

RINGS ALL WHOSE STABLE RANGE ONE ELEMENTS ARE UNITS

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ABSTRACT. Let $sreg(R)$ denote the set of all the strongly regular elements of a unital ring R , $ureg(R)$ the set of all the unit-regular elements of R and $sr1(R)$ the set of all the stable range one elements of R . It is well-known that $U(R) \cup \{0\} \subseteq sreg(R) \subseteq ureg(R) \subseteq sr1(R)$, where $U(R)$ denotes the group of units of R . In this note we characterize rings for which some of these inclusions become equalities, either individually or collectively.

1. INTRODUCTION

All rings considered here are associative, nonzero and unital, meaning they include an identity element $1 \neq 0$. We denote by $U(R)$ the group of all units of a ring R , by $Id(R)$ the set of all idempotents of R , by $N(R)$ the set of all nilpotent elements of R , by $J(R)$ the Jacobson radical of R and by $reg(R)$ the set of all (von Neumann) regular elements of R .

An element a in R has *left stable range one*, denoted $lsr(a) = 1$, if, whenever $Ra + Rb = R$ for some $b \in R$, there exists $y \in R$ such that $a + yb \in U(R)$. Equivalently, $lsr(a) = 1$ if for every $x \in R$ there exists $y \in R$ such that $a + y(1 - xa) \in U(R)$. Here, y will be called a *unitizer* of a depending on x . *Right stable range one* elements are defined symmetrically.

Recently, the left-right symmetry of stable range one elements was established (see [4]), so the distinction between left and right stable range one elements is no longer necessary. Thus, in the remainder of the paper, we drop the qualifier "left" and refer only to stable range one ($sr1$, for short) elements.

Recall that an element a in a ring R is called *strongly (von Neumann) regular* if $a = a^2R \cap Ra^2$ and *unit-regular* if there exists a unit u such that $aua = a$. Equivalently, a is strongly regular if and only if there exists some $b \in R$ such that $a^2b = a = ba^2$.

We denote by $sreg(R)$ the set of all strongly regular elements of R , by $ureg(R)$ the set of all unit-regular elements of R and by $sr1(R)$ the set of all stable range one elements of R .

It is well known that $U(R) \cup \{0\} \subseteq sreg(R) \subseteq ureg(R) \subseteq sr1(R)$.

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In this note, we characterize rings for which some of these inclusions become equalities, either individually or collectively. Specifically, we determine:

(**A1**) the rings in which every nonzero strongly regular element is a unit,
 (**A2**) the rings in which every unit-regular element is strongly regular,
 and combining these,

(**A**) the rings in which every nonzero unit-regular element is a unit.

Additionally, we provide conditions when we can determine:

(**B**) the rings in which every stable range one element is unit-regular, and combining these,

(**C**) the rings in which every nonzero stable range one element is a unit.

It is easy to verify that properties **A**, **B**, and **C** are left-right symmetric. Clearly, a ring satisfies property **C** if and only if it satisfies both properties **A** and **B**.

For example, since the only sr1 elements of \mathbb{Z} are the units $\{\pm 1\}$ and zero, \mathbb{Z} satisfies property **C**, and therefore also satisfies properties **A** and **B**.

As a nonexample, the ring $\mathbb{M}_2(\mathbb{Z})$ does not satisfy property **C**. In fact, as shown in [2], a matrix $A \in \mathbb{M}_2(\mathbb{Z})$ has sr1 if and only if $\det(A) \in \{-1, 0, 1\}$. However, all matrices with zero determinant (including idempotents and nilpotents) are not units. This example is further explored in Section 2.

Our results are summarized in the following theorem.

Theorem 1.1. (i) a) A ring satisfies property **A1** iff it is connected (i.e., it has only the trivial idempotents 0, 1),

b) A ring satisfies property **A2** iff it is Abelian (i.e., all idempotents are central),

c) A ring has property **A** iff it is connected.

(ii) (a) If a ring has property **B** then it is semiprimitive.

(b) The regular rings satisfy property **B**.

(c) If a commutative ring has property **B** then it is reduced.

(iii) (a) A semipotent ring (e.g., an exchange ring) has property **C** iff it is a division ring

(b) If a ring has property **C** then it is connected (and so indecomposable) and semiprimitive.

(c) If a commutative ring has property **C** then it is reduced.

As it turns out, property **C** is quite strong. For large classes of rings, including semi-potent rings and, in particular exchange rings, property **C** implies that these rings are division rings. The most challenging aspects of these characterizations involve property **B**.

Throughout this discussion, the term "regular" will refer exclusively to von Neumann regular elements and rings. We denote by E_{ij} the $n \times n$ matrix with all entries zero except for the (i, j) -entry, which is 1. Additionally, we use the well-known property (see [1], Lemma 17): finite products of sr1 elements are themselves sr1.

2. THE **A** EQUALITIES

An equivalent condition for an element a in a ring R to be strongly regular (see [8]) is $a = ue = eu$, for some idempotent $e \in R$ and some unit $u \in U(R)$.

Notably, this condition shows that strongly regular elements are unit-regular, since these can also be expressed as products of an idempotent and a unit (not necessarily commuting).

In the next theorem, we characterize the **A** equalities.

Theorem 2.1. (i) *Every nonzero strongly regular element of a ring is a unit (**A1**) iff the ring is connected.*

(ii) *Every unit-regular element of a ring is strongly regular (**A2**) iff the ring is Abelian.*

(iii) *Every nonzero unit-regular element of a ring is a unit (**A**) iff the ring is connected.*

Proof. (i) By the equivalent condition mentioned above, since for idempotent e and unit u , eu has to be a unit, it follows that $e = 1$. The converse also follows from the above mentioned equivalent condition.

(ii) Just recall that in a ring, units commute with idempotents iff the ring is Abelian (see [5], exercise 12.7).

(iii) One way is obvious, since $wreg(R) = Id(R)U(R)$. Conversely, if $U(R) \cup \{0\} = wreg(R)$ then $Id(R) \subseteq wreg(R) = U(R) \cup \{0\}$ so $Id(R) = \{0, 1\}$. \square

Corollary 2.2. *No matrix ring $\mathbb{M}_n(R)$ with $n \geq 2$ satisfies property **A** or property **C**.*

Proof. Indeed, these matrix rings are not connected due to the presence of nontrivial idempotents (like E_{11}), precluding property **A** and so **C**. \square

3. THE PROPERTY **B**

As a first example, since \mathbb{Z} satisfies **C** it also satisfies **B**. As for nonexamples, *any sr1 ring which is not unit-regular does not satisfy property **B**.*

From [3] (see Theorem 4.12) we first recall, stated elementwise

Theorem 3.1. *A regular element has left (or right) stable range one iff it is unit-regular.*

Proof. One way is covered in [3] as follows. Suppose $a = ara$. As $sr(a) = 1$, there is y such that $a + y(1 - ra) = v \in U(R)$. Hence

$$a = ara = av^{-1}[a + y(1 - ra)]ra = av^{-1}ara = av^{-1}a.$$

Conversely, since unit-regular elements are products of an idempotent and a unit - each having sr1 - the multiplicative closure of elements with sr1 implies the claim. \square

Note that Theorem 3.1 can be written as $reg(R) \cap sr1(R) = wreg(R)$, whence property **B** is equivalent to $sr1(R) \subseteq reg(R)$.

As for the reverse inclusion, recall that a ring R was termed *IC* (internally cancellable) if whenever $R_R = A \oplus B = A' \oplus B'$, $A \cong A'$ it follows $B \cong B'$. It is well-known that R is IC iff $\text{reg}(R) = \text{ureg}(R)$. Hence, R is IC iff $\text{reg}(R) \subseteq \text{sr1}(R)$.

From Theorem 3.1, or from the equivalent inclusion above, it follows that

Corollary 3.2. *The regular rings satisfy property **B**.*

As another consequence of the previous corollary, it follows that the matrix rings which have property **B** are precisely the regular ones.

Theorem 3.3. *Let $n \geq 2$ be a positive integer. For any ring R , the following conditions are equivalent.*

- (i) *the matrix ring $\mathbb{M}_n(R)$ has property **B**.*
- (ii) *R is regular.*
- (iii) *$\mathbb{M}_n(R)$ is regular.*

Proof. For (i) \Rightarrow (ii), suppose R is not regular and let $a \notin aRa$ and $b \in R$. Since any multiple of a matrix E_{ij} has sr1 (see for example Proposition 3 (b) in [2]) and $E_{11} + bE_{12}$ is an idempotent, the matrix

$$M =: aE_{11} + abE_{12} = aE_{11}(E_{11} + bE_{12})$$

has sr1, as product of sr1 matrices. However, the equality $MUM = M$, for any matrix $U = [u_{ij}] \in \mathbb{M}_n(R)$, amounts to $a = a(u_{11} + bu_{21})a$, a contradiction which shows that **B** fails for $\mathbb{M}_n(R)$.

The implication (ii) \Rightarrow (iii) is well-known.

The implication (iii) \Rightarrow (i) follows from the previous corollary. \square

It is easy to see that

Proposition 3.4. *The only (unit-)regular element which belongs to the Jacobson radical is zero.*

Since for any ring R , $\text{Id}(R), U(R) \subseteq \text{ureg}(R) \subseteq \text{sr1}(R)$, $J(R) \subseteq \text{sr1}(R)$, $U(R) \cap J(R) = \emptyset$ and $\text{Id}(R) \cap J(R) = \{0\}$, this suggests an important necessary condition for rings that have property **B**.

Proposition 3.5. *If a ring R has property **B** then it is semiprimitive (i.e., $J(R) = 0$).*

Proof. Since $J(R) \subseteq \text{sr1}(R) = \text{ureg}(R)$, by the previous proposition it follows $J(R) = J(R) \cap \text{ureg}(R) = \{0\}$. \square

Indeed, both \mathbb{Z} and the regular rings (including the regular matrix rings) are semiprimitive.

Motivated by the example of the integers, one might ask whether *every ring with only two units necessarily has property **B***. However, the ring \mathbb{Z}_4 , which is both local and commutative, provides a counterexample. Indeed, the nilpotent element $2 \in N(\mathbb{Z}_4)$ has stable range 1 - since for any x , the

element $2 + y(1 - 2x)$ belongs to $U(\mathbb{Z}_4)$ when we choose $y = 1$, making it a unitizer. However, 2 is not unit-regular, as $2 \cdot u \cdot 2 = 0 \neq 2$ for all units u .

The example also shows that *commutative local rings need not satisfy property B*.

Actually more can be proved.

Proposition 3.6. *Let R be a commutative ring. If R satisfies **B** then R is reduced. The converse fails.*

Proof. According to Proposition 3.5, rings that have property **B** have zero Jacobson radical. If the ring is commutative, $N(R) \subseteq J(R)$ so $N(R) = \{0\}$.

To see that the converse fails, it suffices to provide an example of a commutative reduced ring containing a stable range 1 element that is not unit-regular. Consider the localization $\mathbb{Z}_{(2)} = \{\frac{m}{n} \in \mathbb{Q} : 2 \nmid n\}$, a local, commutative, and reduced subring of \mathbb{Q} with stable range 1. The set of non-units in $\mathbb{Z}_{(2)}$ is precisely $2\mathbb{Z}_{(2)} = J(\mathbb{Z}_{(2)})$, the Jacobson radical. The element 2 clearly lies in this radical and is not unit-regular, as there exists no unit u such that $2 \cdot u \cdot 2 = 2$. Thus, $\mathbb{Z}_{(2)}$ does not satisfy property **B** - and consequently, it also fails to satisfy property **C**. \square

Remarks. 1) In the case of commutative rings, this demonstrates that 'local' cannot serve as a sufficient additional condition for 'reduced' to imply property **B**.

2) The rings \mathbb{Z}_n , being Artinian, all have sr1. However, these are not suitable as counterexamples, since \mathbb{Z}_n is unit-regular iff n is square-free iff \mathbb{Z}_n is reduced iff \mathbb{Z}_n has property **B**.

Further

Proposition 3.7. *A direct product has property **B** iff all the components have property **B**.*

Proof. The statement reduces to $ureg(\prod R_i) = \prod ureg(R_i)$ and $sr1(\prod R_i) = \prod sr1(R_i)$, equalities which are verified componentwise. \square

Remark. Motivated by the previous two propositions, one might wonder to what extent subdirect products play a role in characterizing commutative rings that have property **B**. Unfortunately, as well-known examples show, the subdirect products are not preserved under the functors $ureg$ and $sr1$. For instance, consider the subdirect product $R = \{(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_4 : \phi(a) = \eta(b)\}$ with $\phi = 1_{\mathbb{Z}_2}$ and η the natural projection (mod 2), that is, R is the pullback $\{(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_4 : a \equiv b \pmod{2}\}$. Both \mathbb{Z}_2 and \mathbb{Z}_4 have stable range 1, but only \mathbb{Z}_2 is unit-regular. It is easy to show that $(0, 2)$ is not unit-regular in R .

To motivate what follows, we provide a **direct proof** of the converse in Theorem 3.1 - namely, that unit-regular elements have sr1.

Proof. Suppose $a = aua$ with $u \in U(R)$ and x arbitrary in R . We show that $a + (u^{-1} - a)(1 - xa) \in U(R)$, that is, a unitizer for a (independent of x) is $u^{-1} - a$.

It suffices to replace a with auu^{-1} :

$$\begin{aligned} a + (u^{-1} - a)(1 - xa) &= auu^{-1} + (u^{-1} - auu^{-1})(1 - xa) = \\ &= auu^{-1} + (1 - au)u^{-1}(1 - xa) = u^{-1} - (1 - au)u^{-1}xa = \\ &= [1 - (1 - au)u^{-1}x]u^{-1} \in U(R), \text{ since } au \text{ is idempotent and so} \\ &(1 - au)u^{-1}x \text{ is square-zero.} \quad \square \end{aligned}$$

Remarks. 1) Just taking $u = 1$ in the above proof shows that *idempotents have sr1*. That is, a unitizer for an idempotent e is its complementary idempotent $1 - e$.

2) Notice that *the unitizer y provided in the proof above is independent of x* . In general, if a stable range 1 element a has a unitizer y which is independent of x , the unitizer has a special form. If $a + y(1 - xa) \in U(R)$ for every x , this holds also for $x = 0$. Hence $a + y \in U(R)$, that is, $y = v - a$ for a unit v . For a unit-regular element $a = aua$, we have precisely $v = u^{-1}$.

3) Recall that an element in a ring R is *strongly regular* if $a = a^2r = ra^2$ for some $r \in R$. Notice that $ar = ra$ follows (multiplying $a = ra^2$ by r on the right) and so *strongly regular elements are regular*. Consequently, ar is an idempotent and one can easily show that the complementary idempotent $1 - ar$ is a unitizer for a (independent of x).

Indeed, this follows from the equation

$$a + (1 - ar)(1 - xa) = [1 - (1 - ar)(a + 1 - ar)x(ar)](a + 1 - ar) \in U(R)$$

since $(1 - ar)a = a(1 - ar) = 0$, $(1 - ar)(a + 1 - ar)x(ar)$ is square-zero and $a + 1 - ar$ is a unit (with right inverse $ar^2 + 1 - ar$). To check it is also a left inverse, we use $ar^2a = ar$ (which follows as $ar = ra$).

Prompted by the second remark above, one might introduce a new type of sr1 element, in order to characterize the rings that satisfy property **B**

Definition. An element $a \in R$ has *independent unitizer* if there is a unit $v_a \in U(R)$ such that $a + (v_a - a)(1 - xa) \in U(R)$, for every $x \in R$. We denote by $IndUniz(R)$ the set of all elements of R with an independent unitizer. Then

$$ureg(R) \subseteq IndUniz(R) \subseteq sr1(R).$$

To obtain a characterization of the rings that satisfy property **B**, one might refine it into two sub-properties:

B1: Rings R such that $ureg(R) = IndUniz(R)$.

B2 : Rings R such that $IndUniz(R) = sr1(R)$.

4. THE PROPERTY **C**

Let \mathcal{B} be a class of rings which satisfy property **B**. According to Proposition 2.1, *the connected rings from class \mathcal{B} satisfy property **C***.

Obviously, division rings have **C** and \mathbb{Z} has **C**.

Moreover, *the only sr1 rings which have property \mathbf{C} are the division rings.*

Therefore, semilocal rings, unit-regular rings, strongly π -regular rings and exchange rings with Artinian primitive factors *do not satisfy property \mathbf{C}* , whenever these are not division rings.

Since (see [9]) any Artinian ring has sr1, *the only Artinian rings (in particular, finite rings) which have property \mathbf{C} are the division rings.*

We now record some special classes of rings that either possess or fail to possess property \mathbf{C} .

Proposition 4.1. *(i) The connected regular rings have property \mathbf{C} ; however, such rings are precisely the division rings.*

(ii) A polynomial ring $S[x]$ over any commutative domain S has property \mathbf{C} .

(iii) No direct product of rings with property \mathbf{C} has property \mathbf{C} .

(iv) Factor rings of rings with property \mathbf{C} may not themselves have property \mathbf{C} .

(v) The property \mathbf{C} is not Morita invariant.

Proof. (i) For every element a in a regular ring R , the left ideal Ra is generated by an idempotent. If R is also connected, $Ra \in \{0, R\}$, that is $a = 0$ or a has a left inverse. Similarly, if not zero, a has a right inverse and so $a \in U(R)$.

(ii) The set of stable range one elements in $S[x]$ is just $\{0\} \cup U(S)$.

(iii) Being connected, the rings which have property \mathbf{C} are indecomposable.

(iv) Since \mathbb{Z} has property \mathbf{C} , but \mathbb{Z}_n is decomposable whenever n has at least two distinct prime divisors, it follows from (iii) that \mathbb{Z}_n does not have property \mathbf{C} .

More generally, for commutative rings, this failure occurs for the factor ring R/I , whenever R has property \mathbf{C} and I is not a maximal ideal.

(v) As already observed, the matrix rings, not being connected, do not have property \mathbf{C} . \square

Since the elements of the Jacobson radical of any ring have sr1, it follows that

Proposition 4.2. *Only semiprimitive rings (i.e., those with Jacobson radical $J(R) = 0$) can have property \mathbf{C} . Moreover, being connected, such rings are indecomposable.*

In fact, this also follows directly from Proposition 3.5.

Lemma 4.3. *A left-invertible element which is not a unit has not sr1.*

Proof. Suppose $ca = 1 \neq ac$. By definition, $sr(a) = 1$ means that for every $x \in R$ there exists $y \in R$ such that $a + y(1 - xa) \in U(R)$.

Since for $x = c$ we have $y(1 - xa) = 0$ for any y and $a \notin U(R)$, $sr(a) \neq 1$ follows. \square

Clearly, a left-right symmetric statement also holds. We have the following consequence.

Corollary 4.4. *Rings which have property **C** may not be Dedekind finite.*

Recall that an element a of a ring R is called *exchange* (or *suitable*, in [7]) if there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. Among several other equivalent definitions, a ring is called *exchange* if so are all its elements.

The class of exchange rings is broad enough to have attracted significant attention from researchers, particularly when defining new classes of rings. For any newly introduced class, it is common practice to determine whether it is included in the class of the exchange rings and, if so, how they are situated within it.

Recall that a ring is *semi-potent* (also called I_0 -ring by Nicholson in [7]), if every principal left (resp. right) ideal, not contained in the Jacobson radical, contains a nonzero idempotent. For the class of rings that satisfy property **C**, we prove the following result.

Theorem 4.5. *A (non-zero) semi-potent ring has property **C** iff it is a division ring.*

Proof. Assume R has property **C** and is a semi-potent ring. By the previous results (Proposition 2.1 and Proposition 4.2), R has zero Jacobson radical and only trivial idempotents. As R is semi-potent, every nonzero left ideal of R contains a nonzero idempotent. Therefore 0 and R are the only left ideals of R and so R is a division ring. The converse is obvious. \square

Corollary 4.6. *An exchange ring has property **C** iff it is a division ring.*

Proof. We just mention (see [7], Proposition 1.9) that every exchange ring is semi-potent. \square

Since \mathbb{Z} has property **C** and is not exchange nor a division ring, we cannot remove the semi-potent (or exchange) hypothesis from the previous theorem or corollary.

For commutative rings which have property **C**, Proposition 3.6 has the following consequence.

Corollary 4.7. *Any commutative ring that has property **C** is reduced.*

This also follows from Proposition 4.2, because *rings having property **C** are semiprimitive and so semiprime* (see **10.17**, (e) [6]). As in the category of *commutative* rings, *the semiprime rings are the reduced rings*, the statement follows.

In conclusion, we highlight two open questions for future research.

Q1. Characterize the rings that have property **B1** or **B2**, and consequently, property **B**.

Q2. Are the commutative rings that have property **C**, (left) primitive and so actually fields (see **11.8** [6])?

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