# STRONGLY $\pi$-REGULAR $2 \times 2$ MATRICES OVER INTEGRAL DOMAINS

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In this note, $R$ denotes an associative ring with identity. We denote by $U(R)$ the set of all the units of $R$ and by $N(R)$ the set of all the nilpotents of $R$.

An element $a$ of a ring $R$ is called *clean* if $a = e + u$ with $e^2 = e$ and a unit $u$. If we want to emphasize the idempotent, we say that $a$ is *e-clean*. If this idempotent is trivial (i.e., $e \in \{0, 1\}$ we say $a$ is *trivially* clean. Otherwise, it is *nontrivial(ly)* clean.

A clean element is called *strongly* clean if the idempotent and the unit commute, i.e., $eu = ue$. Notice that an $e$-clean element is strongly clean iff $ea = ae$. Also notice that trivially clean elements are strongly clean.

**Definition**. An element $a$ of a ring $R$ is called *strongly $\pi$-regular* if there exists a positive integer $n$ (depending of $a$) and an element $b \in R$ such that $a^n = a^{n+1}b$. It was proved in [3] (Proposition 2.5) that

**Proposition 1.** *$a$ is strongly $\pi$-regular iff there exist an idempotent $e$ and a unit $u$ such that $a = e + u$, $ea = ae$ and $eae = ea = ae$ is nilpotent.*

Thus, strongly $\pi$-regular elements are a special type of strongly clean elements.

In this short note we characterize the strongly $\pi$-regular $2 \times 2$ matrices over integral domains.

First observe that *every unit in any ring is* 0-clean, so strongly clean and even *$\pi$-regular* since $0 \cdot a = 0$ is nilpotent.

Next, any sum $a = 1 + u$ in any ring is 1-clean, so strongly clean. However, in order to be also strongly $\pi$-regular, since $1 \cdot a = a$, $a$ should be nilpotent. Hence *nilpotents are strongly $\pi$-regular* (indeed, if $a \in N(R)$ then $a = 1 + (a - 1)$ with unit $a - 1$).

Therefore the strongly $\pi$-regular elements in a ring are the units, the nilpotents and the strongly nontrivial $e$-clean elements $a$ for which $ea \in N(R)$.

In what follows, we deal with *nontrivial* clean (i.e., the idempotent $e \notin \{0, 1\}$) $2 \times 2$ matrices over an integral domain $D$.

Since nontrivial idempotent $2 \times 2$ matrices over integral domains are of form $\begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ with $x^2 - x + yz = 0$ (i.e., trace equal 1 and zero determinant), we first recall the following known characterization

**Theorem 2.** *A $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over an integral domain is nontrivial clean iff the system*

$x^2 - x + yz = 0$ *(1)*

$(a - d)x + cy + bz + \det(A) - a \in U(R)$ *(2)*

*with unknowns $x$, $y$, $z$, has at least one solution over $D$.*

*Proof.* Indeed, condition (1) assures that $E = \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ is an idempotent and condition (2) assures that $\det(A-E) \in U(R)$. □

As a simple consequence we have

**Corollary 3.** *A $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over an integral domain is nontrivial strongly clean iff the system*
$x^2 - x + yz = 0$ *(1)*
$(a-d)x + cy + bz + \det(A) - a \in U(R)$ *(2)*
$(a-d)y = b(2x-1)$, $(a-d)z = c(2x-1)$, $bz = cy$ *(3)*
*with unknowns $x$, $y$, $z$, has at least one solution over $D$.*

*Proof.* Indeed, with the notations in the previous proof, condition (3) assures $AE = EA$. □

Finally

**Theorem 4.** *A $2\times2$ nontrivial strongly clean matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over an integral domain is nontrivial strongly $\pi$-regular iff* $\det(A-I_2) - 1 \in U(R)$.

*Proof.* Since $\det(E) = 0$ we also have $\det(AE) = 0$. Thus, for nilpotent $AE$, we only need $\mathrm{Tr}(AE) = ax + cy + bz + d(1-x) = 0$.

Hence $(a-d)x + cy + bz + d = 0$ and using (2), $\det(A) - \mathrm{Tr}(A) \in U(R)$. Equivalently, $\det(A-I_2) - 1 \in U(R)$. □

**Remarks**. 1) In order to have nilpotent matrices characterized by the vanishing of all coefficients in the characteristic polynomial, excepting the first, we do not need $R$ to be an integral domain.

Indeed, as early as 1973, G. Almkvist proved that over any commutative ring $R$, an $n \times n$ matrix is nilpotent iff all coefficients in the characteristic polynomial, excepting the first, are nilpotent.

Therefore, if $R$ is any *reduced commutative* ring, an $n \times n$ matrix is nilpotent iff all coefficients in the characteristic polynomial, excepting the first, equal zero (i.e., the characteristic polynomial is $t^n$). Hence, the above theorem holds over any reduced commutative ring.

2) In the first theorem, we have used the fact that any nontrivial idempotent $2\times2$ matrix $E$ is characterized, over any *integral domain*, by $\det(E) = 0$ and $\mathrm{Tr}(E) = 1$. If $R$ is only a commutative ring, we can only use Cayley-Hamiton's theorem, which if $E^2 = E$ amounts to $[\mathrm{Tr}(E) - 1]E = \det(E)I_2$, not necessarily $\det(E) = 0$ and $\mathrm{Tr}(E) = 1$.

**Example**. Over $\mathbb{Z}_{12}$ consider $E = 4I_2$. Since $4^2 = 4$, $E^2 = E$ is nontrivial idempotent. However, $\mathrm{Tr}(E) = 8$, $\det(E) = 4$ and $[\mathrm{Tr}(E) - 1]E = 4I_2 = \det(E)I_2$.

3) $\det(A-I_2) - 1 \in U(R)$ means that $\det(A-I_2)$ is quasiregular in $R$.

Since all conditions in Proposition 1 are invariant to conjugation, strongly $\pi$-regular elements (and in particular, square matrices) are invariant to conjugation (resp. similarity). Also notice that the rôle of the entries $a$ and $d$ is symmetric since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ are similar under $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Following Steger [4], we say that a ring $R$ is an *ID* ring if every idempotent matrix over $R$ is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

In the sequel we assume $R$ is an ID ring. Hence every nontrivial idempotent $2 \times 2$ matrix is similar to $E_{11}$ (or $E_{22}$; these two are conjugate under $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$). Therefore, in order to give another characterization of nontrivial strongly $\pi$-regular $2 \times 2$ matrices over ID rings, up to similarity, it suffices to describe the $E_{11}$-strongly $\pi$-regular matrices, that is, to take $x = 1$, $y = z = 0$ in the above characterizations. This way we obtain

**Theorem 5.** *Up to similarity, a* $2 \times 2$ *matrix* $A$ *over a commutative ID ring* $R$*,*

*(i) is nontrivial clean iff* $\det(A) - d \in U(R)$ *or* $\det(A) - d \in U(R)$*;*

*(ii) is nontrivial strongly clean iff* $b = c = 0$ *(i.e.,* $A$ *is diagonal) and* $(a-1)d \in U(R)$ *or* $a(d-1) \in U(R)$*;*

*(iii) is nontrivial strongly* $\pi$*-regular iff* $A$ *is diagonal and both* $(a-1)d, ad-a-d \in U(R)$ *or* $a(d-1), ad-a-d \in U(R)$*.*

*Proof.* Special case of Theorem 2, for (i), of Corollary 3, for (ii), and for Theorem 4, for (iii). $\square$

**Examples**. 1) Suppose $U(R) = \{1\}$ (e.g., char$R = 2$). Then, up to similarity, only $E_{22}$ (or $E_{11}$) is strongly $\pi$-regular over $R$.

2) Suppose $U(R) = \{\pm 1\}$ (e.g., $R = \mathbb{Z}$). A matrix $E + U$ is strongly clean (see also [2]) iff it is either of form $E \pm I_2$ or else of form $E \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, with diagonal idempotent $E$ (i.e. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b \in \{0, 1\}$).

Up to similarity, only $\pm E_{22}$ (or $\pm E_{11}$) or $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are $\pi$-regular over $R$.

3) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = E_{11} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is (nontrivial) strongly clean (see also [2]) but not strongly $\pi$-regular.

## References


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