Traces of powers of 2×2 matrices over commutative rings.

Grigore Călugăreanu

1 Introduction

There are formulas which give the coefficients of the characteristic polynomial in terms of traces, but we were not able to find in the linear algebra literature, formulas which give the traces of the positive integer powers of 2×2 matrices over commutative rings (and in particular, over domains or even over integers) in terms of the trace and determinant of the given matrix.

Since such formulas are nice and notable we give them in this short note.

2 Prerequisites

0. In the sequel, A denotes a 2×2 matrix over a commutative ring R.

First recall that by Cayley-Hamilton theorem

$$A^2 = \mathrm{Tr}(A) \cdot A - \det(A) I_2 \qquad (*)$$

Computing the trace of both sides we get $\mathrm{Tr}(A^2) = \mathrm{Tr}(A)^2 - 2\det(A)$.

Multiplying (*) by A and computing the traces we obtain $\mathrm{Tr}(A^3) = \mathrm{Tr}(A).\underline{\mathrm{Tr}(A^2)} - \det(A).\mathrm{Tr}(A) = \mathrm{Tr}(A)^3 - 3\mathrm{Tr}(A)\det(A)$.

In general, multiplying (*) by A^{n-2} we obtain a recurrence formula

$$\mathrm{Tr}(A^n) = \mathrm{Tr}(A).\mathrm{Tr}(A^{n-1}) - \det(A).\mathrm{Tr}(A^{n-2}) \qquad (**)$$

This shows that traces of powers may be expressed in terms of the trace and determinant of the initial matrix.

In order to state and prove a general formula for this, we next recall some undergraduate algebra (e.g. see [1], [2], [3]).

A. The recurrence $a_n = aa_{n-1} + ba_{n-2}$, is an *order-2 homogeneous linear recurrence with (two) constant coefficients*, and the standard (known) solution for this recurrence amounts to the *characteristic equation* of the recurrence $r^2 - ar - b = 0$. We solve for r to obtain the two roots λ_1, λ_2: these roots are known as the characteristic roots or eigenvalues of the characteristic equation. Different solutions are obtained depending on the nature of the roots: If these roots are distinct, we have the general solution $a_n = C\lambda_1^n + D\lambda_2^n$, while if they are identical (when $a^2 + 4b = 0$), we have $a_n = C\lambda^n + Dn\lambda^n$. This is the most general solution; the two constants C and D can be chosen based on two given initial conditions a_0 and a_1 to produce a specific solution.

B. For each $k \geq 0$, the *complete symmetric polynomial* is the sum of all monomials of degree k: $h_k(x_1, ..., x_m) = \sum\limits_{d_1+...+d_m=k} x_{d_1}...x_{d_m}$.

In particular $h_0(x_1, ..., x_n) = 1$.

For example, if $m = 2$ and $k = n$, we have $h_n(x_1, x_2) = x_1^n + x_1^{n-1}x_2 + ... + x_1x_2^{n-1} + x_2^n$.

A special case of the *fundamental theorem of symmetric polynomials* refers to such complete symmetric polynomials.

It can be proved that denoting by e_1, e_2, ... e_k the *elementary symmetric polynomials*, the following holds

$$h_k = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{k-1} & e_k \\ 1 & e_1 & e_2 & \dots & e_{k-2} & e_{k-1} \\ 0 & 1 & e_1 & \dots & e_{k-3} & e_{k-2} \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & 1 & e_1 & e_2 \\ 0 & 0 & 0 & 0 & 1 & e_1 \end{vmatrix}.$$

Here we have used the convention that $e_i(x_1, ..., x_m) = 0$ for $i < 0$ or $i > m$.

3 The formula

Theorem 1 *Let R be a commutative ring with identity, $A \in \mathcal{M}_2(R)$ and let n be a positive integer. Then the trace of A^n can be expressed in terms of*

$\mathrm{Tr}(A)$ *and* $\det(A)$ *in the form*

$$\mathrm{Tr}(A^n) = \mathrm{Tr}(A)\begin{vmatrix} \mathrm{Tr}(A) & \det(A) & 0 & ... & 0 & 0 \\ 1 & \mathrm{Tr}(A) & \det(A) & ... & 0 & 0 \\ 0 & 1 & \mathrm{Tr}(A) & ... & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & 1 & \mathrm{Tr}(A) & \det(A) \\ 0 & 0 & 0 & 0 & 1 & \mathrm{Tr}(A) \end{vmatrix}^{(n-1)} -$$

$$-2\det(A)\begin{vmatrix} \mathrm{Tr}(A) & \det(A) & 0 & ... & 0 & 0 \\ 1 & \mathrm{Tr}(A) & \det(A) & ... & 0 & 0 \\ 0 & 1 & \mathrm{Tr}(A) & ... & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & 1 & \mathrm{Tr}(A) & \det(A) \\ 0 & 0 & 0 & 0 & 1 & \mathrm{Tr}(A) \end{vmatrix}^{(n-2)}$$

where the exponent (i) *of each determinant denotes the number of rows (or columns) and* $|*|^{(0)} = 1$.

If R *is a domain,* $\mathrm{char}(R) \neq 2$ *and* $\mathrm{Tr}(A)^2 = 4\det(A)$ *then*

$$\mathrm{Tr}(A^n) = 2(\frac{\mathrm{Tr}(A)}{2})^n.$$

Proof. Consider the recurrence $a_n = aa_{n-1} + ba_{n-2}$ with $a_0 = 2$ and $a_1 = a$. We are interested in finding the general term a_n expressed as a polynomial in a and b over R. By repeatedly replacement, it is clear that such a formula can be found and that this formula *does not depend on the nature of the roots* of the characteristic equation of the recurrence.

Using **A** we distinguish two cases.

Case 1: $\Delta \neq 0$. From the initial conditions $C+D = 2$ and $C\lambda_1+D\lambda_2 = a$, so $C = \dfrac{2\lambda_2 - a}{\lambda_2 - \lambda_1}$ and $D = \dfrac{a - 2\lambda_2}{\lambda_2 - \lambda_1}$.

Thus the general formula becomes

$$a_n = \frac{2\lambda_2 - a}{\lambda_2 - \lambda_1}\lambda_1^n + \frac{a - 2\lambda_2}{\lambda_2 - \lambda_1}\lambda_2^n = a\frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1} + 2b\frac{\lambda_2^{n-1} - \lambda_1^{n-1}}{\lambda_2 - \lambda_1}.$$

Since $\lambda_1 + \lambda_2 = a$ and $\lambda_1\lambda_2 = -b$ (these are precisely the *elementary symmetric polynomials* in two variables), using the fundamental theorem of symmetric polynomials, $\dfrac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1} = \lambda_2^{n-1} + \lambda_2^{n-2}\lambda_1 + ... + \lambda_2\lambda_1^{n-2} + \lambda_1^{n-1}$ can be expressed as a polynomial in a and b.

Now using **B**, these are complete symmetric polynomials and since $\lambda_1 + \lambda_2 = a = \mathrm{Tr}(A)$, and $\lambda_1\lambda_2 = -b = \det(A)$, the statement follows.

Case 2: $\Delta = 0$. We use the second formula, $a_n = C\lambda^n + Dn\lambda^n$. In this case $C = a_0 = 2$ and $(C + D)\lambda = a_1 = a$ which gives $D = \dfrac{a - 2\lambda}{\lambda}$ since $\lambda \neq 0$ (otherwise $a = b = 0$, the trivial zero recurrence). But if $\Delta = 0$ then (if $\mathrm{char}(R) \neq 2$) $\lambda = \dfrac{a}{2}$ and so $D = 0$. Hence $a_n = 2(\dfrac{a}{2})^n$ (notice that if $b \in R$ then $\dfrac{a}{2} \in R$).

Therefore, if $\mathrm{Tr}(A)^2 = 4\det(A)$ then

$$\mathrm{Tr}(A^n) = 2(\frac{\mathrm{Tr}(A)}{2})^n.$$

■

Examples. $\mathrm{Tr}(A^2) = \mathrm{Tr}(A)^2 - 2\det(A)$;

$\mathrm{Tr}(A^3) = \mathrm{Tr}(A).\begin{vmatrix} \mathrm{Tr}(A) & \det(A) \\ 1 & \mathrm{Tr}(A) \end{vmatrix} - 2\det(A) = \mathrm{Tr}(A)^3 - \mathrm{Tr}(A).\det(A) - 2\det(A).\mathrm{Tr}(A) = \mathrm{Tr}(A)^3 - 3\mathrm{Tr}(A).\det(A)$;

$\mathrm{Tr}(A^4) = \mathrm{Tr}(A).\begin{vmatrix} \mathrm{Tr}(A) & \det(A) & 0 \\ 1 & \mathrm{Tr}(A) & \det(A) \\ 0 & 1 & \mathrm{Tr}(A) \end{vmatrix} - 2\det(A).\begin{vmatrix} \mathrm{Tr}(A) & \det(A) \\ 1 & \mathrm{Tr}(A) \end{vmatrix} =$

$= \mathrm{Tr}(A)[\mathrm{Tr}(A)^3 - 2\mathrm{Tr}(A).\det(A)] - 2\det(A)[\mathrm{Tr}(A)^2 - \det(A)] = \mathrm{Tr}(A)^4 - 4\mathrm{Tr}(A)^2\det(A) + 2\det(A)^2$.

Remarks. 1) It is easy to show that replacing $\det(A) = (\dfrac{\mathrm{Tr}(A)}{2})^2$ in the formula obtained in Case 1, we get the special formula in Case 2. So actually (as previously noticed) the formula obtained in Case 1 covers also Case 2, and so *does not depend on the nature of the roots of the characteristic equation.*

2) The form of the general term of the recurrence $a_n = aa_{n-1} + ba_{n-2}$ with $a_0 = 2$ and $a_1 = a$ depends on the parity of n. We obtain

$$a_{2k} = a^{2k} + 2ka^{2k-2}b + s_2^{(k)}a^{2k-4}b^2 + s_3^{(k)}a^{2k-6}b^3 + ... + s_{k-1}^{(k)}a^2b^{k-1} + 2b^k$$

respectively

$$a_{2k+1} = a^{2k+1} + (2k+1)a^{2k-1}b + d_2^{(k)}a^{2k-3}b^2 + d_3^{(k)}a^{2k-5}b^3 + ... + d_{k-1}^{(k)}a^2b^{k-1} + (2k+1)ab^k$$

where the coefficients s_i, d_i satisfy the recurrences

$s_2^{(k)} = s_2^{(k-1)} + 4k - 5$ beginning with $s_2^{(2)} = 2$, $d_2^{(k)} = d_2^{(k-1)} + 4k - 3$ beginning with $d_2^{(1)} = 0$,

$s_3^{(k)} = s_2^{(k-1)} + 2k - 1$, $d_3^{(k)} = s_3^{(k)} + d_2^{(k-1)} = d_2^{(k-1)} + d_2^{(k-2)} + 4k - 5$ and so on.

These recurrences may be easily solved. For instance we obtain

$s_2^{(k)} = k(2k-3)$ for $k \geq 2$, $d_2^{(k)} = 2k^2 - k - 1$ for $k \geq 1$, $s_3^{(k)} = 2k^2 - 5k + 4$ for $k \geq 4$, $d_3^{(k)} = 2(2k^2 - 5k + 3)$ for $k \geq 4$, and so on.

References

[1] Hazewinkel, M. *Recurrence relation*, ed. (2001) Encyclopedia of Mathematics, Springer.

[2] Macdonald, I.G. *Symmetric Functions and Hall Polynomials*, second ed. (1995), Oxford: Clarendon Press.

[3] Stanley, R. P. *Enumerative Combinatorics*, Vol. 2. (1999), Cambridge: Cambridge University Press.