able to prove the following theorem, connecting the conditions $\mathcal{C}_5(\mathcal{K})$ and $\mathcal{C}_3(\mathcal{K})$, but only for a natural class $\mathcal{K}$.

Theorem 5.3.11 *Let $\mathcal{K}$ be a natural class. If the condition $(*)$ holds, then $\mathcal{C}_5(\mathcal{K}) \Longrightarrow \mathcal{C}_3(\mathcal{K})$.*

Proof. Suppose that $I_1 \subset I_2 \subset \dots$ is a strictly ascending chain of τ-dense left ideals of $H_{\mathcal{K}}(R)$. Then $I_{j+1}/I_j \in \mathcal{K}$ for each j. By Lemma 5.3.10, there exist U_j and V_{j+1} such that $I_j \subseteq U_j \subset V_{j+1} \subseteq I_{j+1}$ and V_{j+1}/U_j is a cyclic $\mathcal{K}$-cocritical module. Since I_j is τ-dense in R, V_{j+1} is τ-dense in R, so that V_{j+1}/U_j is τ-dense in R/U_j. Now let $\alpha_j : V_{j+1}/U_j \to E_\tau(V_{j+1}/U_j)$ be the inclusion homomorphism for each j. By the τ-injectivity of $E_\tau(V_{j+1}/U_j)$, there exists a homomorphism $\beta_j : R/U_j \to E_\tau(V_{j+1}/U_j)$ that extends α_j. Denote $I = \bigcup_{j=1}^{\infty} I_j$ and $A = \bigoplus_j E_\tau(V_{j+1}/U_j)$. Since $E_\tau(V_{j+1}/U_j) \in \mathcal{K}$, we have $A \in \mathcal{K}$. We may define

$$f : I \to A\,, \quad f(x) = (\beta_j(x + U_j))_j\,.$$

It is easy to check that f is a well-defined homomorphism. Let

$$Q = Q_\tau(A) = \sum\{h(A) \mid h \in \mathrm{End}_R(E_\tau(A))\}$$

be the τ-quasi-injective hull of A (see Theorem 5.2.4). We have $E_\tau(A) \in \mathcal{K}$, hence $Q \in \mathcal{K}$. It follows that Q is τ-injective. Since I is τ-dense in R, there exists a homomorphism $g : R \to Q$ such that the following diagram, where the unspecified homomorphisms are inclusions, is commutative:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[d, dashed, "g"] \\ & A \arrow[r] & Q \end{tikzcd}$$

Then we have

$$g(1) \in N = \sum_{k=1}^{t}\sum_{j=1}^{s} h_k(E_\tau(V_{j+1}/U_j))$$

for some t and s. It follows that for every $x \in I$, $f(x) = g(x) = g(1)x \in N$, hence $f(I) \subseteq N$. By Lemma 5.3.9, $h_k(V_{j+1}/U_j) \cong V_{j+1}/U_j$ is a cyclic $\mathcal{K}$-cocritical module. Moreover,

$$E_\tau(h_k(V_{j+1}/U_j)) = h_k(E_\tau(V_{j+1}/U_j))\,.$$

By Lemma 5.3.8 and again by Lemma 5.3.9, N has finite uniform dimension.

On the other hand, $E_\tau(f(V_2)) = E_\tau(V_2/U_1)$ and

$$f(V_2) \subseteq f(V_3) \subseteq E_\tau(V_2/U_1) \oplus V_3/U_2\,.$$

Since $f(V_3) \not\subseteq E_\tau(f(V_2))$ and all V_{j+1}/U_j are uniform, it follows that

$$E_\tau(f(V_3)) = E_\tau(V_2/U_1) \oplus E_\tau(V_3/U_2)\,.$$

Similarly, for each n we have

$$E_\tau(f(V_n)) = E_\tau(V_2/U_1) \oplus E_\tau(V_3/U_2) \oplus \cdots \oplus E_\tau(V_{n+1}/U_n)\,.$$

But this means that $E(f(I))$ and thus $f(I)$ has infinite uniform dimension, a contradiction. Therefore $\mathcal{C}_3(\mathcal{K})$ holds. $\square$

Proposition 5.3.12 *Let σ be a hereditary torsion theory such that $\tau \leq \sigma$ and let $\mathcal{K}$ be the class of all σ-torsion modules. Then $\mathcal{C}_5(\mathcal{K}) \Longrightarrow \mathcal{C}_3(\mathcal{K})$.*

Proof. Clearly, $\mathcal{K}$ is a τ-natural class. Note also that the set of all σ-dense left ideals of R is exactly $H_{\mathcal{K}}(R)$, hence the condition $(*)$ holds for $\mathcal{K}$. Let A be a $\mathcal{K}$-cocritical module. If there exists a non-zero submodule B of A, then A/B is σ-torsion, i.e. $A/B \in \mathcal{K}$, a contradiction. Hence A is simple and thus uniform. Therefore every $\mathcal{K}$-cocritical module is simple.

We mention that Lemma 5.3.10 holds for this particular τ-natural class $\mathcal{K}$, the proofs being identical. Note also that since every τ-torsion module is σ-torsion, the set of τ-dense left ideals of R is contained in $H_{\mathcal{K}}(R)$. Now the result follows by the same arguments as in the proof of Theorem 5.3.11. □

Theorem 5.3.13 *Let $\mathcal{K}$ be a τ-natural class. Then $\mathcal{C}_7(\mathcal{K}) \Longrightarrow \mathcal{C}_3(\mathcal{K})$.*

Proof. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of τ-dense left ideals of R such that each $I_j \in \mathcal{H}_{\mathcal{K}}(R)$. Denote $E_j = E_\tau(R/I_j)$ and $A = \bigoplus_{j=1}^{\infty} E_j$. Clearly each $E_j \in \mathcal{K}$, hence $A \in \mathcal{K}$. Let $p_j : A \to E_j$ be the canonical projection and consider the following diagram where the unspecified homomorphisms are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & E_j \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[dl, "p_j"] & E_\tau(A) \arrow[dll, dashed, "q_j"] \\
& E_j & &
\end{tikzcd}$$

Since A is τ-dense in $E_\tau(A)$ and E_j is τ-injective, there exists a homomorphism $q_j : E_\tau(A) \to E_j$ that extends p_j. It follows immediately that E_j is a direct summand of $E_\tau(A)$, hence $E_\tau(A) = E_j \oplus C_j$ for some submodule C_j of $E_\tau(A)$. We have

$$(E_\tau(A))^{(\mathbb{N})} \cong \bigoplus_{j=1}^{\infty}(E_j \oplus C_j) = (\bigoplus_{j=1}^{\infty} E_j) \oplus (\bigoplus_{j=1}^{\infty} C_j) = A \oplus (\bigoplus_{j=1}^{\infty} C_j) .$$

By $\mathcal{C}_7(\mathcal{K})$, $(E_\tau(A))^{(\mathbb{N})}$ is τ-quasi-injective, hence A is τ-quasi-injective. Denote $I = \bigcup_{j=1}^{\infty} I_j$. For each j define a homomorphism

$$f_j : I/I_1 \to E_j\,, \quad f_j(x + I_1) = x + I_j$$

for every $x \in I$. Then we may define a homomorphism

$$f : I/I_1 \to A\,, \quad f(x + I_1) = (f_j(x))_j$$

for every $x \in I$. It is easy to check that f is well-defined. Consider the following diagram, where the unspecified homomorphisms are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & I/I_1 \arrow[r] \arrow[d, "f"'] & R/I_1 \arrow[r] & E_1 \arrow[d] \\
& A & & A \arrow[ll, dashed, "g"]
\end{tikzcd}$$

Note that I is τ-dense in R, hence I/I_1 is τ-dense in R/I_1. Clearly, R/I_1 is τ-dense in E_1. Further, $A/E_1 \cong \bigoplus_{j=2}^{\infty} E_j$ is τ-torsion because each $E_j = E_\tau(R/I_j)$ is τ-torsion. Hence E_1 is τ-dense in A. It follows that I/I_1 is τ-dense in A. Now since A is τ-quasi-injective, there exists a homomorphism $g : A \to A$ that extends f. It follows that $f(I/I_1) \subseteq g(R/I_1) \subseteq A$. Since $a = g(1 + I_1) \in A$, we have

$$f(I/I_1) \subseteq Ra \subseteq \bigoplus_{j=1}^{n} E_j$$

for some n. Then $I_{n+1} = I_{n+2} = \cdots = I$. Therefore $\mathcal{C}_3(\mathcal{K})$ holds. $\square$

Theorem 5.3.14 *Let $\mathcal{K}$ be a τ-natural class. Then $\mathcal{C}_6(\mathcal{K}) \Longrightarrow \mathcal{C}_7(\mathcal{K})$.*

Proof. Let A be a τ-quasi-injective module in $\mathcal{K}$ and let I be a set. By hypothesis $(E_\tau(A))^{(I)}$ is τ-injective. Then by Lemma 5.1.10, $A^{(I)}$ is τ-quasi-injective. Hence A is $\sum$-τ-quasi-injective. Therefore $\mathcal{C}_7(\mathcal{K})$ holds. $\square$

Theorem 5.3.15 *Let $\mathcal{K}$ be a natural class. Then $\mathcal{C}_8(\mathcal{K}) \Longleftrightarrow \mathcal{C}_9(\mathcal{K})$.*

Proof. Suppose that $\mathcal{C}_8(\mathcal{K})$ holds and let A be a τ-quasi-injective module in $\mathcal{K}$. By Theorem 5.1.3, A is a fully invariant submodule of $E_\tau(A)$. But $E_\tau(A) = E(A)$. Hence A is a fully invariant submodule of $E(A)$, i.e. A is quasi-injective. Therefore $\mathcal{C}_9(\mathcal{K})$ holds.

Suppose that $\mathcal{C}_9(\mathcal{K})$ holds and let A be a τ-injective module in $\mathcal{K}$. Then A is a τ-quasi-injective module in $\mathcal{K}$, hence A is quasi-injective by hypothesis. Clearly, $A \oplus E(A) \in \mathcal{K}$. Moreover, $A \oplus E(A)$ is τ-injective, hence τ-quasi-injective. By hypothesis, $A \oplus E(A)$ is quasi-injective. Now by Proposition 5.1.9 applied for the improper torsion theory, it follows that A is injective. Therefore $\mathcal{C}_8(\mathcal{K})$ holds. $\square$

References: P. Bland [12], [13], S. Crivei [29], [35], J. Dauns [37], S.S. Page, Y. Zhou [90], [91].

Notes on Chapter 5

The literature on τ-quasi-injective modules seems to be rather poor, some basic properties appearing only in the work of P. Bland (1990, 1998). He showed that every module has a τ-quasi-injective hull, which is unique up to an isomorphism. Some other properties of quasi-injective modules have torsion-theoretic versions. The concept of natural class of modules originates into the work of J. Dauns on saturated classes from the early 1990's. Afterwards, this was developed by S.S. Page and Y. Zhou (1994), who coined the terminology of natural class. Later on, Y. Zhou continued their study, establishing results especially on the lattice of natural classes (1996). We have used here the more general context of a τ-natural class in the study of some injectivity-related properties.

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