Strongly nil-clean rings are Dedekind finite

George Bergman

October 3, 2025

A ring R is called *clean* if every element is a sum of an idempotent and a unit, and *strongly clean* if the idempotent and the unit commute.

A ring R is called *nil-clean* if every element is a sum of an idempotent and a nilpotent, and *strongly nil-clean* if the idempotent and the unit commute.

It is easy to check that every (strongly) nil-clean ring is (strongly) clean.

A longstanding open question is whether every strongly clean ring is Dedekind-finite (i.e., for any $a, b \in R$, if $ab = 1$ then $ba = 1$).

Since every strongly nil-clean ring is strongly clean, an apparently easier question would be whether every strongly nil-clean ring is Dedekind- finite.

In what follows we present an affirmative answer, as (privately) communicated by George Bergman.

Theorem 1 *Every strongly nil-clean ring is Dedekind finite.*

Proof. In a non-Dedekind finite ring, if we take elements x, y such that $xy = 1$ but $yx \neq 1$, we see that x, though not left invertible, is not a left zero-divisor. On the other hand, I claim that *a strongly nil-clean element of any ring is either 2-sided invertible or a 2-sided zero-divisor.*

To see this, consider a strongly nil-clean element $e + t$. If $e = 1$, then this is 2-sided invertible. On the other hand, suppose $e \neq 1$, and take an n such that $t^n = 0$.

Then expanding $(e + t)^n$, the summand t^n is zero, while all the other summands have a factor e, and since e and t commute, these are all annihilated both on the right

and the left by the nonzero element $1 - e$. Hence $(e + t)^n$ is a 2-sided zero-divisor (possibly equal to 0), and it is easy to deduce that $e + t$ is also a 2-sided zero-divisor.

(Detail: If $(e+t)^m$ $(m \geq 0)$ is the largest power of $e+t$ that is not annihilated by $1 - e$, then $e + t$ annihilates $(1 - e)(e + t)^m$.) ■