Rings that are unions of known subsets

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1 Elementary

Hereafter, R denotes a nonzero associative ring with identity. As is customary, we denote by $U(R)$ the set of all the units of a ring R, by $N(R)$ the set of all the nilpotents of R and by $Id(R)$ the set of all idempotents of R.

Lemma 1 *For a ring R, $R = Id(R) \cup N(R)$ iff R is Boolean (i.e., $R = Id(R)$).*

Proof. If $R = Id(R) \cup N(R)$ then $U(R) = \{1\}$. Hence R is reduced (i.e., $N(R) = \{0\}$). Otherwise, if $0 \neq t \in N(R)$ then $1 \neq 1 + t \in U(R)$. Therefore $R = Id(R)$, so R is Boolean. Conversely, if R is Boolean, then $R = Id(R) \subseteq Id(R) \cup N(R)$ implies the equality. ■

Lemma 2 *For a ring R, $R = Id(R) \cup U(R)$ iff R is either a division ring or a Boolean ring.*

Proof. It is easy to see that such a ring is reduced (as $N(R) \cap Id(R) = \{0\}$ and $N(R) \cap U(R) = \varnothing$). Hence it is Abelian (so the idempotents are central).

I. If R is indecomposable then R is connected (i.e., $Id(R) = \{0, 1\}$) and so R is a division ring.

II. If R is decomposable, write $R = A \oplus B$. For any $0 \neq a \in A$, $(a, 0) \in R$ cannon be a unit, so it is an idempotent. Hence a is an idempotent and $A = Id(A)$. Thus A is Boolean, so is B a finally so is R.

The converse is obvious as $Id(R) \cap U(R) = \{1\}$. ■

Lemma 3 *For a ring R, $R = U(R) \cup N(R)$ iff R is local with nil Jacobson radical.*

Proof. Let a be a nilpotent element with degree of nilpotency n in R, and r be any other element in R. The product ar cannot be a unit, for if $ar = u$, then $0 = a^n r = a^{n-1}u$, but since a^{n-1} is nonzero by assumption, multiplying it by a unit would be nonzero too. Therefore ar is nilpotent. It follows that $1 - ar$ is a unit for any r, and by a well-known characterization of the Jacobson radical, $a \in J(R)$. So $J(R)$ contains all the nilpotent elements, and consisting of nonunits itself, it is a nil ideal, so it is in fact equal to the set of all nilpotent elements.

By another well-known exercise, when the nonunits are closed under addition (as is the case here, the nonunits = nilpotents are closed under addition), R is a local ring. ∎

Remark. A ring with $R = U(R) \cup N(R)$, is local, but the converse fails.

In a local ring (R, m), it is true that every element is either a unit or a nonunit, and the nonunits are exactly the elements of the maximal ideal m.

But elements of the maximal ideal need not be nilpotent.

Counterexample. Let $R = k[[x]]$, the ring of formal power series over a field k.

R is a nonzero local ring, its maximal ideal is $m = (x)$. The element $x \in m$ is not a unit. But x is not nilpotent (no power of x is zero).

If R is a local *Artinian* ring, then every element is either a unit or nilpotent.

Indeed, in a local Artinian ring, the maximal ideal m is nilpotent, so every nonunit (i.e., every element of m) is nilpotent.

2 Further reading

In [1], p.33 it is proved that

(3.16) **Theorem.** A ring R consists entirely of units, idempotents, and nilpotent elements if and only if it satisfies one (or more) of the following.

(1) R is a boolean ring;

(2) R is a local ring with a nil Jacobson radical;

(3) R is isomorphic to the full matrix ring $\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})$;

(4) R is isomorphic to the ring of a Morita context with zero pairings where both of the underlying rings are $\mathbb{Z}/2\mathbb{Z}$.

References

[1] Nicholas A. Immormino *Clean rings and clean group rings*. PhD Thesis, 2013.