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RELATIONAL STRUCTURES, MULTIALGEBRAS AND INVERSE LIMITS

By

COSMIN PELEA

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1. Introduction

The study of multialgebras (hyperstructures) began more than 65 years ago and they were used in different areas of mathematics and in some applied sciences (see [2]). Multialgebras are particular relational systems which generalize the universal algebras. The starting point of this paper is in [6], [7] and [15]. The papers [6] and [7] present the construction of the inverse limit for particular inverse systems for some cases of (semi)hypergroups. The construction is a generalization for the same construction made for corresponding universal algebras as well as a particularization of the construction mentioned in [4] for relational systems. In [15] is proved that the category of multialgebras (with the homomorphisms obtained from the relational systems homomorphisms) has finite products and equalizers, hence it is finitely complete. With no major changes in the proof it can be shown that the category studied in [15] is a category with products, hence this is a complete category. But the notion of multioperation used in [15] does not exclude the empty set from its range and the proof that these multistructures form a category with equalizers uses this fact. Such a multistructure does not satisfy the characterization theorem given by G. Grätzer in [3]. A natural question is if the subcategory whose objects are the multialgebras characterized by Grätzer is closed under the formation of limits in the category studied in [15]. We will show that the answer is negative by proving that the subcategory of the multialgebras of type $(n_\gamma)_{\gamma<o(\tau)}$ is not closed under the formation of (inverse) limits of inverse systems in the category of the relational systems of type $(n_\gamma+1)_{\gamma<o(\tau)}$. We will also examine the conditions under which the inverse

limit of an inverse system of multialgebras (considered in the category of relational systems) is a multialgebra.

The main results of this paper are presented in Section 4 and Section 5. The first important result in Section 4 is Proposition 4.2. This proposition states that if we consider an inverse system of multialgebras for which the carrier has a cofinal subset for which the resulting inverse system of multialgebras has an inverse limit which is a multialgebra, then the inverse limit of the given inverse system of multialgebras is also a multialgebra and the two inverse limit multialgebras are isomorphic. As we will see, some of the results presented in [2], [6] and [7] can be easily obtained from this proposition. In the same section we will prove that a class of multialgebras closed under the formation of the isomorphic images and under the formation of the inverse limits of inverse systems with well-ordered carriers is closed under the formation of the inverse limits of arbitrary inverse systems.

As it is shown in [14], an important tool in the hyperstructure theory is the fundamental relation of a multialgebra. The factorization of a multialgebra with the fundamental relation gives a universal algebra. As we have seen in [10], this way we obtain a functor from the category of the multialgebras of a given type into the category of the universal algebras of this type. Another purpose of this paper is to find some conditions for this functor to preserve the inverse limits of inverse systems of multialgebras. Some results in this direction will be presented in Section 5.

2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence over $\mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = \big(\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)}\big)$.

Let A be a set and let $P^*(A)$ be the set of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. An empty set A is admissible if there are no nullary multioperations among the multioperations f_γ, $\gamma < o(\tau)$. Of course, any universal algebra is a multialgebra (we can identify a one element set with its element).

We remind the reader that if $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ is a multialgebra and $B \subseteq A$, we say that $\mathfrak{B} = (B, (f_\gamma)_{\gamma<o(\tau)})$ is a submultialgebra of $\mathfrak{A}$ if for any

$\gamma < o(\tau)$ and $b_0, \dots, b_{n_\gamma - 1} \in B$, we have $f_\gamma(b_0, \dots, b_{n_\gamma - 1}) \subseteq B$. If $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $b_0, \dots, b_{n-1} \in B$ then $p(b_0, \dots, b_{n-1}) \subseteq B$ (see [1]).

As in [13], we can see the multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ as a relational system $(A, (r_\gamma)_{\gamma < o(\tau)})$ if we consider that, for any $\gamma < o(\tau)$, r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \dots, a_{n_\gamma - 1}, a_{n_\gamma}) \in r_\gamma \;\Leftrightarrow\; a_{n_\gamma} \in f_\gamma(a_0, \dots, a_{n_\gamma - 1}). \tag{1}$$

Of course, if we keep unchanged some of the multioperations which are operations, we can see $\mathfrak{A}$ as a first order structure. It is clear that a submultialgebra of the multialgebra $\mathfrak{A}$ is a substructure of the first order structure obtained this way, but not any substructure of $\mathfrak{A}$ is a submultialgebra. Such an example can be found in Section 4 (but it is not difficult to find easier examples).

We mention that the objects of the categories studied in [15] are obtained by seeing each relation r_γ of a relational system $(A, (r_\gamma)_{\gamma < o(\tau)})$ of type $(n_\gamma + {} + 1)_{\gamma < o(\tau)}$ as a function $A^{n_\gamma} \to P(A)$ by using (1).

If we define for any $\gamma < o(\tau)$ and for any $A_0, \dots, A_{n_\gamma - 1} \in P^*(A)$

$$f_\gamma(A_0, \dots, A_{n_\gamma - 1}) = \bigcup \{ f_\gamma(a_0, \dots, a_{n_\gamma - 1}) \mid \ a_i \in A_i,\ i \in \{0, \dots, n_\gamma - 1\}\},$$

we obtain a universal algebra on $P^*(A)$ (see [11]). We denote this algebra by $\mathfrak{P}^*(A)$. As in [4], we can construct the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$ for any $n \in \mathbb{N}$.

The *fundamental relation* of the multialgebra $\mathfrak{A}$ is the transitive closure α^* of the relation α given on A as follows: for $x, y \in A$, $x \ \alpha \ y$ if and only if

$$x, y \in p(a_0, \dots, a_{n-1}) \text{ for some } n \in \mathbb{N},\ \mathbf{p} \in \mathbf{P}^{(n)}(\tau) \text{ and } a_0, \dots, a_{n-1} \in A,$$

where $p \in P^{(n)}(\mathfrak{P}^*(A))$ is the term function induced by $\mathbf{p}$ on $\mathfrak{P}^*(A)$. The relation α^* is the smallest equivalence relation on A with the property that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra (see [9] and [10]). The universal algebra $\mathfrak{A}/\alpha^*$ is called the *fundamental algebra* of the multialgebra $\mathfrak{A}$. We denote by $\overline{\mathfrak{A}}$ the fundamental algebra of $\mathfrak{A}$. We also denote by φ_A the canonical projection of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$ and by $\overline{a}$ the class $\alpha^*\langle a \rangle = \varphi_A(a)$ of the element $a \in A$.

We remind that for an equivalence relation ρ on A we obtain a multialgebra on A/ρ by defining the multioperations in the factor multialgebra $\mathfrak{A}/\rho$ as follows: for any $\gamma < o(\tau)$, $f_\gamma(\rho\langle a_0\rangle, \dots, \rho\langle a_{n_\gamma - 1}\rangle)$ is the set

$$\{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \dots, b_{n_\gamma - 1}),\ a_i \rho b_i,\ i \in \{0, \dots, n_\gamma - 1\}\}$$

($\rho\langle x\rangle$ denotes the class of x modulo ρ (see [3])).

A map $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of type τ is called homomorphism if for any $\gamma < o(\tau)$ and $a_0, \dots, a_{n_\gamma -1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma -1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma -1})).$$

It is easy to see that the multialgebra homomorphisms are obtained from the homomorphisms of relational systems using (1). The definition of the multioperations of $\mathfrak{A}/\rho$ allows us to see the canonical map from A to A/ρ as a homomorphism of multialgebras. The map h is called an ideal homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma -1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma -1})) = f_\gamma(h(a_0), \dots, h(a_{n_\gamma -1})).$$

A bijective map h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it shown in [11], the multialgebra isomorphisms can be characterized as being those bijective homomorphisms which are ideal.

REMARK 1. From the steps of construction of a term (function) it follows that for a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \dots, a_{n-1} \in A$ then

$$h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1})).$$

If h is an ideal homomorphism then

$$h(p(a_0, \dots, a_{n-1})) = p(h(a_0), \dots, h(a_{n-1})).$$

In [10] we proved the following:

THEOREM 2.1. *If $\mathfrak{A}$, $\mathfrak{B}$ are multialgebras and $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively are their fundamental algebras and if $f : A \to B$ is an ideal homomorphism then there exists only one homomorphism of universal algebras $\overline{f} : \overline{A} \to \overline{B}$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ \overline{A} \arrow[r, "\overline{f}"] & \overline{B} \end{tikzcd} \quad . \tag{2}$$

The proof uses only the fact that f is a homomorphism, so the statement holds if we drop the property of f being ideal.

COROLLARY 2.2. *a) If $\mathfrak{A}$ is a multialgebra then* $\overline{1_A} = 1_{\overline{A}}$.

b) If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of the same type τ and if $f : A \to B$, $g : B \to C$ are homomorphisms, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

We can easily construct the category of multialgebras of the same type τ where the morphisms are considered to be the homomorphisms and the composition of two morphisms is the usual map composition. It is known that the universal algebras of the same type τ together with the homomorphisms between them form a category which is, obviously, a full subcategory in the category of the multialgebras introduced above. We will denote by **Malg**(τ) the category of the multialgebras of type τ and by **Alg**(τ) the category of the universal algebras of type τ.

REMARK 2. From Corollary 2.2 it results that we can define a functor F from **Malg**(τ) into **Alg**(τ) as follows: $F(\mathfrak{A}) = \overline{\mathfrak{A}}$, for any multialgebra $\mathfrak{A}$ of type τ, and $F(f) = \overline{f}$ which makes diagram (2) commutative for any homomorphism f between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of type τ.

3. Direct products of multialgebras

If we consider a family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras of type τ using (1) and the definition of the direct product of a family of relational systems presented in [4], we can organize the Cartesian product $\prod_{i\in I} A_i$ as a relational system which is a multialgebra $\prod_{i\in I} \mathfrak{A}_i$ of type τ with the multioperations defined as follows:

$$f_\gamma\left((a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma - 1})_{i\in I}\right) = \prod_{i\in I} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma - 1}\right),$$

for any $\gamma < o(\tau)$. This multialgebra is called the *direct product* of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. We observe that the projections of the product, e_i^I, $i \in I$, are multialgebra (ideal) homomorphisms. So, we have:

PROPOSITION 3.1. *The multialgebra $\prod_{i\in I} \mathfrak{A}_i$, with the projections e_i^I, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category* **Malg**(τ).

4. Inverse limits of inverse systems of multialgebras

Let $(I, \leq)$ be a preordered directed set and let us consider an inverse system $(\mathfrak{A}_i \mid i \in I)$ of multialgebras of type τ together with the homomorphisms $(\varphi_j^k : A_j \to A_k \mid j,k \in I,\ j \leq k)$. We remind the reader that for any $i \in I$, $\varphi_i^i = 1_{A_i}$ and if $j \leq k \leq l$, we have $\varphi_j^k \circ \varphi_k^l = \varphi_j^l$. We consider the inverse limit of the inverse system of sets $(A_i \mid i \in I)$

$$A^{\infty} = \Big\{(a_i)_{i\in I} \in \prod_{i\in I} A_i \mid \forall j,k \in I,\ j \leq k,\ \varphi_j^k(a_k) = a_j\Big\}.$$

In [4] it is mentioned that the inverse limits for first order structures are defined the same way as for algebras, as suitable substructures of the direct product. If we see each n_γ-ary multioperation in each $\mathfrak{A}_i$ as an $(n_\gamma + 1)$-ary relation r_γ as in (1), we obtain the definitions for the relations on A^{∞} : given $\gamma < o(\tau)$ and $\big(a_i^0\big)_{i\in I}, \ldots, \big(a_i^{n_\gamma -1}\big)_{i\in I}, (a_i)_{i\in I} \in A^{\infty}$ we have

$$\big((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma -1})_{i\in I}, (a_i)_{i\in I}\big) \in r_\gamma \Leftrightarrow a_i \in f_\gamma\big(a_i^0, \ldots, a_i^{n_\gamma -1}\big),\ \forall i \in I.$$

Since we are dealing with multialgebras our question is whether the relational system obtained in this way is a multialgebra. If the answer were affirmative then, using again (1), it would follow that its multioperations would be defined by:

$$f_\gamma\big((a_i^0)_{i\in I}, \ldots, (a_i^{n_\gamma -1})_{i\in I}\big) = \prod_{i\in I} f_\gamma\big(a_i^0, \ldots, a_i^{n_\gamma -1}\big) \cap A^{\infty}, \tag{3}$$

for every $\gamma < o(\tau)$ and $\big(a_i^0\big)_{i\in I}, \ldots, \big(a_i^{n_\gamma -1}\big)_{i\in I} \in A^{\infty}$.

REMARK 3. Let us remark that the inverse limit of an inverse system of sets $(A_i \mid i \in I)$, A^{∞} is not necessarily a submultialgebra of $\prod_{i\in I} \mathfrak{A}_i$, so, the intersection with A^{∞} cannot be omitted in (3).

EXAMPLE 1. Let us consider the finite set of positive integers $I = \{1, 2\}$ ordered with the usual relation $\leq$, induced from $\mathbb{N}$, let us consider the hypergroupoids $(H_1, \circ)$, $(H_2, \circ)$ defined on $H_1 = H_2 = \{x, y\}$ by

$$x \circ x = x \circ y = y \circ x = y \circ y = \{x, y\}$$

and let us consider the (ideal) homomorphisms $\varphi_1^1 = 1_{H_1}$, $\varphi_2^2 = 1_{H_2}$ and

$$\varphi_1^2 \colon H_2 \to H_1,\ \varphi_1^2(x) = y,\ \varphi_1^2(y) = x.$$

Then $A^{\infty} = \{(x,y),(y,x)\}$ is not a subhypergroupoid in $H_1 \times H_2$.

REMARK 4. The functions f_γ given by (3) are not always multioperations on A^{∞}. Even if $A^{\infty} \neq \emptyset$, the intersection in the second member of the equality can be the empty set. As a matter of fact, since for any $\gamma < o(\tau)$, $\left(a_i^0\right)_{i\in I}, \dots, \left(a_i^{n_\gamma-1}\right)_{i\in I} \in A^{\infty}$ and for any $j,k \in I$, $j \geq k$,

$$\varphi_k^j\left(f_\gamma\left(a_j^0,\dots,a_j^{n_\gamma-1}\right)\right) \subseteq f_\gamma\left(a_k^0,\dots,a_k^{n_\gamma-1}\right),$$

it follows that for a given $\gamma < o(\tau)$, and $\left(a_i^0\right)_{i\in I}, \dots, \left(a_i^{n_\gamma-1}\right)_{i\in I} \in A^{\infty}$, the family

$$\left(f_\gamma\left(a_i^0,\dots,a_i^{n_\gamma-1}\right) \mid i \in I\right)$$

of sets together with the restrictions of the maps φ_k^j to these sets form an inverse system of sets and the second member in (3) is the inverse limit of this inverse system of sets. So $f_\gamma\left(\left(a_i^0\right)_{i\in I}, \dots, \left(a_i^{n_\gamma-1}\right)_{i\in I}\right)$ can be empty even if A^{∞} is not.

EXAMPLE 2. In [5], the authors present an example of an inverse system of (countable) sets, with surjective maps and empty inverse limit: let ω_1 be the first uncountable ordinal and for $\alpha < \omega_1$,

$$E_\alpha = \{\gamma \mid \gamma \leq \alpha\},\ F_\alpha = \{f \in \mathbb{R}^{E_\alpha} \mid f \text{ is strictly increasing}\};$$

and for $\alpha < \beta < \omega_1$, let

$$\theta_\alpha^\beta : F_\beta \to F_\alpha,\ \theta_\alpha^\beta(f) = f|_{E_\alpha} \text{ (the restriction of } f \text{ to } E_\alpha).$$

The authors define by transfinite induction a family of subsets S_α of F_α for which $|S_\alpha| = \aleph_0$ and $\theta_\alpha^\beta(S_\beta) = S_\alpha$ whenever $\alpha < \beta$, such that the inverse system $(S_\alpha \mid \alpha < \omega_1)$ (with the corresponding restrictions of the functions θ_α^β) has the desired property.

Starting from this example, we will consider for each $1 \leq \alpha < \omega_1$,

$$A_\alpha = S_\alpha \cup \{0_{E_\alpha}\},\ \text{where } 0_{E_\alpha} : E_\alpha \to \mathbb{R},\ 0_{E_\alpha}(\gamma) = 0.$$

We define a hyperproduct $\circ$ on A_α by taking

$$f \circ g = \begin{cases} S_\alpha, & \text{if } f = 0_{E_\alpha} = g \text{ or } f \neq 0_{E_\alpha} \neq g \\ 0_{E_\alpha}, & \text{otherwise.} \end{cases}$$

The maps

$$\varphi_\alpha^\beta : A_\beta \to A_\alpha,\ \varphi_\alpha^\beta(f) = f|_{E_\alpha}\ (\alpha < \beta)$$

are (ideal) homomorphisms. Thus we obtain an inverse system of hypergroupoids.

We observe that $\varphi_\alpha^\beta|_{S_\alpha} = \theta_\alpha^\beta|_{S_\alpha}$ and that A^∞ is not empty since $(0_{E_\alpha})_{1\leq\alpha<\omega_1} \in A^\infty$. In fact $A^\infty = \{(0_{E_\alpha})_{1\leq\alpha<\omega_1}\}$, for if A^∞ had an element different from this, it would follow that this element belongs to the inverse limit of the inverse system of the sets $(S_\alpha \mid 1 \leq \alpha < \omega_1)$, which is impossible, since this inverse limit is empty.

Now it is easy to see that (3) is not always the definition of a hyperproduct on A^∞ since in our case $(0_{E_\alpha})_{1\leq\alpha<\omega_1} \circ (0_{E_\alpha})_{1\leq\alpha<\omega_1} = \emptyset$.

REMARK 5. In order to obtain a multialgebra $\mathfrak{A}^\infty$ on $A^\infty \neq \emptyset$ as above it would be required that for any $\gamma < o(\tau)$ and $\left(a_i^0\right)_{i\in I}, \ldots, \left(a_i^{n_\gamma - 1}\right)_{i\in I} \in A^\infty$,

$$f_\gamma\left(\left(a_i^0\right)_{i\in I}, \ldots, \left(a_i^{n_\gamma-1}\right)_{i\in I}\right) = \varprojlim_{i\in I} f_\gamma\left(a_i^0, \ldots, a_i^{n_\gamma-1}\right) \neq \emptyset$$

hold. As it is shown in [4, §21, Theorem 1], a case when this happens is given by the condition that for every $i \in I, \gamma < o(\tau)$ and $a_i^0, \ldots, a_i^{n_\gamma-1} \in A_i$, $f_\gamma\left(a_i^0, \ldots, a_i^{n_\gamma-1}\right)$ is nonempty and finite.

REMARK 6. This is the case of universal algebras for which the sets $f_\gamma\left(a_i^0, \ldots, a_i^{n_\gamma-1}\right)$ are one-element sets. It is also clear that if for some $\gamma < o(\tau)$, f_γ is an operation in all the multialgebras $\mathfrak{A}_i$, then f_γ is an operation in $\mathfrak{A}^\infty$. In this case, the definition of f_γ is the one in [4, §21].

REMARK 7. If $\mathcal{I}$ is the category associated to the preordered set $(I, \leq)$, we can see (as in [12]) the inverse system of multialgebras $(\mathfrak{A}_i \mid i \in I)$, with the homomorphisms $(\varphi_k^j \mid j, k \in I,\ j \geq k)$, as a contravariant functor $G\colon \mathcal{I} \to \mathbf{Malg}(\tau)$.

REMARK 8. If $(A^\infty, (f_\gamma)_{\gamma<o(\tau)})$ is a multialgebra then, for any $j \in I$,

$$\varphi_j^\infty : A^\infty \to A_j,\ \varphi_j^\infty((a_i)_{i\in I}) = a_j$$

is a multialgebra homomorphism.

Indeed, since the map φ_j^∞ is the restriction of e_j^I to A^∞, if we take $\gamma < o(\tau)$, $(a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma -1})_{i\in I} \in A^\infty$ we have

$$\begin{aligned}\varphi_j^\infty\left(f_\gamma\left((a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma-1})_{i\in I}\right)\right) &= \varphi_j^\infty\left(\prod_{i\in I} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma-1}\right) \cap A^\infty\right)\\ &\subseteq e_j^I\left(\prod_{i\in I} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma-1}\right)\right)\\ &= f_\gamma\left(a_j^0, \dots, a_j^{n_\gamma-1}\right)\\ &= f_\gamma\left(\varphi_j^\infty\left((a_i^0)_{i\in I}\right), \dots, \varphi_j^\infty\left((a_i^{n_\gamma-1})_{i\in I}\right)\right).\end{aligned}$$

REMARK 9. As it can be easily seen from the previous examples, the problems which appear when asking for the inverse limit of an inverse system of multialgebras to be a multialgebra are not solved if we are dealing with ideal homomorphisms. Moreover, in this situation new problems arise since the maps φ_j^∞ $(j \in I)$ are not always ideal homomorphisms.

To illustrate this, we built up an example starting once again from the example of Higman and Stone, as in Example 2.

EXAMPLE 3. Take the sets $(A_\alpha \mid \alpha < \omega_1)$, the maps φ_α^β as in Example 2 and define on each A_α, the hyperproduct $\circ$ by

$$f \circ g = A_\alpha.$$

Then, using (3), we obtain a hypergroup(oid) on $A^\infty = \{(0_{E_\alpha})_{\alpha<\omega_1}\}$ with the hyperproduct

$$(0_{E_\alpha})_{\alpha<\omega_1} \circ (0_{E_\alpha})_{\alpha<\omega_1} = (0_{E_\alpha})_{\alpha<\omega_1}.$$

The maps φ_α^β are ideal homomorphisms of hypergroupoids and for any $\beta <$ $< \omega_1$ the map

$$\varphi_\beta^\infty : A^\infty \to A_\beta,\ \varphi_\beta^\infty((0_{E_\alpha})_{\alpha<\omega_1}) = 0_{E_\beta}$$

is not an ideal homomorphism because

$$\begin{aligned}\varphi_\beta^\infty((0_{E_\alpha})_{\alpha<\omega_1} \circ (0_{E_\alpha})_{\alpha<\omega_1}) &= \varphi_\beta^\infty((0_{E_\alpha})_{\alpha<\omega_1}) = 0_{E_\beta} \neq A_\beta = 0_{E_\beta} \circ 0_{E_\beta}\\ &= \varphi_\beta^\infty((0_{E_\alpha})_{\alpha<\omega_1}) \circ \varphi_\beta^\infty((0_{E_\alpha})_{\alpha<\omega_1}).\end{aligned}$$

Note that in Remark 5 we presented a necessary and sufficient condition for a category of multialgebras (of type τ) to be a subcategory of the category of the relational systems (of type $(n_\gamma+1)_{\gamma<o(\tau)}$) closed under the inverse limits of inverse systems. Thus we have:

THEOREM 4.1. *If $\mathfrak{A}^\infty = (A^\infty, (f_\gamma)_{\gamma<o(\tau)})$ is a multialgebra then, together with the homomorphisms $(\varphi_j^\infty \mid j \in I)$, it is the inverse limit of the functor G.*

The first of the following diagrams:

$$\begin{tikzcd}
A_k \arrow[r, "\varphi_j^k"] & A_j \\
& A^\infty \arrow[ul, "\varphi_k^\infty"] \arrow[u, "\varphi_j^\infty"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_k \arrow[r, "\varphi_j^k"] & A_j \\
A' \arrow[u, "\alpha_k"] \arrow[ur, "\alpha_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_j \\
A' \arrow[ur, "\alpha_j"] \arrow[r, "\mu"] & A^\infty \arrow[u, "\varphi_i^\infty"']
\end{tikzcd}$$

is commutative and whenever a multialgebra $\mathfrak{A}' = (A', (f_\gamma)_{\gamma<o(\tau)})$, together with a family $(\alpha_j : A' \to A_j \mid j \in I)$ of homomorphisms make the second diagram commutative, there exists a unique homomorphism $\mu : A' \to A^\infty$, given by the equality $\mu(x) = (\alpha_i(x))_{i\in I}$, such that the third diagram is commutative.

The last three results of this section are generalizations for some results presented for universal algebras in [4, §21].

From now on we will consider that $(I, \leq)$ is a directed partially ordered set (unless we will specify something else). Let $\mathscr{A} = (\mathfrak{A}_i \mid i \in I)$ be an inverse system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also a directed partially ordered set. We will denote by $\mathscr{A}_J$ the inverse system of multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J, \leq)$ and whose homomorphisms are φ_j^i, with $i, j \in J$, $i \leq j$. From the proof of [4, §21, Lemma 7] it results that if we consider the inverse systems of sets $(A_i \mid i \in I)$ and $(A_i \mid i \in J)$, the canonical projection $(a_i)_{i\in I} \mapsto (a_i)_{i\in J}$ furnishes a bijection ψ between $\varprojlim_{i\in I} A_i$ and $\varprojlim_{i\in J} A_i$.

PROPOSITION 4.2. *Let $\mathscr{A}$ be an inverse system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ such that $(J, \leq)$ is a directed partially ordered set cofinal with $(I, \leq)$. The inverse limit $\varprojlim \mathscr{A}$ is a multialgebra if* *and only if* $\varprojlim \mathcal{A}_J$ *is a multialgebra. If this happens, the two multialgebras are isomorphic.*

PROOF. Of course, if the relational systems $\varprojlim \mathcal{A}$ and $\varprojlim \mathcal{A}_J$ are multialgebras, then the isomorphism between them follows from the dual [8, Proposition 2.11, Chapter II]. However, while proving that $\varprojlim \mathcal{A}$ is in **Malg**(τ) if and only if $\varprojlim \mathcal{A}_J$ is in **Malg**(τ) we will find the form of this isomorphism, so, we will also present a proof based on [4, §21, Lemma 7].

Clearly, $\varprojlim_{i\in I} A_i = \emptyset$ if and only if $\varprojlim_{i\in J} A_i = \emptyset$. So, $\varprojlim \mathcal{A}$ and $\varprojlim \mathcal{A}_J$ are multialgebras if and only if for any $\gamma < o(\tau)$, $n_\gamma \neq 0$. If this is the case, they are, trivially, isomorphic.

Let us consider that $\varprojlim_{i\in I} A_i \neq \emptyset \neq \varprojlim_{i\in J} A_i$. The limit $\varprojlim \mathcal{A}$ is a multialgebra whenever for each $\gamma < o(\tau)$ and for all $(a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma -1})_{i\in I} \in A^\infty$, the right member of (3) is not the empty set. According to Remark 4, this happens exactly when the inverse system of the nonempty sets $(f_\gamma(a_i^0, \dots, a_i^{n_\gamma -1}) \mid i \in I)$, together with the restrictions of the maps φ_k^j, $j \geq k$, to these sets, is not empty. It is easy to see from the definition of ψ that its restrictions

$$\varprojlim_{i\in I} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma -1}\right) \to \varprojlim_{i\in J} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma -1}\right)$$

are also bijective.

As for the multialgebra isomorphism between $\mathfrak{A}^\infty$ and $\mathfrak{A}'^\infty = \varprojlim \mathcal{A}_J$, let us verify that ψ is an ideal homomorphism of multialgebras.

Given $\gamma < o(\tau)$ and $(a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma -1})_{i\in I} \in A^\infty$ we have

$$\begin{aligned}
\psi\left(f_\gamma\left((a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma -1})_{i\in I}\right)\right) &= \psi\left(\varprojlim_{i\in I} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma -1}\right)\right)\\
&= \varprojlim_{i\in J} f_\gamma\left(a_i^0, \dots, a_i^{n_\gamma -1}\right)\\
&= f_\gamma\left((a_i^0)_{i\in J}, \dots, (a_i^{n_\gamma -1})_{i\in J}\right)\\
&= f_\gamma\left(\psi\left((a_i^0)_{i\in I}\right), \dots, \psi\left((a_i^{n_\gamma -1})_{i\in I}\right)\right),
\end{aligned}$$

and the proof is finished. ∎

REMARK 10. Using this proposition, the constructions of inverse limits from [2], [6] and [7], which are made for inverse systems of (particular) multialgebras with $(I,\leq)$ directed ordered set which has a maximum, are isomorphic to the member of the system having this maximum as an index. It is clear that such an inverse limit exists and it has all the properties of this member.

Let us consider that the support set I of the carrier $(I,\leq)$ of the inverse system $\mathscr{A}=(\mathfrak{A}_i \mid i\in I)$ of multialgebras can be written as $I=\bigcup_{p\in P} I_p$, where $(I_p,\leq)$ is a directed partially ordered subset of $(I,\leq)$ for each $p\in P$ and $(P,\leq)$ is also a directed partially ordered set such that $I_p\subseteq I_q$, whenever $p,q\in P$, $p\leq q$. We will denote

$$\varprojlim \mathscr{A}=\mathfrak{A}^{\infty}=(A^{\infty},(f_\gamma)_{\gamma<o(\tau)}),\ \varprojlim \mathscr{A}_{I_p}=\mathfrak{A}_p^{\infty}=(A_p^{\infty},(f_\gamma)_{\gamma<o(\tau)})\ (p\in P).$$

For any $p,q\in P,\ p\leq q$ we can define the map

$$\psi_p^q : A_q^{\infty}\to A_p^{\infty},\ \psi_p^q((a_i)_{i\in I_q})=(a_i)_{i\in I_p}.$$

In this way we obtain an inverse system of sets consisting of $(P,\leq)$, the sets A_p^{∞}, and the maps ψ_p^q. We will denote it by $\mathscr{A}/P$.

THEOREM 4.3. *Let us consider that all $\mathfrak{A}_p^{\infty}$, $p\in P$, are multialgebras. Then $\mathscr{A}/P$ is an inverse system of multialgebras and $\varprojlim \mathscr{A}$ is a multialgebra if and only if $\varprojlim \mathscr{A}/P$ is a multialgebra. In this case these multialgebras are isomorphic.*

PROOF. First we will show that for any $p,q\in P$, $p\leq q$, the map ψ_p^q is a multialgebra homomorphism. If we take $\gamma<o(\tau)$, $\left(a_i^0\right)_{i\in I_q}$, …, $\left(a_i^{n_\gamma-1}\right)_{i\in I_q}\in A_q^{\infty}$ then we have

$$\begin{aligned}
\psi_p^q(f_\gamma\left(\left(a_i^0\right)_{i\in I_q},\dots,\left(a_i^{n_\gamma-1}\right)_{i\in I_q}\right)) &= \psi_p^q\left(\prod_{i\in I_q} f_\gamma\left(a_i^0,\dots,a_i^{n_\gamma-1}\right)\cap A_q^{\infty}\right)\\
&\subseteq \prod_{i\in I_p} f_\gamma\left(a_i^0,\dots,a_i^{n_\gamma-1}\right)\cap A_p^{\infty}\\
&= f_\gamma\left(\left(a_i^0\right)_{i\in I_p},\dots,\left(a_i^{n_\gamma-1}\right)_{i\in I_p}\right)\\
&= f_\gamma\left(\psi_p^q\left(\left(a_i^0\right)_{i\in I_q}\right),\dots,\psi_p^q\left(\left(a_i^{n_\gamma-1}\right)_{i\in I_q}\right)\right).
\end{aligned}$$

We denote by A'^{∞} the inverse limit of the inverse system of sets $\mathcal{A}/P$ and we consider the map

$$\psi : A^{\infty} \to A'^{\infty},\ \psi(g) = h_g,$$

where $h_g \in \prod_{p\in P} A_p^{\infty}$ is defined by

$$h_g(p) = g|_{I_p} \in A_p^{\infty}.$$

By [4, §21, Theorem 3] ψ is a bijective map.

Note that $A'^{\infty} = \emptyset$ if and only if $A^{\infty} = \emptyset$. Thus, if $A_p^{\infty} = \emptyset$ for some p then $A'^{\infty} = A^{\infty} = \emptyset$. However, if $A'^{\infty} = A^{\infty} = \emptyset$ then $\varprojlim \mathcal{A}$ and $\varprojlim \mathcal{A}/P$ are multialgebras if and only if $n_\gamma \neq 0$ for any $\gamma < o(\tau)$, and the two multialgebras are trivially isomorphic in this case. So, we can assume that the sets A_p^{∞}, A'^{∞}, A^{∞} are not empty.

The inverse limit $\varprojlim \mathcal{A} = \mathfrak{A}^{\infty}$ is a multialgebra if and only if for any $\gamma < o(\tau)$ and for all $g_0, \dots, g_{n_\gamma-1} \in A^{\infty}$ the inverse limit of the inverse system of sets $(f_\gamma(g_0(i), \dots, g_{n_\gamma-1}(i)) \mid i \in I)$ is nonempty. But the restriction of ψ to this inverse limit of sets is a bijection between

$$f_\gamma\left(g_0, \dots, g_{n_\gamma-1}\right) = \varprojlim_{i\in I} f_\gamma(g_0(i), \dots, g_{n_\gamma-1}(i))$$

and the inverse limit

$$\begin{aligned}\varprojlim_{p\in P}(\varprojlim_{i\in I_p} f_\gamma(g_0(i), \dots, g_{n_\gamma-1}(i))) &= \varprojlim_{p\in P} f_\gamma(g_0|_{I_p}, \dots, g_{n_\gamma-1}|_{I_p})\\ &= \varprojlim_{p\in P} f_\gamma(h_{g_0}(p), \dots, h_{g_{n_\gamma-1}}(p))\\ &= f_\gamma(h_{g_0}, \dots, h_{g_{n_\gamma-1}}).\end{aligned}$$

Since

$$\begin{aligned}f_\gamma(g_0|_{I_p}, \dots, g_{n_\gamma-1}|_{I_p}) &= \varprojlim_{i\in I_p} f_\gamma(g_0|_{I_p}(i), \dots, g_{n_\gamma-1}|_{I_p}(i))\\ &= \varprojlim_{i\in I_p} f_\gamma(g_0(i), \dots, g_{n_\gamma-1}(i)) \neq \emptyset\end{aligned}$$

it follows that f_γ is a multioperation on A^{∞} if and only if f_γ is a multioperation on A'^{∞}. Moreover, it follows that

$$\psi(f_\gamma(g_0, \dots, g_{n_\gamma-1})) = f_\gamma(h_{g_0}, \dots, h_{g_{n_\gamma-1}}) = f_\gamma(\psi(g_0), \dots, \psi(g_{n_\gamma-1}))).$$

Thus ψ is a multialgebra isomorphism. ∎

As in [4], we will use the term of algebraic class for those classes of multialgebras which are closed under the formation of isomorphic images.

We will say that a class of multialgebras is closed under the formation of inverse limits of inverse systems if for any inverse system of multialgebras from this class the inverse limit is a multialgebra from this class. Now we can prove the following theorem:

THEOREM 4.4. *If K is an algebraic class of multialgebras then K is closed under the formation of inverse limits of arbitrary inverse systems if and only if K is closed under the formation of inverse limits of well-ordered inverse systems.*

PROOF. The proof goes as in [4, §21, Theorem 4]. Assume that K is closed under the formation of inverse limits of well-ordered inverse systems. We take an inverse system $\mathcal{A}$ of multialgebras with carrier $(I, \leq)$. If the theorem were not true, then we could choose $\mathfrak{A}$ such that $|I| = \mathfrak{m}$ is the smallest possible with the property that either the inverse limit of $\mathcal{A}$ is not a multialgebra or $\varprojlim \mathcal{A} \notin K$. From Proposition 4.2 we have $\mathfrak{m} \geq \aleph_0$ and using [4, Exercise 44, pp. 73] it follows that we can write $I = \bigcup_{\delta<\alpha} I_\delta$, where α is an ordinal, $(I_\delta, \leq)$ is directed, $I_{\delta_1} \subseteq I_{\delta_2}$ if $\delta_1 \leq \delta_2 < \alpha$, and $|I_\delta| < |I| = \mathfrak{m}$. From the minimality of $\mathfrak{m}$ it follows that $\varprojlim \mathfrak{A}_{I_\delta}$, $\delta < \alpha$, are multialgebras from K, according to our assumption it follows that $\varprojlim \mathcal{A}/P$ is also a multialgebra from K and by Theorem 4.3 $\varprojlim \mathcal{A}$ is a multialgebra from K, a contradiction. ∎

5. On the fundamental algebra of an inverse limit of multialgebras

In general, the fundamental algebra of the product of two multialgebras is not the product of the fundamental algebras of the given multialgebras, as it is shown by the following:

EXAMPLE 4. Let us consider the hypergroupoid $(\mathbb{Z}, \circ)$, where $\mathbb{Z}$ is the set of the integers and the multioperation $\circ$ is defined by

$$x \circ y = \{x + y, x + y + 1\}$$

for any $x, y \in \mathbb{Z}$. It is not difficult to prove that for any $n \in \mathbb{N}^*$, $(\mathbb{Z}^n, \circ)$ is a hypergroup with the fundamental relation $\beta = \mathbb{Z}^n \times \mathbb{Z}^n$. It means that for any $n \in \mathbb{N}^*$ the fundamental group of $(\mathbb{Z}^n, \circ)$ is a one-element group. Now let us consider the product $(\mathbb{Z}^{\mathbb{N}}, \circ)$. The fundamental group of this hypergroup has more than one element. Indeed, $f, g : \mathbb{N} \to \mathbb{Z}$, $f(n) = 0$, $g(n) = n + 1$

($n \in \mathbb{N}$) are not in the same equivalence class of the fundamental relation of the hypergroup $(\mathbb{Z}^{\mathbb{N}}, \circ)$.

EXAMPLE 5. An useful example of inverse limit of multialgebras can be given in a similar way as Grätzer did in [4] for universal algebras. So, let us consider a set I and a family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras of type τ. We can get an inverse system of multialgebras taking $(J, \subseteq)$ to be the set of all the finite nonempty subsets of the set I, ordered with the set inclusion, $\mathfrak{B}_j = \prod_{i \in j} \mathfrak{A}_i$, for any $j \in J$, and the canonical projections $\varphi^{j_0}_{j_1}$ from $\prod_{i \in j_0} \mathfrak{A}_i$ onto $\prod_{i \in j_1} \mathfrak{A}_i$, for any $j_0 \supseteq j_1$ from J. The inverse limit of this inverse system of multialgebras exists and it is isomorphic to $\prod_{i \in I} \mathfrak{A}_i$.

REMARK 11. Using the previous examples we can obtain an example to show that the functor F from Remark 2 does not preserve the arbitrary inverse limits of inverse systems of multialgebras (hypergroupoids, in this case) even if they exist.

Now we will find a condition for this functor F to preserve the inverse limits of inverse systems of multialgebras from an algebraic class K closed under the formation of the inverse limits of inverse systems.

Let $\mathcal{A} = (\mathfrak{A}_i \mid i \in I)$ be an inverse system of multialgebras with the homomorphisms $(\varphi^i_j \mid i, j \in I,\ i \geq j)$. We will denote by $\overline{\mathcal{A}}$ the inverse system of the fundamental algebras $(\overline{\mathfrak{A}_i} \mid i \in I)$ of the multialgebras from $\mathcal{A}$, with the homomorphisms $(\overline{\varphi^i_j} \mid i, j \in I,\ i \geq j)$. So, if we see the inverse system $\mathcal{A}$ as the functor G from Remark 7 then $\overline{\mathcal{A}}$ is the functor FG.

In this section, we will refer to the inverse limit $\varprojlim \mathcal{A}$ of the inverse system $\mathcal{A}$ as the inverse limit $(\mathfrak{A}^{\infty}, (\varphi^{\infty}_i \mid i \in I))$ of G. Clearly, $\varprojlim \overline{\mathcal{A}} = \varprojlim (FG)$. If we denote $(\overline{\mathfrak{A}^{\infty}}, (\overline{\varphi^{\infty}_i} \mid i \in I))$ by $\overline{\varprojlim \mathcal{A}}$ we can state the following:

LEMMA 5.1. *Let $\mathcal{A}$ be an inverse system of multialgebras with the carrier $(I, \leq)$ and let us consider $J \subseteq I$ with $(J, \leq)$ a directed partially ordered set cofinal with $(I, \leq)$. Assume that $\varprojlim \mathcal{A}_J$ is a multialgebra. Under these conditions, $\overline{\varprojlim \mathcal{A}}$ is the inverse limit of the inverse system of universal algebras $\overline{\mathcal{A}}$ if and only if $\overline{\varprojlim \mathcal{A}_J}$ is the inverse limit of the inverse system of universal algebras $\overline{\mathcal{A}_J}$.*

PROOF. From the fact that $\varprojlim \mathcal{A}_J$ is a multialgebra it follows that $\varprojlim \mathcal{A}$ is a multialgebra, and, since they are isomorphic (Proposition 4.2), the functor F induces an isomorphism from $\overline{\varprojlim \mathcal{A}}$ onto $\overline{\varprojlim \mathcal{A}_J}$. Of course, the inverse system $\mathcal{A}_J$ of multialgebras determines an inverse system of universal algebras $\overline{\mathcal{A}_J} = \overline{\mathcal{A}}_J$ and the inverse limit algebras $\varprojlim \overline{\mathcal{A}}$ and $\varprojlim \overline{\mathcal{A}}_J$ are isomorphic. From the universal property that characterizes the inverse limit we obtain the homomorphisms:

$$\overline{\varprojlim \mathcal{A}} \to \varprojlim \overline{\mathcal{A}},\ \overline{(a_i)_{i\in I}} \mapsto (\overline{a_i})_{i\in I};$$

$$\overline{\varprojlim \mathcal{A}_J} \to \varprojlim \overline{\mathcal{A}_J},\ \overline{(a_i)_{i\in J}} \mapsto (\overline{a_i})_{i\in J},$$

which, together with the above mentioned isomorphisms, make the following diagram commutative:

$$\begin{tikzcd}
\overline{\varprojlim \mathcal{A}} \arrow[r] \arrow[d] & \overline{\varprojlim \mathcal{A}_J} \arrow[d] \\
\varprojlim \overline{\mathcal{A}} \arrow[r] & \varprojlim \overline{\mathcal{A}_J}
\end{tikzcd}.$$

Since each one of the conditions for which we want to prove the equivalence makes one of the vertical morphisms an isomorphism, the conclusion of the lemma is immediate. ∎

LEMMA 5.2. *Using the notations from the previous section, let us consider that $\mathfrak{A}_p^\infty$, $p \in P$, and $\mathfrak{A}^\infty$ are multialgebras. Let us also assume that for each $p \in P$, $\overline{\varprojlim \mathcal{A}_{I_p}}$ is the inverse limit of the inverse system $\overline{\mathcal{A}_{I_p}}$ of universal algebras.*

$\overline{\underleftarrow{\mathrm{Lim}}\,\mathcal{A}}$ is the inverse limit of $\overline{\mathcal{A}}$ if and only if $\overline{\varprojlim \mathcal{A}/P}$ is the inverse limit of $\overline{\mathcal{A}/P}$.

PROOF. Using the fact that for any $p \in P$, $\overline{\varprojlim \mathcal{A}_{I_p}} = \varprojlim \overline{\mathcal{A}_{I_p}}$ it follows that $\overline{\mathcal{A}}/P = \overline{\mathcal{A}/P}$. From [4, §21, Theorem 3] it results an isomorphism from $\varprojlim \overline{\mathcal{A}}$ onto

$$\varprojlim \overline{\mathcal{A}}/P = \varprojlim \overline{\mathcal{A}/P}.$$

Using Theorem 4.3 and the universal property of the inverse limit, we obtain the following commutative diagram:

$$\begin{tikzcd}
\overline{\varprojlim \mathcal{A}} \arrow[r] \arrow[d] & \overline{\varprojlim \mathcal{A}/P} \arrow[d] \\
\varprojlim \overline{\mathcal{A}} \arrow[r] & \varprojlim \overline{\mathcal{A}/P}
\end{tikzcd}.$$

The conclusion follows as in the previous lemma. ∎

If K is a class of multialgebras of type τ then we can obtain a subcategory $\mathcal{K}$ of $\mathbf{Malg}(\tau)$ if we consider morphisms only those homomorphisms which are defined between two multialgebras from K. Of course, the composition FU of the functor F with the inclusion functor $U : \mathcal{K} \longrightarrow \mathbf{Malg}(\tau)$. Knowing the definition of U, in the next theorem we may use F instead of FU.

THEOREM 5.3. *Let K be an algebraic class of multialgebras closed under the formation of inverse limits of well-ordered inverse systems. Then F preserves the inverse limits of arbitrary inverse systems of multialgebras from K if and only if F preserves the inverse limits of well-ordered inverse systems of multialgebras from K.*

PROOF. Clearly, $\mathcal{K}$ is a subcategory of the category of relational systems of type $(n_\gamma+1)_{\gamma<o(\tau)}$ closed under the inverse limits of inverse systems. Let us assume that F preserves the inverse limits of well-ordered inverse systems of multialgebras from K. If F would not preserve the inverse limits of arbitrary inverse systems of multialgebras from K then we could choose an inverse system $\mathcal{A}$ of multialgebras from K with the carrier $(I, \leq)$ such that $\overline{\varprojlim \mathcal{A}}$ is not $\varprojlim \overline{\mathcal{A}}$ and $|I| = \mathfrak{m}$ is the smallest possible with this property. We will continue as in the proof of Theorem 4.4, so we will use the same notations as there. First we obtain (using Lemma 5.1) that $\mathfrak{m} \geq \aleph_0$ and then, using the minimality of $\mathfrak{m}$, our assumption and Lemma 5.2 it will result that

$$\overline{\varprojlim \mathcal{A}_{I_\delta}} = \varprojlim \overline{\mathcal{A}_{I_\delta}},$$

for any $\delta < \alpha$, and

$$\overline{\varprojlim \mathcal{A}} = \overline{\varprojlim \mathcal{A}/P} = \varprojlim \overline{\mathcal{A}/P} = \varprojlim \overline{\mathcal{A}}/P = \varprojlim \overline{\mathcal{A}},$$

contradiction. ∎

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Cosmin Pelea

Babeş–Bolyai University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu nr. 1
RO–3400 Cluj–Napoca, Romania
cpelea@math.ubbcluj.ro